On a vector-valued singular perturbation problem on the sphere

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1 Introduction

Let Γ_1 and Γ_2 be two disjoint smooth, simple closed curves in $\mathbb{R}^2$ of lengths $l(\Gamma_1)$ and $l(\Gamma_2)$, respectively, such that Γ_1 lies inside Γ_2 and the origin 0 lies inside Γ_1. Let $W : \mathbb{R}^2 \to [0, \infty)$ be a smooth function (i.e., at least of class C^4) satisfying

$$W > 0 \text{ on } \mathbb{R}^2 \setminus (\Gamma_1 \cup \Gamma_2) \text{ and } W = 0 \text{ on } \Gamma_1 \cup \Gamma_2, . \tag{H_1}$$

Since W attains its minimal value zero on $\Gamma_1 \cup \Gamma_2$ we have clearly $W_n = 0$ on Γ_j, $j = 1, 2$, where W_n denotes the derivative in the direction of the exterior normal to Γ_j. We assume then that we are in the generic case, i.e., that

$$W_{nn} > 0 \text{ on } \Gamma_1 \cup \Gamma_2 \,. \tag{H_2}$$

Finally, we add the following coercivity assumption on the behavior of W at infinity: there exist constants $R_0 > 0$ and $C_0 > 0$ such that

$$W(x) \geq C_0|x| \text{ for } |x| \geq R_0 \,. \tag{H_3}$$

Let G be either a bounded smooth domain in $\mathbb{R}^N$, or a smooth N-dimensional manifold. For each $\varepsilon > 0$ consider the energy functional

$$E_\varepsilon(u) = \int_G |\nabla u|^2 + \frac{W(u)}{\varepsilon^2} \tag{1.1}$$

for $u \in H^1(G, \mathbb{R}^2)$. Let R_c be a positive number such that the circle $S_{R_c} = \{|x| = R_c\}$ *separates* the two curves Γ_1 and Γ_2. We shall assume w.l.o.g. that Γ_1 lies inside S_{R_c} which

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lies inside Γ_2. The number R_c represents the constraint in the following minimization problem that we shall study:

$$\min\{E_\varepsilon(u)\,:\,u\in H^1(G,\mathbb{R}^2),\,\fint_G |u|=R_c\}\,, \tag{P_ε}$$

where $\fint_G |u| := \frac{1}{\mu(G)}\int_G |u|$. Denoting by u_ε a minimizer in (P_ε), we are interested in the asymptotic behavior of the minimizers $\{u_\varepsilon\}$ and their energies $E_\varepsilon(u_\varepsilon)$, as ε goes to 0.

A first study of this problem was carried out by Sternberg [7]. He proved a Γ-convergence result, which has the following consequences:

(i) If a subsequence $\{u_{\varepsilon_n}\}$ converges to a limit u_0, then u_0 belongs to the set

$$\mathcal{S} := \{u\in BV(G,\Gamma_1\cup\Gamma_2)\,:\,\fint_G |u|=R_c\}, \tag{1.2}$$

and

$$\mathrm{Per}_G\{u_0\in\Gamma_1\}=\min\{\mathrm{Per}_G\{u\in\Gamma_1\}\,:\,u\in\mathcal{S}\}\,. \tag{1.3}$$

(ii) The asymptotic expansion for the energy $E_\varepsilon(u_\varepsilon)$, as $\varepsilon\to 0$, is

$$\varepsilon E_\varepsilon(u_\varepsilon)=2D\min_{u\in\mathcal{S}}\mathrm{Per}_G\{u\in\Gamma_1\}+o(1)\,. \tag{1.4}$$

We refer to the books [5, 1] for the definition of the perimeter and other notions from the theory of BV-functions, that we shall use in the sequel.

The constant D which appears in (1.4) is a certain "distance" between the two curves Γ_1 and Γ_2 which we shall now define. First, for any pair of points $x,y\in\mathbb{R}^2$ we set

$$d_W(x,y)=\inf_{\substack{\gamma\in\mathrm{Lip}([0,1],\mathbb{R}^2),\\ \gamma(0)=0,\gamma(1)=y}} L(\gamma)\,, \tag{1.5}$$

where

$$L(\gamma)=\int_0^1 \big(W(\gamma(t))\big)^{1/2}\,|\gamma'(t)|\,dt\,. \tag{1.6}$$

Then, we define

$$D:=\inf_{\substack{\gamma\in\mathrm{Lip}([0,1],\mathbb{R}^2),\\ \gamma(0)\in\Gamma_1,\gamma(1)\in\Gamma_2}} L(\gamma)\,. \tag{1.7}$$

It was proved by Sternberg (see [7, Lemma 9]) that there exists a geodesic $\underline{\gamma}$ realizing the infimum D in (1.7). There may be of course more than one such geodesic; their number may be infinite (as is the case of Γ_1,Γ_2 which are concentric circles and W radially symmetric). We denote the set of all these geodesics by

$$\mathcal{G}=\{\underline{\gamma}^{(i)}\,:\,i\in\mathcal{I}\}\,, \tag{1.8}$$

where $\mathcal{I}$ is some set of indices. For each $i \in \mathcal{I}$ we denote by $\zeta_1^{(i)} = \underline{\gamma}^{(i)}(0) \in \Gamma_1$ and $\zeta_2^{(i)} = \underline{\gamma}^{(i)}(1) \in \Gamma_2$ the endpoints of the geodesic $\underline{\gamma}^{(i)}$, and then set $Z_j = \{\zeta_j^{(i)}\}_{i\in\mathcal{I}}$, for $j = 1, 2$.

The results of Sternberg left some important questions unresolved:
(1) Existence of a converging subsequence is not known, i.e., a compactness result is missing.
(2) Even if we assume convergence of a subsequence towards a limit u_0, which is a map in $\mathcal{S}$ satisfying (1.3), we cannot say *where* on $\Gamma_1 \cup \Gamma_2$, u_0 takes its values.

In [3] we made some progress on this problem in the case where G is a domain in $\mathbb{R}^N$. In particular, we demonstrated the major role played by the geometry of G in determining the asymptotic behavior of $\{u_\varepsilon\}$. In fact, when G is *convex*, and under some additional technical assumptions, the limit u_0 takes only two values, one in Γ_1 and the other one in Γ_2. On the other hand, when G is *nonconvex*, the limit u_0 may be more complicated (i.e., the restriction of u_0 to $\{x \in G : u_0(x) \in \Gamma_j\}$, $j = 1, 2$, is not necessarily identically constant). However, there is no complete answer to the above questions in general.

In the present article we shall concentrate on the special case where G is the sphere S^2. Thanks to the symmetry properties of the minimizers $\{u_\varepsilon\}$ in this case, we are able to give a quite complete analysis for both the asymptotic behavior of the minimizers and their energies. We believe that the case of the sphere will give some indication on the expected behavior of the minimizers in the case of a convex domain G. First, notice that from the symmetrization method of O. Lopes, see [6][Theorem IV.3], it follows that for each ε, the minimizer u_ε is axially symmetric with respect to some axis. We shall assume in the sequel, without loss of generality, a common axis of symmetry for $\{u_\varepsilon\}$, i.e.,

$$\text{each } u_\varepsilon \text{ is symmetric w.r.t. the } \boldsymbol{e}_3 \text{ axis.} \tag{1.9}$$

In view of (1.9) we can view each u_ε as a function of a single variable ϕ, i.e.,

$$u_\varepsilon = u_\varepsilon(\phi), \quad \phi \in [-\pi/2, \pi/2]\,. \tag{1.10}$$

Next we introduce some notation needed in order to state our results. Set

$$m_j = \min_{x\in\Gamma_j} |x| \quad \text{and} \quad M_j = \max_{x\in\Gamma_j} |x| \ (j = 1, 2)\,, \tag{1.11}$$

and

$$\mathcal{M}_j = \{x \in \Gamma_j : |x| = m_j\}\,,\ j = 1, 2\,.$$

We shall also assume the following:

$$\text{both } \mathcal{M}_1 \text{ and } \mathcal{M}_2 \text{ consist of a finite number of points}\,. \tag{H_4}$$

Consider any $v \in \mathcal{S}$ and let $G_j = \{x \in S^2 \,:\, v(x) \in \Gamma_j\}$, $j = 1,2$. Denoting by μ the standard measure on the sphere, we have

$$\mu(G_1)m_1 + \mu(G_2)m_2 \le \mu(S^2)R_c = \int_{S^2} |v| \le \mu(G_1)M_1 + \mu(G_2)M_2\,, \tag{1.12}$$

Clearly, (1.12) and (1.11) imply that

$$\frac{m_2 - R_c}{m_2 - m_1} \le \mu(G_1) \le \frac{M_2 - R_c}{M_2 - M_1}\,. \tag{1.13}$$

Recall now the well-known solution to the isoperimetric problem on the sphere. For a given value $t \in (0, 4\pi)$, the domain on S^2 of area surface t with minimal perimeter is a disk on the sphere with surface t. The perimeter of this disk is given by the function

$$I(t) = \sqrt{4\pi t - t^2}\,, \tag{1.14}$$

which is a concave function on $(0, 4\pi)$, symmetric about the middle point 2π. Therefore, from (1.13) we deduce that in order to obtain G_1 with minimal perimeter we must have:

$$\text{either} \quad (i)\, \mu(G_1)/\mu(S^2) = \frac{m_2 - R_c}{m_2 - m_1} \quad \text{or} \quad (ii)\, \mu(G_1)/\mu(S^2) = \frac{M_2 - R_c}{M_2 - M_1}\,. \tag{1.15}$$

In order to know which of the two possibilities in (1.15) is preferable we should check which possibility realizes the minimum in $\min\left(\frac{m_2 - R_c}{m_2 - m_1}, 1 - \frac{M_2 - R_c}{M_2 - M_1}\right)$. Without loss of generality, we shall assume in the sequel that possibility (i) holds in (1.15), i.e., that

$$\alpha := \frac{m_2 - R_c}{m_2 - m_1} < 1 - \frac{M_2 - R_c}{M_2 - M_1}\,. \tag{1.16}$$

For α as defined in (1.16), we denote by $\phi_0 \in (0, \pi/2)$ the angle satisfying

$$2\pi\Big(\int_{\phi_0}^{\pi/2} \cos\phi \, d\phi\Big)/4\pi = \alpha\,. \tag{1.17}$$

In other words, the area of the disc on the sphere given, in spherical coordinates, by $\{(\theta, \phi) \,:\, \phi \in [\phi_0, \pi/2]\}$, is $4\pi\alpha$. Moreover, thanks to (1.16) we have:

$$(1 - \sin\phi_0)m_1 + (1 + \sin\phi_0)m_2 = 2R_c\,. \tag{1.18}$$

Finally, on each of the curves Γ_j, $j = 1,2$ we introduce a kind of a distance function to the set $\mathcal{M}_j$ by

$$\delta_j(x, \mathcal{M}_j) = \inf\Big\{\int_0^\infty \big(|w'|^2 + |w| - m_j\big)\, dt \,:\, w \in H^1([0,\infty), \Gamma_j) \text{ s.t. } w(0) = x\Big\}\,,\ x \in \Gamma_j\,. \tag{1.19}$$

Note that the minimization problem in (1.19) is actually scalar, since w takes its values on a curve. In the next lemma we give some of the properties of the solution to (1.19).

Lemma 1.1. *For any $x \in \Gamma_j$ there exists a minimizer w_j that realizes the minimum in* (1.19)*. Moreover, there exists a point $y = y(x) \in \mathcal{M}_j$ such that $\lim_{t\to\infty} w_j(t) = y$ and*

$$\frac{\delta_j(x, \mathcal{M}_j)}{2} = \int_0^\infty |w_j'|^2\, dt = \int_0^\infty (|w_j| - m_j)\, dt\,. \tag{1.20}$$

Proof. The analysis of problem (1.19) is identical to that of scalar problem with a one-well potential. Therefore, all the the assertion of the lemma are known and standard. □

Now we are in position to state our main result.

Theorem 1. *Assume that W satisfies hypotheses $(H_1) - (H_4)$,* (1.16) *and that* (1.9) *holds, i.e., each u_ε is of the form* (1.10)*. Then, up to replacing each u_ε by $u_\varepsilon \circ \mathcal{R}$, where $\mathcal{R}$ is the reflection w.r.t. the $(\boldsymbol{e}_1, \boldsymbol{e}_2)$-plane, we have:*

(i) For a subsequence

$$u_{\varepsilon_n} \to \chi_{(-\pi/2,\phi_0)} x^{(2)} + \chi_{(\phi_0,\pi/2)} x^{(1)}\,, \tag{1.21}$$

uniformly on $(-\frac{\pi}{2}+\delta, \phi_0-\delta) \cup (\phi_0+\delta, \frac{\pi}{2}-\delta)$, for every $\delta > 0$, with $x^{(j)} \in \mathcal{M}_j, j = 1, 2$.
(ii) The asymptotic expansion of the energy is given by

$$\frac{E_\varepsilon(u_\varepsilon)}{2\pi} = \cos\phi_0\Big(\frac{2D}{\varepsilon} + \sqrt{2\beta D}\frac{K}{\varepsilon^{1/2}}\Big) + o(\varepsilon^{-\frac{1}{2}}), \tag{1.22}$$

where

$$K = \min\{\delta_1(\zeta_1^{(i)}, \mathcal{M}_1) + \delta_2(\zeta_2^{(i)}, \mathcal{M}_2) \,:\, i \in \mathcal{I}\}\,, \tag{1.23}$$

and $\beta := \frac{\tan\phi_0}{m_2 - m_1}$.

Note that Theorem 1 provides us with a *criterion* for identifying the limit in (i). Indeed, the points $x^{(1)}, x^{(2)}$ that realize the limit form the pair of points, one in $\mathcal{M}_1$ and the other one in$\mathcal{M}_2$, which are the closest (in an appropriate sense) to a geodesic in $\mathcal{G}$. It is very likely that most of the results of this paper can be extended by the same techniques to more general potentials W, for example, with zero set consisting of two compact surfaces in $\mathbb{R}^3$ separated by a sphere of radius R_c.

2 A first upper-bound

We shall first introduce some more notation that will be needed in the sequel. Using d_W we define the corresponding distance functions to the curves Γ_1, Γ_2 by

$$\Psi_j(\zeta) = d_W(\zeta, \Gamma_j) := \inf_{x\in\Gamma_j} d_W(\zeta, x)\,,\ j = 1, 2\,. \tag{2.1}$$

We also set

$$\widetilde{\Psi} = \min(\Psi_1, \Psi_2)\,. \tag{2.2}$$

It is well known (c.f. [7, 4]) that for $j = 1, 2$, $\Psi_j \in \mathrm{Lip}(\mathbb{R}^2)$ is a solution of the eikonal-type equation

$$|\nabla\Psi_j(\zeta)|^2 = W(\zeta) \quad \text{a.e. on } \mathbb{R}^2 \,. \tag{2.3}$$

It was further shown in [2] that Ψ_j is regular in a neighborhood of Γ_j, i.e.,

$$\exists d_0 > 0 \text{ s.t. } \Psi_j \text{ is of class } C^2 \text{ in } \{x : \Psi_j(x) < d_0\} \,, \tag{2.4}$$

for $j = 1, 2$. Moreover, we have

$$\Psi_j(x) \sim W(x) \sim \mathrm{dist}^2(x, \Gamma_j) \quad \text{on } \{x : \Psi_j(x) < d_0\} \,. \tag{2.5}$$

Clearly, $d_W(x_1, x_2) = D$ for every $x_1 \in \Gamma_1$ and $x_2 \in \Gamma_2$. In order to identify the end points of geodesics from $\mathcal{G}$ we shall use yet another distance function between points from Γ_1 and Γ_2. We denote by Ω the domain lying between Γ_1 and Γ_2.

Recall that in [2] it was shown that for each $x_0 \in \Gamma_j$, $j = 1, 2$, there is a curve $\mathcal{G}^{(j)}_{x_0}$ parametrized on $(-\infty, t(x_0)]$ which satisfies the equation

$$\begin{cases} \dot{\mathcal{G}}^{(j)}_{x_0} = \nabla\Psi(\mathcal{G}^{(j)}_{x_0}), \\ \mathcal{G}^{(j)}_{x_0}(-\infty) = x_0 \,, \end{cases} \tag{2.6}$$

such that the union of these curves (over all $x_0 \in \Gamma_j$) covers without intersections $\{\Psi_j \leq d_0\}^+$, which is the part of $\{\Psi_j \leq d_0\}$ lying between Γ_1 and Γ_2 (see (2.4)). Similarly, the remaining parts $\{\Psi_j \leq d_0\}^-$, $j = 1, 2$, can be covered by an analogous family of curves. Using these curves we can now define a projection map $\tilde{s}_j$ from $\{\Psi_j \leq d_0\}$ to Γ_j which associates to each $x \in \{\Psi_j \leq d_0\}$ the unique point $x_0 = \tilde{s}_j(x) \in \Gamma_j$ for which the curve $\mathcal{G}^{(j)}_{x_0}$ passes by x.

For any small $\delta > 0$ set

$$\Omega_\delta = \{x \in \Omega : \widetilde{\Psi}(x) > \delta\} \,. \tag{2.7}$$

For any $x_1 \in \Gamma_1$ and $x_2 \in \Gamma_2$ we let $y_1, y_2 \in \partial\Omega_\delta$ be the points determined by $d_W(y_j, x_j) = \delta$, $j = 1, 2$. We then define

$$d^{(\delta)}_W(x_1, x_2) = 2\delta + \inf_{\substack{\gamma \in \mathrm{Lip}([0,1], \bar{\Omega}_\delta), \\ \gamma(0) = y_1, \gamma(1) = y_2}} L(\gamma) \,, \tag{2.8}$$

where $L(\gamma)$ is defined in (1.6). Note that for each $\delta > 0$ we have $d^{(\delta)}_W(x_1, x_2) \geq D$ for every $x_1 \in \Gamma_1$ and $x_2 \in \Gamma_2$ with equality if and only if x_1 and x_2 are the end points of a geodesic in $\mathcal{G}$.

The main result of this section is a simple upper-bound for the energy. It is not optimal, but it gives the exact first term in the energy expansion (of the order $\frac{1}{\varepsilon}$), and the order $\varepsilon^{-1/2}$ of the next term.

Proposition 2.1. *There exists a constant* $C_1 > 0$ *such that*

$$E_\varepsilon(u_\varepsilon) \le 2\pi\cos\phi_0\Big(\frac{2D}{\varepsilon} + \frac{C_1}{\varepsilon^{1/2}}\Big) \tag{2.9}$$

Proof. We shall construct a function $v_\varepsilon = v_\varepsilon(\phi)$ which satisfies the constraint in (P_ε), i.e.,

$$\int_{-\pi/2}^{\pi/2} |v_\varepsilon|\cos\phi\, d\phi = 2R_c\,, \tag{2.10}$$

as well as the bound (2.9). Fix two points $x_j \in \mathcal{M}_j$, $j = 1, 2$ and a geodesic $\gamma := \underline{\gamma}^{(i_0)} \in \mathcal{G}$ (see (1.8)) of length $L = \mathrm{len}(\gamma)$. We shall denote for short by p_1 and p_2 the endpoints $\zeta_1^{(i_0)}$ and $\zeta_2^{(i_0)}$ of γ, respectively, so that choosing the arclength parametrization for γ we have, $\gamma(0) = p_1$ and $\gamma(L) = p_2$. The following function $z(s)$ will be used in a choice of a certain parametrization of the curve γ. It is defined as the solution of the ODE:

$$\frac{dz}{ds} = \sqrt{W(\gamma(z(s)))}\,, \quad z(0) = L/2\,. \tag{2.11}$$

It is easy to see that z is defined on the whole real line and satisfies

$$\lim_{s\to-\infty} z(s) = 0 \text{ and } \lim_{s\to\infty} z(s) = L\,.$$

Furthermore, since

$$\sqrt{W(\gamma(z(s)))} \sim d(\gamma(z(s)),\Gamma_1) \sim z(s) \qquad \text{as } s \to -\infty$$

and

$$\sqrt{W(\gamma(z(s)))} \sim d(\gamma(z(s)),\Gamma_2) \sim L - z(s) \qquad \text{as } s \to \infty\,,$$

where d stands for the euclidean distance, we have:

$$0 \le z(s) \le C_2 e^{c_1 s} \qquad \text{as } s \to -\infty\,, \tag{2.12}$$

$$0 \le L - z(s) \le C_2 e^{-c_2 s} \qquad \text{as } s \to \infty\,, \tag{2.13}$$

for some positive constants c_1, c_2 and C_1, C_2. From (2.11) we deduce that for any function $v(\phi)$ defined on an interval $[\phi_1, \phi_2]$ by

$$v(\phi) = \gamma(z(c \pm \frac{\phi}{\varepsilon}))\,,$$

we have

$$\int_{\phi_1}^{\phi_2}\Big(|v'|^2 + \frac{W(v)}{\varepsilon^2}\Big)\cos\phi\, d\phi = \frac{2}{\varepsilon^2}\int_{\phi_1}^{\phi_2} W(v)\cos\phi\, d\phi = \frac{2}{\varepsilon}\int_{\phi_1}^{\phi_2}\sqrt{W(v)}|v'(\phi)|\cos\phi\, d\phi\,. \tag{2.14}$$

Set $\bar{\phi} = \phi_0 - \tau\varepsilon^{1/2}$, with $\tau = \tau_\varepsilon$ to be determined later. We define $v_\varepsilon(\phi)$ on $[-\pi/2, \pi/2]$ as follows:

(i) On $[\bar{\phi} + \varepsilon^{1/2}, \pi/2]$ we set $v_\varepsilon \equiv x^{(1)}$.
(ii) On $[\bar{\phi}, \bar{\phi} + \varepsilon^{1/2}]$ we follow the curve Γ_1 in a constant velocity from $p_1\,(= \zeta_1^{(i_0)})$ to $x^{(1)}$ (in the shortest way between the two possibilities).
(iii) On $[\bar{\phi} - \varepsilon, \bar{\phi}]$ we let v_ε be the linear function which equals to $\gamma(z(-\frac{\ln 1/\varepsilon}{c_1}))$ at $\phi = \bar{\phi} - \varepsilon$ and to p_1 at $\phi = \bar{\phi}$.
(iv) On $[\bar{\phi} - \varepsilon - (\frac{1}{c_1} + \frac{1}{c_2})\varepsilon \ln \frac{1}{\varepsilon}, \bar{\phi} - \varepsilon]$ we set

$$v_\varepsilon(\phi) = \gamma\Big(z\big(\frac{\bar{\phi} - \varepsilon - \phi}{\varepsilon} - \frac{\ln 1/\varepsilon}{c_1}\big)\Big)\,.$$

(v) On $[\bar{\phi} - 2\varepsilon - (\frac{1}{c_1} + \frac{1}{c_2})\varepsilon \ln \frac{1}{\varepsilon}, \bar{\phi} - \varepsilon - (\frac{1}{c_1} + \frac{1}{c_2})\varepsilon \ln \frac{1}{\varepsilon}]$ we let v_ε to be the linear function which equals to $p_2\,(= \zeta_2^{(i_0)})$ at $\phi = \bar{\phi} - 2\varepsilon - (\frac{1}{c_1} + \frac{1}{c_2})\varepsilon \ln \frac{1}{\varepsilon}$ and to $\gamma(z(\frac{\ln 1/\varepsilon}{c_2}))$ at $\phi = \bar{\phi} - \varepsilon - (\frac{1}{c_1} + \frac{1}{c_2})\varepsilon \ln \frac{1}{\varepsilon}$.
(vi) On $[\bar{\phi} - 2\varepsilon - (\frac{1}{c_1} + \frac{1}{c_2})\varepsilon \ln \frac{1}{\varepsilon} - \varepsilon^{1/2}, \bar{\phi} - 2\varepsilon - (\frac{1}{c_1} + \frac{1}{c_2})\varepsilon \ln \frac{1}{\varepsilon}]$ we follow the curve Γ_2 in a constant velocity from $x^{(2)}$ to p_2 (using the shortest way among the two possibilities).
(vii) On $[-\pi/2, \bar{\phi} - 2\varepsilon - (\frac{1}{c_1} + \frac{1}{c_2})\varepsilon \ln \frac{1}{\varepsilon} - \varepsilon^{1/2}]$ we set $v_\varepsilon \equiv x^{(2)}$.

We shall next compute the contribution to the integral of $|v_\varepsilon|$ from each of the intervals (i)–(vii) and show that a bounded solution for τ is determined by the constraint (2.10). Note first that

$$\begin{aligned}\Big(\int_{\bar{\phi}+\varepsilon^{1/2}}^{\pi/2} \cos\phi\Big)|x^{(1)}| &= \big(1 - \sin(\phi_0 - (\tau - 1)\varepsilon^{1/2})\big)|x^{(1)}| \\ &= \big(1 - \sin\phi_0 + (\tau - 1)\varepsilon^{1/2}\cos\phi_0\big)|x^{(1)}| + O(\varepsilon)\,. \quad (2.15)\end{aligned}$$

Here and in the rest of the proof we denote by $O(f(\varepsilon))$ a function $g(t,\varepsilon)$ satisfying $|g(t,\varepsilon)| \le C(t)f(\varepsilon)$ with $C(t)$ bounded for bounded t. Next, it is easy to verify that

$$\int_{\bar{\phi}}^{\bar{\phi}+\varepsilon^{1/2}} |v_\varepsilon|\cos\phi = \varepsilon^{1/2}\lambda_1(x^{(1)}, p_1)\cos\phi_0 + O(\varepsilon)\,, \qquad (2.16)$$

with

$$\lambda_1(x^{(1)}, p_1) = \frac{\int_{J_1(x^{(1)},p_1)} |z|}{d_{\Gamma_1}(x^{(1)}, p_1)}\,,$$

where $J_1(x^{(1)}, p_1)$ denotes the (shortest) segment of Γ_1 between $x^{(1)}$ and p_1, whose length is denoted then by $d_{\Gamma_1}(x^{(1)}, p_1)$. Clearly

$$\int_{\bar{\phi}-\varepsilon}^{\bar{\phi}} |v_\varepsilon|\cos\phi = O(\varepsilon)\,. \qquad (2.17)$$

Similarly, for the intervals in (iv) and (v) we find, respectively,

$$\int_{\bar\phi-\varepsilon-(\frac{1}{c_1}+\frac{1}{c_2})\varepsilon\ln\frac{1}{\varepsilon}}^{\bar\phi-\varepsilon} |v_\varepsilon|\cos\phi = O(\varepsilon\ln\frac{1}{\varepsilon})\,, \tag{2.18}$$

and

$$\int_{\bar\phi-2\varepsilon-(\frac{1}{c_1}+\frac{1}{c_2})\varepsilon\ln\frac{1}{\varepsilon}}^{\bar\phi-\varepsilon-(\frac{1}{c_1}+\frac{1}{c_2})\varepsilon\ln\frac{1}{\varepsilon}} |v_\varepsilon|\cos\phi = O(\varepsilon)\,. \tag{2.19}$$

As in (2.16) we find

$$\int_{\bar\phi-2\varepsilon-(\frac{1}{c_1}+\frac{1}{c_2})\varepsilon\ln\frac{1}{\varepsilon}-\varepsilon^{1/2}}^{\bar\phi-2\varepsilon-(\frac{1}{c_1}+\frac{1}{c_2})\varepsilon\ln\frac{1}{\varepsilon}} |v_\varepsilon|\cos\phi = \varepsilon^{1/2}\lambda_2(x^{(2)},p_2)\cos\phi_0 + O(\varepsilon)\,, \tag{2.20}$$

where J_2 and λ_2 are defined analogously to J_1 and λ_1. Finally, as in (2.15), we have

$$\Big(\int_{-\pi/2}^{\bar\phi-2\varepsilon-(\frac{1}{c_1}+\frac{1}{c_2})\varepsilon\ln\frac{1}{\varepsilon}-\varepsilon^{1/2}} \cos\phi\Big)|x^{(2)}| = \big(1+\sin\phi_0-(\tau+1)\varepsilon^{1/2}\cos\phi_0\big)|x^{(2)}| + O(\varepsilon\ln\frac{1}{\varepsilon})\,. \tag{2.21}$$

Summing up the integrals in (2.15)–(2.21) and comparing to (1.18) and (2.10) yields

$$\varepsilon^{1/2}\cos\phi_0\Big(m_1(\tau-1)+\lambda_1(x^{(1)},p_1)+\lambda_2(x^{(2)},p_2)-m_2(\tau+1)\Big) = O(\varepsilon\ln\frac{1}{\varepsilon})\,,$$

from which we can solve for $\tau=\tau_\varepsilon$ that satisfies

$$\tau_\varepsilon = \frac{\lambda_1(x^{(1)},p_1)+\lambda_2(x^{(2)},p_2)-m_1-m_2}{m_2-m_1} + O(\varepsilon^{1/2}\ln\frac{1}{\varepsilon})\,.$$

Clearly τ_ε remains bounded as ε goes to zero.

Next we compute the contribution to the energy $E_\varepsilon(v_\varepsilon)$ from each of the segments (i) to (vii). We denote these contributions by $I_1 - I_7$, respectively. Clearly we have

$$I_1 = I_7 = 0\,, \tag{2.22}$$

and

$$I_2 + I_6 = O(\varepsilon^{-1/2})\,. \tag{2.23}$$

From (2.12)–(2.13) we infer that

$$|p_1-\gamma(z(-\frac{\ln 1/\varepsilon}{c_1}))| = O(\varepsilon) \quad\text{and}\quad |p_2-\gamma(z(\frac{\ln 1/\varepsilon}{c_2}))| = O(\varepsilon)\,,$$

which implies that

$$I_3 + I_5 = O(1)\,. \tag{2.24}$$

Finally, for the contribution I_4 from (iv), we find using (2.14)

$$\frac{I_4}{2\pi} \le \frac{2D}{\varepsilon} \cos\Big(\bar{\phi} - \varepsilon - (\frac{1}{c_1} + \frac{1}{c_2})\varepsilon \ln \frac{1}{\varepsilon}\Big) \le \frac{2D}{\varepsilon}(\cos\phi_0 + C\varepsilon^{1/2}) \,. \tag{2.25}$$

Combining (2.22)–(2.25) we are led to the desired result

$$E_\varepsilon(v_\varepsilon) \le 2\pi \cos\phi_0 \big(\frac{2D}{\varepsilon} + \frac{C_1}{\varepsilon^{1/2}}\big) \,.$$

□

Note that an immediate consequence of the upper bound (2.9) is that for any $\alpha > 0$ there exists $\beta > 0$ such that

$$\mu\big(\{x \in S^2 \,:\, W(u_\varepsilon(x)) \ge \alpha\varepsilon^{1/2}\}\big) \le \beta\varepsilon^{1/2} \,. \tag{2.26}$$

3 A key proposition

This section is devoted to the proof of Proposition 3.1 below which establishes a partition of the sphere to two parts, on each of them u_ε is close to either Γ_1 or Γ_2. For any ϕ_1, ϕ_2 satisfying $-\frac{\pi}{2} \le \phi_1 < \phi_2 \le \frac{\pi}{2}$ we define the spherical annulus

$$A_{\phi_1,\phi_2} := \{x \in S^2 \,:\, \phi_1 < \phi(x) < \phi_2\} \,,$$

where $\phi(x)$ and $\theta(x)$ denote the spherical coordinates of the point x, i.e.,

$$x = (x_1, x_2, x_3) = (\cos\phi(x)\cos\theta(x), \cos\phi(x)\sin\theta(x), \sin\phi(x)) \,.$$

Proposition 3.1. *For each small $\varepsilon > 0$ there exists $\gamma_1 = \gamma_1(\varepsilon)$ satisfying either:*

$$|\gamma_1 - \phi_0| \le C_1\varepsilon^{1/2} \quad \textit{and} \quad \mu(\{x \in S^2 \,:\, \Psi_1(u_\varepsilon(x)) \le \varepsilon^{1/2}\}\Delta A_{\gamma_1,\frac{\pi}{2}}) \le C_2\varepsilon \,, \tag{3.1}$$

or

$$|\gamma_1 + \phi_0| \le C_1\varepsilon^{1/2} \quad \textit{and} \quad \mu(\{x \in S^2 \,:\, \Psi_1(u_\varepsilon(x)) \le \varepsilon^{1/2}\}\Delta A_{-\frac{\pi}{2},-\gamma_1}) \le C_2\varepsilon \,, \tag{3.2}$$

where C_1 and C_2 are positive constants, independent of ε.

Proof. In the sequel we shall denote by c and C different positive constants which do not depend on ε. The upper bound (2.9) combined with (2.5) implies (for small ε) that

$$\int_0^{C\varepsilon^{1/2}} \widetilde{\Psi}(u_\varepsilon(\phi)) \cos\phi \, d\phi \le C\varepsilon \,.$$

Therefore, choosing an appropriate C yields the existence of $\gamma_0 = \gamma_0^\varepsilon \in (0, C\varepsilon^{1/2})$ satisfying

$$\widetilde{\Psi}(u_\varepsilon(\gamma_0)) \le \varepsilon^{1/2} \,. \tag{3.3}$$

We shall assume in the sequel that (3.3) holds with $\widetilde{\Psi}(u_\varepsilon(\gamma_0)) = \Psi_2(u_\varepsilon(\gamma_0))$ (this implies, of course, that $u_\varepsilon(\gamma_0)$ is close to Γ_2).

Next, define, if it exists,

$$\phi_1^+ = \inf\{\phi \ge \gamma_0 \,:\, \Psi_1(u_\varepsilon(\phi)) \le \varepsilon^{1/2}\} \,,$$

and then

$$\overline{\phi}_1^+ = \sup\{\phi \le \phi_1^+ \,:\, \Psi_2(u_\varepsilon(\phi)) \le \varepsilon^{1/2}\} \,.$$

In general, for an even j let

$$\begin{aligned} \phi_j^+ &= \inf\{\phi \ge \phi_{j-1}^+ \,:\, \Psi_2(u_\varepsilon(\phi)) \le \varepsilon^{1/2}\} \,, \\ \overline{\phi}_j^+ &= \sup\{\phi \le \phi_j^+ \,:\, \Psi_1(u_\varepsilon(\phi)) \le \varepsilon^{1/2}\} \,, \end{aligned}$$

while for an odd j:

$$\begin{aligned} \phi_j^+ &= \inf\{\phi \ge \phi_{j-1}^+ \,:\, \Psi_1(u_\varepsilon(\phi)) \le \varepsilon^{1/2}\} \,, \\ \overline{\phi}_j^+ &= \sup\{\phi \le \phi_j^+ \,:\, \Psi_2(u_\varepsilon(\phi)) \le \varepsilon^{1/2}\} \,, \end{aligned}$$

We stop at $j = k^+ - 1$ if $j = k^+$ is the first index for which either ϕ_j^+ does not exist, or $\phi_j^+ > \frac{\pi}{2} - \varepsilon^{1/2}$. For each $j = 1, \dots, k^+ - 1$ we have (see (2.14))

$$\frac{1}{2\pi} E_\varepsilon(u_\varepsilon, A_{\overline{\phi}_j^+, \phi_j^+}) \ge \frac{2}{\varepsilon} \cos\phi_j^+ \int_{\overline{\phi}_j^+}^{\phi_j^+} \sqrt{W(u_\varepsilon)} |u_\varepsilon'| \ge \frac{2}{\varepsilon}(D - 2\varepsilon^{1/2}) \cos\phi_j^+ \,. \tag{3.4}$$

In particular, we deduce

$$\frac{1}{2\pi} E_\varepsilon(u_\varepsilon, A_{\overline{\phi}_j^+, \phi_j^+}) \ge \frac{2}{\varepsilon}(D - 2\varepsilon^{1/2}) \sin\varepsilon^{1/2} \ge \frac{C}{\varepsilon^{1/2}} \,, \tag{3.5}$$

which together with the upper bound (2.9) implies that the process of selection of pairs $(\overline{\phi}_j^+, \phi_j^+)$ must terminate, with the bound $k^+ \le \frac{C}{\varepsilon^{1/2}}$.

Similarly, set, if it exists,

$$\phi_1^- = \sup\{\phi \le \gamma_0 \,:\, \Psi_1(u_\varepsilon(\phi)) \le \varepsilon^{1/2}\} \,,$$

and then

$$\overline{\phi}_1^- = \inf\{\phi \ge \phi_1^- \,:\, \Psi_2(u_\varepsilon(\phi)) \le \varepsilon^{1/2}\} \,.$$

In general, for an even j let

$$\begin{aligned}\phi_j^- &= \sup\{\phi \le \phi_{j-1}^- \,:\, \Psi_2(u_\varepsilon(\phi)) \le \varepsilon^{1/2}\}\,,\\ \overline{\phi}_j^- &= \inf\{\phi \ge \phi_j^- \,:\, \Psi_1(u_\varepsilon(\phi)) \le \varepsilon^{1/2}\}\,,\end{aligned}$$

while for an odd j:

$$\begin{aligned}\phi_j^- &= \sup\{\phi \le \phi_{j-1}^- \,:\, \Psi_1(u_\varepsilon(\phi)) \le \varepsilon^{1/2}\}\,,\\ \overline{\phi}_j^- &= \inf\{\phi \ge \phi_j^- \,:\, \Psi_2(u_\varepsilon(\phi)) \le \varepsilon^{1/2}\}\,.\end{aligned}$$

We stop at $j = k^- - 1$ if $j = k^-$ is the first index for which either ϕ_j^- does not exist, or $\phi_j^- < -\frac{\pi}{2} + \varepsilon^{1/2}$. The same computation as in (3.4)–(3.5) gives

$$\frac{1}{2\pi} E_\varepsilon(u_\varepsilon, A_{\phi_j^-,\overline{\phi}_j^-}) \ge \frac{2}{\varepsilon}\cos\phi_j^-(D - 2\varepsilon^{1/2}) \ge \frac{C}{\varepsilon^{1/2}}\,, \tag{3.6}$$

which implies the bound $k^- \le \frac{C}{\varepsilon^{1/2}}$. It will be convenient to set also

$$\phi_{k+}^+ = \overline{\phi}_{k+}^+ = \frac{\pi}{2} - \varepsilon^{1/2} \quad\text{and}\quad \phi_{k-}^- = \overline{\phi}_{k-}^- = -\frac{\pi}{2} + \varepsilon^{1/2}\,.$$

Since on each $A_{\overline{\phi}_i^+,\phi_i^+}$ and $A_{\phi_i^-,\overline{\phi}_i^-}$ we have $\widetilde{\Psi}(u_\varepsilon) \ge \varepsilon^{1/2}$, it follows from (2.5) and (2.26) that

$$\mu\Big(\bigcup_{i=1}^{k^+-1} A_{\overline{\phi}_i^+,\phi_i^+} \cup \bigcup_{i=1}^{k^--1} A_{\phi_i^-,\overline{\phi}_i^-}\Big) \le C\varepsilon^{1/2}\,. \tag{3.7}$$

Put

$$V_\varepsilon = \bigcup_{j\ge 1} A_{\phi_{2j-1}^+,\phi_{2j}^+} \cup \bigcup_{j\ge 1} A_{\phi_{2j}^-,\phi_{2j-1}^-}\,. \tag{3.8}$$

In the above only j's which do not go out of range are taken into account, i.e., those satisfying $2j \le k^+$ in the first union and those satisfying $2j \le k^-$ in the second. On the one hand we have $\Psi_1(u_\varepsilon) > \varepsilon^{1/2}$ on each $A_{\phi_{2j}^+,\phi_{2j+1}^+}$ and $A_{\phi_{2j+1}^-,\phi_{2j}^-}$, which implies that

$$\{\Psi_1(u_\varepsilon) \le \varepsilon^{1/2}\} \cap A_{\varepsilon^{1/2}-\frac{\pi}{2},\frac{\pi}{2}-\varepsilon^{1/2}} \subset V_\varepsilon\,. \tag{3.9}$$

On the other hand, since $\Psi_2(u_\varepsilon) > \varepsilon^{1/2}$ on each $A_{\phi_{2j-1}^+,\phi_{2j}^+}$ and $A_{\phi_{2j}^-,\phi_{2j-1}^-}$, we have

$$V_\varepsilon \subset \{\Psi_1(u_\varepsilon) \le \varepsilon^{1/2}\} \cup \{\widetilde{\Psi}(u_\varepsilon) \ge \varepsilon^{1/2}\}\,. \tag{3.10}$$

Note that from (2.26) it follows that there exists $\alpha_\varepsilon \in (0,1)$ such that

$$|\mu(\{\Psi_1(u_\varepsilon) \le \varepsilon^{1/2}\}) - 4\pi\alpha_\varepsilon| \le C\varepsilon^{1/2} \quad\text{and}\quad |\mu(\{\Psi_2(u_\varepsilon) \le \varepsilon^{1/2}\}) - 4\pi(1-\alpha_\varepsilon)| \le C\varepsilon^{1/2}\,. \tag{3.11}$$

Note that on the sets $\{\Psi_j(u_\varepsilon) < d_0\}$ we have $|u_\varepsilon - \tilde{s}_j(u_\varepsilon)|^2 \sim W(u_\varepsilon)$ (see (2.5)), hence

$$\Big| \int_{\{\Psi_j(u_\varepsilon)\leq\varepsilon^{1/2}\}} |u_\varepsilon| - \int_{\{\Psi_j(u_\varepsilon)\leq\varepsilon^{1/2}\}} |\tilde{s}_j(u_\varepsilon)| \Big| \leq \int_{\{\Psi_j(u_\varepsilon)\leq\varepsilon^{1/2}\}} |u_\varepsilon - \tilde{s}_j(u_\varepsilon)| \\ \leq C\Big(\int_{\{\Psi_j(u_\varepsilon)\leq\varepsilon^{1/2}\}} W(u_\varepsilon)\Big)^{1/2} \leq C\varepsilon^{1/2} . \quad (3.12)$$

By (H_3), (2.5) and (2.26) we also have

$$\int_{\{\widetilde{\Psi}(u_\varepsilon)>\varepsilon^{1/2}\}} |u_\varepsilon| \leq R_0 \mu\big(\{\widetilde{\Psi}(u_\varepsilon) > \varepsilon^{1/2}\} \cap \{|u_\varepsilon| \leq R_0\}\big) + \frac{1}{C_0} \int_{\{\widetilde{\Psi}(u_\varepsilon)>\varepsilon^{1/2}\}\cap\{|u_\varepsilon|>R_0\}} W(u_\varepsilon) \leq C\varepsilon^{1/2} . \quad (3.13)$$

Set

$$m_\varepsilon^{(j)} = \frac{1}{\mu(\{\Psi_j(u_\varepsilon) \leq \varepsilon^{1/2}\})} \int_{\{\Psi_j(u_\varepsilon)\leq\varepsilon^{1/2}\}} |\tilde{s}_j(u_\varepsilon)| , \quad \text{for } j = 1, 2 . \quad (3.14)$$

Using (3.12)–(3.14) and the constraint we obtain

$$m_\varepsilon^{(1)} \mu(\{\Psi_1(u_\varepsilon) \leq \varepsilon^{1/2}\}) + m_\varepsilon^{(2)} \mu(\{\Psi_2(u_\varepsilon) \leq \varepsilon^{1/2}\}) \\ = \int_{\{\Psi_1(u_\varepsilon)\leq\varepsilon^{1/2}\}} |\tilde{s}_1(u_\varepsilon)| + \int_{\{\Psi_2(u_\varepsilon)\leq\varepsilon^{1/2}\}} |\tilde{s}_2(u_\varepsilon)| \\ = \int_{\{\Psi_1(u_\varepsilon)\leq\varepsilon^{1/2}\}} |u_\varepsilon| + \int_{\{\Psi_2(u_\varepsilon)\leq\varepsilon^{1/2}\}} |u_\varepsilon| + O(\varepsilon^{1/2}) = 4\pi R_c + O(\varepsilon^{1/2}) . \quad (3.15)$$

Form (3.15) and (3.12) it follows that there exists a number K, which is uniformly bounded, such that

$$m_\varepsilon^{(1)} (\alpha_\varepsilon + K\varepsilon^{1/2}) + m_\varepsilon^{(2)} (1 - \alpha_\varepsilon - K\varepsilon^{1/2}) = R_c . \quad (3.16)$$

Since $m_\varepsilon^{(j)} \in [m_j, M_j]$, $j = 1, 2$, it follows from (3.16), in particular, that α_ε is bounded away from 0 and 1. Therefore, by the definitions of α and I (see (1.14)),

$$I(4\pi(\alpha_\varepsilon + K\varepsilon^{1/2})) \geq I(4\pi\alpha) = 2\pi \cos\phi_0 . \quad (3.17)$$

By (3.9)–(3.10),(3.17) and the Lipschitz property of I we conclude that

$$2\pi\Big(\sum_{j=1}^{2[\frac{k^+}{2}]} \cos\phi_j^+ + \sum_{j=1}^{2[\frac{k^-}{2}]} \cos\phi_j^- \Big) = \operatorname{Per} V_\varepsilon \geq I(\mu(V_\varepsilon)) \geq 2\pi \cos\phi_0 - C\varepsilon^{1/2} . \quad (3.18)$$

On the other hand, summing-up the inequalities in (3.4) and (3.6) yields

$$\frac{1}{2\pi} E_\varepsilon\big(u_\varepsilon, \bigcup_{j=1}^{k^+-1} A_{\overline{\phi_j^+},\phi_j^+} \cup \bigcup_{j=1}^{k^--1} A_{\phi_j^-,\overline{\phi_j^-}}\big) \geq \frac{2D}{\varepsilon}\big(\sum_{j=1}^{k^+-1} \cos\phi_j^+ + \sum_{j=1}^{k^--1} \cos\phi_j^- \big) - C\varepsilon^{-1/2} . \quad (3.19)$$

Combining (3.19) with the upper-bound (2.9) we obtain

$$\sum_{j=1}^{k^+} \cos \phi_j^+ + \sum_{j=1}^{k^-} \cos \phi_j^- \le \cos \phi_0 + C\varepsilon^{1/2} . \qquad (3.20)$$

Note that we used the fact that $\cos \phi_{k^+}^+ = \cos \phi_{k^-}^- = \sin \varepsilon^{1/2} = O(\varepsilon^{1/2})$. Therefore, combining (3.18) with (3.20) we get

$$\Big| \sum_{j=1}^{k^+} \cos \phi_i^+ + \sum_{j=1}^{k^-} \cos \phi_i^- - \cos \phi_0 \Big| \le C\varepsilon^{1/2} . \qquad (3.21)$$

Our next objective is to show that V_ε consists of "essentially" one annulus.

Claim: We have:

$$\text{either } \mu(V_\varepsilon \Delta A_{\phi_j^+, \phi_{j+1}^+}) = O(\varepsilon), \text{ or } \mu(V_\varepsilon \Delta A_{\phi_{j+1}^-, \phi_j^-}) = O(\varepsilon), \text{ for some } j. \qquad (3.22)$$

Proof of the Claim: If V_ε consists of only one annulus, then either $V_\varepsilon = A_{\phi_1^+, \phi_2^+}$ or $V_\varepsilon = A_{\phi_2^-, \phi_1^-}$ (see (3.8)) and the claim holds. It remains to treat the case where V_ε consists of more than one annulus. Note that since I is concave and $I(0) = 0$, we have

$$I(a+b) \le I(a) + I(b) , \qquad (3.23)$$

for all admissible values of a and b. By assumption, we may write V_ε as a *disjoint* union $V_\varepsilon = B_\varepsilon \cup C_\varepsilon$ with $\mu(V_\varepsilon)/2 \le \mu(B_\varepsilon) < \mu(V_\varepsilon)$. From (1.14) it is clear that there exists $\delta_0 > 0$ such that

$$I(\delta) \ge 2\delta I'(\mu(V_\varepsilon) - \delta) , \quad \forall \delta \le \delta_0 . \qquad (3.24)$$

We distinguish two cases:

(i) $\mu(C_\varepsilon) > \delta_0$.
(ii) $\mu(C_\varepsilon) \le \delta_0$.

In case (i) we have, for some constant $\eta > 0$, a stronger form of (3.23), namely

$$I(\mu(V_\varepsilon)) + \eta \le I(\mu(B_\varepsilon)) + I(\mu(C_\varepsilon)) . \qquad (3.25)$$

From (3.25) and (3.21) it follows that

$$I(\mu(V_\varepsilon)) + \eta \le I(\mu(B_\varepsilon)) + I(\mu(C_\varepsilon)) \le \mathrm{Per}(B_\varepsilon) + \mathrm{Per}(C_\varepsilon) = \mathrm{Per}(V_\varepsilon) \le 2\pi \cos \phi_0 + C\varepsilon^{1/2} ,$$

which clearly contradicts (3.18), for ε small enough. Therefore, only case (ii) should be considered.

Notice the following simple consequence of the concavity of I:

$$\mu(C_\varepsilon) I'(\mu(B_\varepsilon)) \ge I(\mu(V_\varepsilon)) - I(\mu(B_\varepsilon)) . \qquad (3.26)$$

By (3)–(3.26) and (3.24) we get

$$2\pi\cos\phi_0 + C\varepsilon^{1/2} \geq I(\mu(B_\varepsilon)) + I(\mu(C_\varepsilon)) \geq I(\mu(V_\varepsilon)) - \mu(C_\varepsilon)I'(\mu(B_\varepsilon)) + I(\mu(C_\varepsilon))$$
$$\geq I(\mu(V_\varepsilon)) + \frac{1}{2}I(\mu(C_\varepsilon)) \geq 2\pi\cos\phi_0 - C\varepsilon^{1/2} + \frac{1}{2}I(\mu(C_\varepsilon))\,,$$

which yields, $I(\mu(C_\varepsilon)) \leq C\varepsilon^{1/2}$, i.e.,

$$\mu(C_\varepsilon) \leq C\varepsilon\,. \tag{3.27}$$

To conclude the proof of the claim, we shall show that one of the components of V_ε has measure larger than $\mu(V_\varepsilon) - C\varepsilon$. By the above computation it is enough to consider the case where V_ε consists of $r \geq 3$ components (i.e., annuli) that we shall now denote by $A_1, A_2, \ldots, A_r$ with

$$\mu(A_1) \geq \mu(A_2) \geq \cdots \geq \mu(A_r)\,.$$

Furthermore, if $\mu(A_1) \geq \frac{\mu(V_\varepsilon)}{2}$ then the conclusion follows from the argument that led to (3.27). Thus assume that

$$\mu(A_1) < \frac{\mu(V_\varepsilon)}{2}\,. \tag{3.28}$$

We shall see that this is impossible. Indeed, (3.28) implies the existence of $j_0 \geq 1$ which is the *largest* index for which

$$\mu(A_{j_0+1}) + \mu(A_{j_0+2}) + \cdots + \mu(A_r) \geq \frac{\mu(V_\varepsilon)}{2}\,.$$

Setting

$$B_\varepsilon = A_{j_0+1} \cup A_{j_0+2} \cup \cdots \cup A_r \quad \text{and} \quad C_\varepsilon = A_1 \cup A_2 \cup \cdots \cup A_{j_0}\,,$$

we obtain from the argument that led to (3.27) that $\mu(C_\varepsilon) \leq C\varepsilon$. But from the definition of j_0 it is easy to see that we must have then $\mu(B_\varepsilon) \leq \frac{3}{4}\mu(V_\varepsilon)$, i.e., $\mu(V_\varepsilon) = \mu(B_\varepsilon) + \mu(C_\varepsilon) \leq \frac{3}{4}\mu(V_\varepsilon) + C\varepsilon$, which is a contradiction (for ε small enough). The proof of the claim is then complete.

Consider now the "fat component" A_1, and assume without loss of generality that it lies in the upper hemisphere, i.e., $A_1 = A_{\phi_j^+,\phi_{j+1}^+}$ for some j. Under this assumption we shall show that (3.1) holds, for $\gamma_1 = \phi_j^+$ (the other possibility leads to (3.2) by an identical argument). By (3.21) we deduce that

$$|\cos\phi_j^+ + \cos\phi_{j+1}^+ - \cos\phi_0| \leq C\varepsilon^{1/2}\,, \tag{3.29}$$

while by (3.18) we have

$$|I(\mu(V_\varepsilon)) - 2\pi\cos\phi_0| \leq C\varepsilon^{1/2}\,. \tag{3.30}$$

Using the Lipschitzity of the (local) inverse of the function I we deduce from (3.30) and (3.22) that

$$\sin\phi_{j+1}^{+}-\sin\phi_{j}^{+}=1-\sin\phi_0+O(\varepsilon^{1/2})\,. \tag{3.31}$$

It is easy to conclude from (3.29) and (3.31) that

$$|\phi_j^{+}-\phi_0|\le C\varepsilon^{1/2}\quad\text{and}\quad |\phi_{j+1}^{+}-\frac{\pi}{2}|\le C\varepsilon^{1/2}\,.$$

Finally, we must have $j=1$, since otherwise we will get a contradiction to (3.21). We can therefore set $\gamma_1=\phi_1^{+}$ and (3.1) follows. □

In the sequel we shall assume, without loss of generality, that (3.1) holds. We then set

$$\gamma_2=\gamma_2(\varepsilon)=\sup\{\phi\le\gamma_1\,:\,\Psi_2(u_\varepsilon(\phi))=\varepsilon^{1/2}\}\,. \tag{3.32}$$

Note that by (3.1) we have

$$E_\varepsilon(u_\varepsilon,A_{\gamma_2,\gamma_1})\ge\frac{2\cos\gamma_1}{\varepsilon}(D-2\varepsilon^{1/2})\ge\frac{2D\cos\phi_0}{\varepsilon}-C\varepsilon^{1/2}\,. \tag{3.33}$$

The next lemma provides pointwise estimates that roughly speaking show that $u_\varepsilon(\phi)$ is "close" to Γ_2 for ϕ below γ_2, while $u_\varepsilon(\phi)$ is "close" to Γ_1 for ϕ above γ_1.

Lemma 3.1. *There exists $c_3>0$ such that*

$$\Psi_1(u_\varepsilon(\phi))\cos\phi\le c_3\varepsilon^{1/2}\,,\quad\forall\phi\ge\gamma_1\,, \tag{3.34}$$

$$\Psi_2(u_\varepsilon(\phi))\cos\phi\le c_3\varepsilon^{1/2}\,,\quad\forall\phi\le\gamma_2\,. \tag{3.35}$$

Proof. By (2.9) and (3.33), for each $\tilde\phi\ge\gamma_1$ we have

$$\begin{aligned}\frac{C}{\varepsilon^{1/2}}\ge E_\varepsilon(u_\varepsilon,A_{\gamma_1,\tilde\phi})&\ge\Big|\frac{2}{\varepsilon}\int_{\gamma_1}^{\tilde\phi}\cos\phi\frac{d}{d\phi}\big(\Psi_1(u_\varepsilon(\phi))\big)\,d\phi\Big|\\&\ge\frac{2}{\varepsilon}\Big(\Psi_1(u_\varepsilon(\tilde\phi))\cos\tilde\phi-\Psi_1(u_\varepsilon(\gamma_1))\cos\gamma_1+\int_{\gamma_1}^{\tilde\phi}\sin\phi\,\Psi_1(u_\varepsilon(\phi))\,d\phi\Big)\\&\ge\frac{2}{\varepsilon}\Big(\cos\tilde\phi\,\Psi_1(u_\varepsilon(\tilde\phi))-\cos\gamma_1\,\varepsilon^{1/2}\Big)\,,\end{aligned}$$

and (3.34) follows.

Next, for each $\tilde\phi\le\gamma_2$ we have by the same computation as above

$$\begin{aligned}\frac{C}{\varepsilon^{1/2}}\ge E_\varepsilon(u_\varepsilon,A_{\tilde\phi,\gamma_2})&\ge\Big|\frac{2}{\varepsilon}\int_{\tilde\phi}^{\gamma_2}\cos\phi\frac{d}{d\phi}\big(\Psi_2(u_\varepsilon(\phi))\big)\,d\phi\Big|\\&\ge\frac{2}{\varepsilon}\Big(-\cos\gamma_2\Psi_2(u_\varepsilon(\gamma_2))+\cos\tilde\phi\,\Psi_2(u_\varepsilon(\tilde\phi))-\int_{\tilde\phi}^{\gamma_2}\sin\phi\,\Psi_2(u_\varepsilon(\phi))\,d\phi\Big)\,.\end{aligned} \tag{3.36}$$

Denoting $\tilde{\phi}_+ = \max(\tilde{\phi}, 0)$, we have

$$\int_{\tilde{\phi}}^{\gamma_2} \sin\phi\, \Psi_2(u_\varepsilon(\phi))\, d\phi = \int_{\tilde{\phi}}^{\tilde{\phi}_+} \sin\phi\, \Psi_2(u_\varepsilon(\phi))\, d\phi + \int_{\tilde{\phi}_+}^{\gamma_2} \sin\phi\, \Psi_2(u_\varepsilon(\phi))\, d\phi$$
$$\leq \int_{\tilde{\phi}_+}^{\gamma_2} \sin\phi\, \Psi_2(u_\varepsilon(\phi))\, d\phi\,. \quad (3.37)$$

Since for $\phi \in [0, \gamma_2]$ we have $\sin\phi \leq C\cos\phi$ and since by (3.33) and (2.9),

$$\frac{2}{\varepsilon} \int_{\tilde{\phi}_+}^{\gamma_2} \cos\phi\, \Psi_2(u_\varepsilon(\phi))\, d\phi \leq \frac{C}{\varepsilon^{1/2}}\,,$$

we obtain from (3.36)–(3.37) that

$$\frac{C}{\varepsilon^{1/2}} \geq \frac{2}{\varepsilon} \cos\tilde{\phi}\, \Psi_2(u_\varepsilon(\tilde{\phi})) - \frac{C}{\varepsilon^{1/2}}\,,$$

and (3.35) follows as well. □

4 Proof of the lower-bound

In this section we prove the lower-bound energy estimate of Theorem 1. The next lemma provides a crucial estimate for the distance between γ_1 and γ_2.

Lemma 4.1. *We have* $\gamma_1 - \gamma_2 \sim \varepsilon \ln \frac{1}{\varepsilon}$.

Proof. Put

$$\bar{\phi}_2 = \sup\{\phi \in (\gamma_2, \gamma_1) \,:\, \Psi_2(u_\varepsilon(\phi)) = \frac{d_0}{2}\}\,, \quad (4.1)$$

$$\bar{\phi}_1 = \inf\{\phi \in (\gamma_2, \gamma_1) \,:\, \Psi_1(u_\varepsilon(\phi)) = \frac{d_0}{2}\}\,. \quad (4.2)$$

Note that $\widetilde{\Psi}(u_\varepsilon) \geq \frac{d_0}{2}$ on $[\bar{\phi}_2, \bar{\phi}_1]$ (see (2.2)). Therefore, also $W(u_\varepsilon) \geq c > 0$ on $[\bar{\phi}_2, \bar{\phi}_1]$ for some constant c, by (2.5). From the simple upper-bound

$$\frac{1}{\varepsilon^2} \int_{A_{\bar{\phi}_2, \bar{\phi}_1}} W(u_\varepsilon) \leq \frac{C}{\varepsilon}$$

we then easily conclude that

$$\bar{\phi}_1 - \bar{\phi}_2 \leq C\varepsilon\,. \quad (4.3)$$

It is therefore sufficient to prove the following:

$$\bar{\phi}_2 - \gamma_2 \sim \varepsilon \ln \frac{1}{\varepsilon} \quad \text{and} \quad \gamma_1 - \bar{\phi}_1 \sim \varepsilon \ln \frac{1}{\varepsilon}\,. \quad (4.4)$$

Clearly it is enough to prove the first estimate in (4.4), as the proof of the second one is identical.

We define the function $\tau_2(\phi)$ by the equation

$$u_\varepsilon(\phi) = \mathcal{G}_{\tilde{s}_2(u_\varepsilon(\phi))}(\tau_2(\phi)) , \quad \phi \in [\gamma_2, \gamma_1] . \tag{4.5}$$

We can write $u'_\varepsilon(\phi)$ as a sum of two orthogonal components, the first, $(u'_\varepsilon)_\nu$ in the direction $\boldsymbol{\nu} := \nabla\psi_2$ and the second, $(u'_\varepsilon)_\sigma$ in the direction of $\nabla\tilde{s}_2$. We therefore have

$$|u'_\varepsilon|^2 = |(u'_\varepsilon)_\nu|^2 + |(u'_\varepsilon)_\sigma|^2 . \tag{4.6}$$

It is easy to verify that the first component is given by

$$(u'_\varepsilon)_\nu = \nabla\Psi_2(u_\varepsilon(\phi))\tau'_2(\phi) . \tag{4.7}$$

Therefore,

$$\begin{aligned} \int_{\gamma_2}^{\bar\phi_2} \Big(|u'_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon^2}\Big)\cos\phi &\geq \int_{\gamma_2}^{\bar\phi_2} \Big(|(u'_\varepsilon)_\nu|^2 + \frac{W(u_\varepsilon)}{\varepsilon^2}\Big)\cos\phi \\ = \int_{\gamma_2}^{\bar\phi_2} \Big(|\nabla\Psi_2(u_\varepsilon)|^2(\tau'_2)^2 + \frac{W(u_\varepsilon)}{\varepsilon^2}\Big)\cos\phi &= \int_{\gamma_2}^{\bar\phi_2} W(u_\varepsilon)\big((\tau'_2)^2 + \frac{1}{\varepsilon^2}\big)\cos\phi \\ &= \int_{\gamma_2}^{\bar\phi_2} \Big(W(u_\varepsilon)(\tau'_2 - \frac{1}{\varepsilon})^2 + \frac{2}{\varepsilon}\frac{d}{d\phi}\big(\Psi_2(u_\varepsilon)\big)\Big)\cos\phi . \end{aligned} \tag{4.8}$$

We also have

$$\int_{\bar\phi_2}^{\gamma_1} \Big(|u'_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon^2}\Big)\cos\phi \geq \int_{\bar\phi_2}^{\gamma_1} \frac{2}{\varepsilon}\frac{d}{d\phi}\big(\Psi_2(u_\varepsilon)\big)\cos\phi . \tag{4.9}$$

Note that by (3.1) we have

$$\int_{\gamma_2}^{\gamma_1} \frac{d}{d\phi}\big(\Psi_2(u_\varepsilon)\big)\cos\phi = (D-\varepsilon^{1/2})\cos\gamma_1 - \varepsilon^{1/2}\cos\gamma_2 + \int_{\gamma_2}^{\gamma_1} \Psi_2(u_\varepsilon)\sin\phi \geq D\cos\phi_0 - C\varepsilon^{1/2} . \tag{4.10}$$

Combining (4.8)–(4.10) with (3.33) yields

$$\int_{\gamma_2}^{\bar\phi_2} W(u_\varepsilon)(\tau'_2 - \frac{1}{\varepsilon})^2 \leq C\varepsilon^{-1/2} . \tag{4.11}$$

Since $W(u_\varepsilon) \geq C\varepsilon^{1/2}$ on $[\gamma_1, \gamma_2]$ we obtain from (4.11) that

$$\int_{\gamma_2}^{\bar\phi_2} (\tau'_2 - \frac{1}{\varepsilon})^2 \leq \frac{C}{\varepsilon} .$$

Applying the last estimate together with the Cauchy-Schwarz inequality yields

$$|\tau_2(\bar{\phi}_2) - \tau_2(\gamma_2) - \frac{\bar{\phi}_2 - \gamma_2}{\varepsilon}| \le \int_{\gamma_2}^{\bar{\phi}_2} |\tau_2' - \frac{1}{\varepsilon}| \le C\big(\frac{\bar{\phi}_2 - \gamma_2}{\varepsilon}\big)^{1/2}. \tag{4.12}$$

From (2.6) we obtain that

$$\tau_2(\bar{\phi}_2) = O(1) \quad \text{and} \quad \tau_2(\gamma_2) \sim -\ln\frac{1}{\varepsilon}. \tag{4.13}$$

Plugging (4.13) in (4.12) yields the first estimate in (4.4). □

The next lemma provides an estimate for the distance between γ_2 (or γ_1) and ϕ_0 in terms of the two averages

$$\overline{m}_{1,\varepsilon} = \frac{\int_{\gamma_1}^{\pi/2} |u_\varepsilon| \cos\phi}{1 - \sin\gamma_1} \quad \text{and} \quad \overline{m}_{2,\varepsilon} = \frac{\int_{-\pi/2}^{\gamma_2} |u_\varepsilon| \cos\phi}{1 + \sin\gamma_2}. \tag{4.14}$$

Lemma 4.2. *There exist two constants $\alpha_1, \alpha_2 > 0$ (independent of ε) such that*

$$\cos\gamma_2 - \cos\phi_0 = \alpha_1(\overline{m}_{1,\varepsilon} - m_1) + \alpha_2(\overline{m}_{2,\varepsilon} - m_2) + O(\varepsilon\ln\frac{1}{\varepsilon}). \tag{4.15}$$

Proof. Because of the constraint and Lemma 4.1 we have

$$\Big(\int_{-\pi/2}^{\phi_0} \cos\phi\Big) m_2 + \Big(\int_{\phi_0}^{\pi/2} \cos\phi\Big) m_1 = \Big(\int_{-\pi/2}^{\gamma_2} \cos\phi\Big)\overline{m}_{2,\varepsilon} + \Big(\int_{\gamma_2}^{\pi/2} \cos\phi\Big)\overline{m}_{1,\varepsilon} + O(\varepsilon\ln\frac{1}{\varepsilon}),$$

which can be rewritten as

$$(\sin\phi_0 - \sin\gamma_2)(m_2 - m_1) = (\sin\gamma_2 + 1)(\overline{m}_{2,\varepsilon} - m_2) + (1 - \sin\gamma_2)(\overline{m}_{1,\varepsilon} - m_1) + O(\varepsilon\ln\frac{1}{\varepsilon}). \tag{4.16}$$

Since $\sin\gamma_2 - \sin\phi_0 = O(\varepsilon^{1/2})$ (see (3.1) and Lemma 4.1), it follows from (4.16) that also

$$\overline{m}_{2,\varepsilon} - m_2 = O(\varepsilon^{1/2}) \text{ and } \overline{m}_{1,\varepsilon} - m_1 = O(\varepsilon^{1/2}). \tag{4.17}$$

Therefore,

$$\sin\gamma_2 - \sin\phi_0 = -\frac{1}{m_2 - m_1}(\sin\phi_0 + 1)(\overline{m}_{2,\varepsilon} - m_2) - \frac{1}{m_2 - m_1}(1 - \sin\phi_0)(\overline{m}_{1,\varepsilon} - m_1) + O(\varepsilon\ln\frac{1}{\varepsilon}). \tag{4.18}$$

Finally, a simple computation using Taylor formula leads from (4.18) to (4.15) with

$$\alpha_1 = \frac{\tan\phi_0(1 - \sin\phi_0)}{m_2 - m_1} \quad \text{and} \quad \alpha_2 = \frac{\tan\phi_0(1 + \sin\phi_0)}{m_2 - m_1}. \tag{4.19}$$

□

Next we shall introduce some more notation. For ϕ's for which $\Psi_j(u_\varepsilon(\phi)) \leq d_0$ we define $v_j = v_{j,\varepsilon}(\phi) = \tilde{s}_j(u_\varepsilon(\phi))$, $j = 1, 2$. In particular we set

$$p_1^\varepsilon = v_1(\gamma_1) = \tilde{s}_1(u_\varepsilon(\gamma_1)) \quad \text{and} \quad p_2^\varepsilon = v_2(\gamma_2) = \tilde{s}_2(u_\varepsilon(\gamma_2))\,. \tag{4.20}$$

Recall the distance function $d_W^{(\delta)}$ that was defined in (2.8). We shall next use it for $\delta = \varepsilon^{1/2}$.

Lemma 4.3. *We have*

$$\begin{aligned} \frac{1}{2\pi} E_\varepsilon(u_\varepsilon) \geq & \int_{-\pi/2+\varepsilon^{1/3}}^{\gamma_2} \Big(|v_2'|^2 + \frac{2\beta d_W^{(\varepsilon^{1/2})}(p_1^\varepsilon, p_2^\varepsilon)}{\varepsilon} (|v_2(\phi)| - m_2) \Big) \cos\phi \\ & + \int_{\gamma_1}^{\pi/2-\varepsilon^{1/3}} \Big(|v_1'|^2 + \frac{2\beta d_W^{(\varepsilon^{1/2})}(p_1^\varepsilon, p_2^\varepsilon)}{\varepsilon} (|v_1(\phi)| - m_1) \Big) \cos\phi \\ & + \frac{2 d_W^{(\varepsilon^{1/2})}(p_1^\varepsilon, p_2^\varepsilon) \cos\phi_0}{\varepsilon} - C\varepsilon^{-5/12}\,, \end{aligned} \tag{4.21}$$

where $\beta = \frac{\tan\phi_0}{m_2 - m_1}$.

Proof. Note first that for $\phi \in [-\pi/2 + \varepsilon^{1/3}, \gamma_2]$ we have $\cos\phi \geq C\varepsilon^{1/3}$, so by Lemma 3.1

$$\Psi_2(u_\varepsilon(\phi)) \leq C\varepsilon^{1/6}\,, \tag{4.22}$$

hence v_2 is well-defined. Similarly, v_1 is well-defined for $\phi \in [\gamma_1, \pi/2 - \varepsilon^{1/3}]$. By the definition of γ_1 and γ_2, (4.15) and (4.17) we have

$$\begin{aligned} & \frac{1}{2\pi} E_\varepsilon(u_\varepsilon, A_{\gamma_2,\gamma_1}) \geq 2\frac{\cos\gamma_1}{\varepsilon} (d_W^{(\varepsilon^{1/2})}(p_1^\varepsilon, p_2^\varepsilon) - 2\varepsilon^{1/2}) \\ & \geq \frac{2}{\varepsilon} \Big(\cos\phi_0 + \alpha_1(\overline{m}_{1,\varepsilon} - m_1) + \alpha_2(\overline{m}_{2,\varepsilon} - m_2) + O(\varepsilon \ln\frac{1}{\varepsilon}) \Big) (d_W^{(\varepsilon^{1/2})}(p_1^\varepsilon, p_2^\varepsilon) - 2\varepsilon^{1/2}) \\ & \geq \frac{2 d_W^{(\varepsilon^{1/2})}(p_1^\varepsilon, p_2^\varepsilon)}{\varepsilon} \big(\cos\phi_0 + \alpha_1(\overline{m}_{1,\varepsilon} - m_1) + \alpha_2(\overline{m}_{2,\varepsilon} - m_2) \big) - \frac{4\cos\phi_0}{\varepsilon^{1/2}} + O(|\ln\varepsilon|)\,. \end{aligned} \tag{4.23}$$

From (4.23) and the upper bound (2.9) we deduce that

$$E_\varepsilon(u_\varepsilon, S^2 \setminus A_{\gamma_1,\gamma_2}) \leq \frac{C}{\varepsilon^{1/2}}\,. \tag{4.24}$$

By (4.24)

$$\int_0^{\gamma_2} \Psi_2(u_\varepsilon)\cos\phi \leq C \int_0^{\gamma_2} W(u_\varepsilon)\cos\phi \leq C\varepsilon^2 E_\varepsilon(u_\varepsilon, S^2 \setminus A_{\gamma_1,\gamma_2}) \leq C\varepsilon^2 \varepsilon^{-1/2} = C\varepsilon^{3/2}\,,$$

so there exists $\theta_2 \in (0, \gamma_2)$ such that

$$\Psi_2(u_\varepsilon(\theta_2)) \le C\varepsilon^{3/2} . \tag{4.25}$$

From (4.25) we get by the same computation as in (3.36) the lower bound for the *normal* energy (see (4.7)–(4.8)), i.e.,

$$\begin{aligned} \frac{1}{2\pi} E^\nu_\varepsilon(u_\varepsilon, A_{\theta_2,\gamma_2}) := \int_{\theta_2}^{\gamma_2} \Big(|(u'_\varepsilon)_\nu|^2 + \frac{W(u_\varepsilon)}{\varepsilon^2} \Big) \cos\phi \\ \ge \frac{2}{\varepsilon} \Big(\cos\gamma_2\, \varepsilon^{1/2} - \cos\theta_1\, \Psi_2(u_\varepsilon(\theta_2)) + \int_{\theta_1}^{\gamma_2} \sin\phi\, \Psi_2(u_\varepsilon) \Big) \\ \ge \frac{2\cos\gamma_2}{\varepsilon^{1/2}} - C\varepsilon^{1/2} \ge \frac{2\cos\phi_0}{\varepsilon^{1/2}} - C . \end{aligned} \tag{4.26}$$

Similarly, there exists $\theta_1 \in (\gamma_1, \gamma_1 + \varepsilon^{1/2})$ such that $\Psi_1(u_\varepsilon(\theta_1)) \le C\varepsilon$. Hence,

$$\begin{aligned} \frac{1}{2\pi} E^\nu_\varepsilon(u_\varepsilon, A_{\gamma_1,\theta_1}) \ge \frac{2}{\varepsilon} \int_{\gamma_1}^{\theta_1} (\cos\phi_0 - C\varepsilon^{1/2}) \Big| \frac{d}{d\phi} \big(\Psi_1(u_\varepsilon)\big) \Big| \\ \ge \frac{2}{\varepsilon} (\cos\phi_0 - C\varepsilon^{1/2})(\varepsilon^{1/2} - C\varepsilon) \ge \frac{2\cos\phi_0}{\varepsilon^{1/2}} - C . \end{aligned} \tag{4.27}$$

Next we take into account also the contribution from the "tangential" energy, i.e., of $(u'_\varepsilon)_\sigma$. As in [2] we have

$$(u'_\varepsilon)_\sigma = u'_\varepsilon \cdot \frac{\nabla \tilde{s}_2(u_\varepsilon)}{|\nabla \tilde{s}_2(u_\varepsilon)|} = \frac{\frac{d}{d\phi}(\tilde{s}_2(u_\varepsilon))}{|\nabla \tilde{s}_2(u_\varepsilon)|} = \frac{v'_2}{|\nabla \tilde{s}_2(u_\varepsilon)|} , \tag{4.28}$$

at points where $\Psi_2(u_\varepsilon) < d_0$. A simple modification of the argument of the proof of [2, Lemma 3.3] shows that (4.28) implies that

$$|(u'_\varepsilon)_\sigma|^2 \ge |v'_2|^2 (1 - cd(v_2, \Gamma_2)) \ge |v'_2|^2 (1 - \kappa \Psi_2^{1/2}(u_\varepsilon)) , \tag{4.29}$$

for some constant $\kappa > 0$. Combining (4.26) with (4.29) in conjunction with (4.22) and (4.24) leads to

$$\begin{aligned} \frac{1}{2\pi} E_\varepsilon(u_\varepsilon, A_{-\pi/2+\varepsilon^{1/3},\gamma_2}) \ge \frac{2\cos\phi_0}{\varepsilon^{1/2}} + \int_{-\pi/2+\varepsilon^{1/3}}^{\gamma_2} \cos\phi (1 - \kappa\Psi_2^{1/2}(u_\varepsilon)) |v'_2|^2 - C \\ \ge \frac{2\cos\phi_0}{\varepsilon^{1/2}} + \int_{-\pi/2+\varepsilon^{1/3}}^{\gamma_2} |v'_2|^2 \cos\phi - C\varepsilon^{-5/12} . \end{aligned} \tag{4.30}$$

Similarly,

$$\frac{1}{2\pi} E_\varepsilon(u_\varepsilon, A_{\gamma_1,\pi/2-\varepsilon^{1/3}}) \ge \frac{2\cos\phi_0}{\varepsilon^{1/2}} + \int_{\gamma_1}^{\pi/2-\varepsilon^{1/3}} |v'_1|^2 \cos\phi - C\varepsilon^{-5/12} . \tag{4.31}$$

Adding together (4.23) with (4.30) and (4.31) leads to

$$\frac{1}{2\pi}E_\varepsilon(u_\varepsilon) \geq \frac{2d_W^{(\varepsilon^{1/2})}(p_1^\varepsilon, p_2^\varepsilon)\cos\phi_0}{\varepsilon} + \frac{2d_W^{(\varepsilon^{1/2})}(p_1^\varepsilon, p_2^\varepsilon)}{\varepsilon}\big(\alpha_1(\overline{m}_{1,\varepsilon} - m_1) + \alpha_2(\overline{m}_{2,\varepsilon} - m_2)\big)$$
$$+ \int_{-\pi/2+\varepsilon^{1/3}}^{\gamma_2} |v_2'|^2\cos\phi + \int_{\gamma_1}^{\pi/2-\varepsilon^{1/3}} |v_1'|^2\cos\phi - C\varepsilon^{-5/12}\,. \quad (4.32)$$

Next we estimate the second term on the r.h.s. of (4.32). First, write

$$\overline{m}_{1,\varepsilon}(1-\sin\gamma_1) = \int_{\gamma_1}^{\pi/2} |u_\varepsilon|\cos\phi$$
$$= \int_{\pi/2-\varepsilon^{1/3}}^{\pi/2} |u_\varepsilon|\cos\phi + \int_{\gamma_1}^{\pi/2-\varepsilon^{1/3}} |v_1|\cos\phi + \int_{\gamma_1}^{\pi/2-\varepsilon^{1/3}} (|u_\varepsilon| - |v_1|)\cos\phi\,. \quad (4.33)$$

Since (H_3) implies that $|u_\varepsilon| \leq C(1+W(u_\varepsilon))$ we get, using also (4.24), that

$$\int_{\pi/2-\varepsilon^{1/3}}^{\pi/2} |u_\varepsilon|\cos\phi \leq C(1-\cos(\varepsilon^{1/3})) + C\int_{\pi/2-\varepsilon^{1/3}}^{\pi/2} W(u_\varepsilon)\cos\phi \leq C(\varepsilon^{2/3}+\varepsilon^{3/2})\,. \quad (4.34)$$

Using (2.5), (4.24) and the Cauchy-Schwarz inequality we obtain

$$\Big|\int_{\gamma_1}^{\pi/2-\varepsilon^{1/3}} (|u_\varepsilon| - |v_1|)\cos\phi\Big| \leq C\int_{\gamma_1}^{\pi/2-\varepsilon^{1/3}} \Psi_1^{1/2}(u_\varepsilon)\cos\phi \leq C\varepsilon^{3/4}\,. \quad (4.35)$$

From (4.33)–(4.35) we get

$$\overline{m}_{1,\varepsilon}(1-\sin\gamma_1) = \int_{\gamma_1}^{\pi/2-\varepsilon^{1/3}} |v_1|\cos\phi + O(\varepsilon^{2/3})\,,$$

from which we obtain that

$$\overline{m}_{1,\varepsilon} - m_1 = \frac{1}{1-\sin\gamma_1}\int_{\gamma_1}^{\pi/2-\varepsilon^{1/3}} (|v_1| - m_1)\cos\phi + O(\varepsilon^{2/3})\,. \quad (4.36)$$

A similar argument yields

$$\overline{m}_{2,\varepsilon} - m_2 = \frac{1}{1+\sin\gamma_2}\int_{-\pi/2+\varepsilon^{1/3}}^{\gamma_2} (|v_2| - m_2)\cos\phi + O(\varepsilon^{2/3})\,. \quad (4.37)$$

Since $|\gamma_2 - \phi_0|, |\gamma_1 - \phi_0| \leq C\varepsilon^{1/2}$ (see (3.1) and Lemma 4.1), we finally deduce (4.21) from (4.32) and (4.36)–(4.37). □

The following lemma is an immediate consequence of Lemma 4.3.

Lemma 4.4. *There exists a constant C, independent of ε, such that*

$$d_W^{(\varepsilon^{1/2})}(p_1^\varepsilon, p_2^\varepsilon) \le D + C\varepsilon^{1/2} \,. \tag{4.38}$$

Proof. It suffices to combine the upper bound (2.9) with (4.21) (taking into account only the third term on the r.h.s.). □

The next lemma provides a lower bound for the two integrals on the right hand side of (4.21).

Lemma 4.5. *We have*

$$\int_{\gamma_1}^{\pi/2-\varepsilon^{1/3}} \Big(|v_1'|^2 + \frac{2\beta d_W^{(\varepsilon^{1/2})}(p_1^\varepsilon, p_2^\varepsilon)}{\varepsilon}(|v_1(\phi)| - m_1)\Big)\cos\phi \ge \frac{\cos\phi_0}{\varepsilon^{1/2}}\sqrt{2\beta d_W^{(\varepsilon^{1/2})}(p_1^\varepsilon, p_2^\varepsilon)}\,\delta_1(p_1^\varepsilon, \mathcal{M}_1) + o_\varepsilon(1) \,. \tag{4.39}$$

and

$$\int_{-\pi/2+\varepsilon^{1/3}}^{\gamma_2} \Big(|v_2'|^2 + \frac{2\beta d_W^{(\varepsilon^{1/2})}(p_1^\varepsilon, p_2^\varepsilon)}{\varepsilon}(|v_2(\phi)| - m_2)\Big)\cos\phi \ge \frac{\cos\phi_0}{\varepsilon^{1/2}}\sqrt{2\beta d_W^{(\varepsilon^{1/2})}(p_1^\varepsilon, p_2^\varepsilon)}\,\delta_2(p_2^\varepsilon, \mathcal{M}_2) + o_\varepsilon(1) \,. \tag{4.40}$$

Proof. Clearly it suffices to prove (4.39). First, note that

$$J_1(\varepsilon) := \varepsilon^{1/2}\int_{\gamma_1}^{\pi/2-\varepsilon^{1/3}} \Big(|v_1'|^2 + \frac{2\beta d_W^{(\varepsilon^{1/2})}(p_1^\varepsilon, p_2^\varepsilon)}{\varepsilon}(|v_1(\phi)| - m_1)\Big)\cos\phi \le C \,, \tag{4.41}$$

by the upper-bound (2.9) and Lemma 4.3. We introduce a new variable t by $\phi = \gamma_1 + t\varepsilon^{1/2}$ and then define a new function by

$$w(t) = v_1(\gamma_1 + t\varepsilon^{1/2}) \,, \quad 0 \le t \le \frac{\pi/2 - \varepsilon^{1/3} - \gamma_1}{\varepsilon^{1/2}} \,. \tag{4.42}$$

We have

$$\begin{aligned} J_1(\varepsilon) &\ge \varepsilon^{1/2}\int_{\gamma_1}^{\gamma_1+\varepsilon^{1/4}} \Big(|v_1'|^2 + \frac{2\beta d_W^{(\varepsilon^{1/2})}(p_1^\varepsilon, p_2^\varepsilon)}{\varepsilon}(|v_1(\phi)| - m_1)\Big)\cos\phi\, d\phi \\ &\ge \varepsilon^{1/2}(\cos\gamma_1 - c\varepsilon^{1/4})\int_{\gamma_1}^{\gamma_1+\varepsilon^{1/4}} \Big(|v_1'|^2 + \frac{2\beta d_W^{(\varepsilon^{1/2})}(p_1^\varepsilon, p_2^\varepsilon)}{\varepsilon}(|v_1(\phi)| - m_1)\Big)\, d\phi \\ &= (\cos\gamma_1 - c\varepsilon^{1/4})\int_0^{\varepsilon^{-1/4}} \Big(|w'|^2 + 2\beta d_W^{(\varepsilon^{1/2})}(p_1^\varepsilon, p_2^\varepsilon)(|w(t)| - m_1)\Big)\, dt \,. \end{aligned} \tag{4.43}$$

By (4.41) we have

$$\int_0^{\varepsilon^{-1/4}} \Big(|w'|^2 + 2\beta d_W^{(\varepsilon^{1/2})}(p_1^\varepsilon, p_2^\varepsilon)(|w(t)| - m_1)\Big)\, dt \le C\,.$$

Hence, there exists $t_\varepsilon \in (0, \varepsilon^{-1/4}/2)$ such that $y_\varepsilon := w(t_\varepsilon)$ satisfies

$$|y_\varepsilon| - m_1 \le C\varepsilon^{1/4}\,. \tag{4.44}$$

From (4.44) it follows that there exists $z_1^\varepsilon \in \mathcal{M}_1$ such that $|z_1^\varepsilon - y_\varepsilon| = o(1)$. It follows then that there exists a function $\delta(\varepsilon) = o_\varepsilon(1)$ such that

$$|z| - m_1 \le \delta(\varepsilon)\,,\ \forall z \in J_1(y_\varepsilon, z_1^\varepsilon)\,, \tag{4.45}$$

see after (2.16) for the definition of J_1. We shall next define a new function $\widetilde{w}$ on $[0,\infty)$. First, on the interval $[0, t_\varepsilon]$ we set $\widetilde{w} = w$. Then, on the interval $(t_\varepsilon, t_\varepsilon + (\delta(\varepsilon))^{-1/2}]$ we let $\widetilde{w}$ go from $w(t_\varepsilon)$ to z_1^ε along Γ_1 in constant velocity. Finally, we set $\widetilde{w}(t) \equiv z_1^\varepsilon$ on $(t_\varepsilon + (\delta(\varepsilon))^{-1/2}, \infty)$. It is clear from the above construction that $\widetilde{w}$ satisfies

$$\int_{t_\varepsilon}^{\infty} \Big(|\widetilde{w}'|^2 + 2\beta d_W^{(\varepsilon^{1/2})}(p_1^\varepsilon, p_2^\varepsilon)(|\widetilde{w}(t)| - m_1)\Big)\, dt = o_\varepsilon(1)\,. \tag{4.46}$$

From (4.46) and (1.19) it follows that

$$\int_0^{\varepsilon^{-1/4}} \Big(|w'|^2 + 2\beta d_W^{(\varepsilon^{1/2})}(p_1^\varepsilon, p_2^\varepsilon)(|w(t)| - m_1)\Big)\, dt \ge \sqrt{2\beta d_W^{(\varepsilon^{1/2})}(p_1^\varepsilon, p_2^\varepsilon)}\, \delta_1(p_1^\varepsilon, \mathcal{M}_1) + o_\varepsilon(1)\,. \tag{4.47}$$

Combining (4.47) with (4.43) we are led to (4.39). □

In order to complete the proof of the lower-bound part of (1.22) we need to identify the limit of the points p_1^ε and p_2^ε. This is the purpose of the next lemma.

Lemma 4.6. *Suppose that for a subsequence,* $\lim p_1^{\varepsilon_n} = p_1$ *and* $\lim p_2^{\varepsilon_n} = p_2$*. Then* p_1 *and* p_2 *are the end points of some geodesic from* $\mathcal{G}$*.*

Proof. Consider a constant unit velocity geodesic γ_n realizing $d_W^{(\varepsilon_n^{1/2})}(p_1^{\varepsilon_n}, p_2^{\varepsilon_n})$, which passes between the points $y_1^{\varepsilon_n} = u_{\varepsilon_n}(\gamma_1)$ and $y_2^{\varepsilon_n} = u_{\varepsilon_n}(\gamma_2)$ (see (4.20)). For each $\delta > 0$ we let $p_{1,n}^\delta$ be the last point on that geodesic where $\Psi_1(p_{1,n}^\delta) = \delta$. Similarly, let $p_{2,n}^\delta$ be the first point on that geodesic where $\Psi_2(p_{2,n}^\delta) = \delta$. Denote by γ_n^δ the part of γ_n between $p_{1,n}^\delta$ and $p_{2,n}^\delta$. Passing to a subsequence if necessary we obtain the convergence of γ_n^δ towards a limiting geodesic with end points p_1^δ and p_2^δ which must satisfy $d_W(p_1^\delta, p_2^\delta) = D - 2\delta$ (because of (4.38)). Repeating this process with a sequence $\delta_m \to 0$ and passing to a diagonal subsequence, we deduce (for a further subsequence) that γ_n converges to a curve $\underset{\sim}{\gamma}$ joining a point $q_1 \in \Gamma_1$ to a point $q_2 \in \Gamma_1$ with the following property: for each small enough $\delta > 0$ we have $d_W(p_1^\delta, p_2^\delta) = D - 2\delta$, where p_j^δ, $j = 1, 2$, are the points on that geodesic satisfying $\Psi_j(p_1^\delta) = \delta$, respectively. It follows that $\underline{\tilde{\gamma}} \in \mathcal{G}$. The result of the lemma would follow once we show that $q_1 = p_1$ and $q_2 = p_2$.

Looking for a contradiction, assume for example that $q_1 \neq p_1$. Let δ be a small positive number that we shall fix later. We may choose n large enough such that $|p_{1,n}^\delta - p_1^\delta| < \delta$. We now denote by $\tilde{\gamma}_n^\delta$ a re-parametrization on the interval $[0, 1]$ of the part of γ_n between $y_1^{\varepsilon_n}$ and $p_{1,n}^\delta$ such that

$$\sqrt{W(\tilde{\gamma}_n^\delta)}\,|(\tilde{\gamma}_n^\delta)'| \equiv const \quad \text{on } [0, 1]\,.$$

Using the same notations as in the proof of Lemma 4.1 we can decompose $(\tilde{\gamma}_n^\delta)'$ as a sum of two orthogonal components: $(\tilde{\gamma}_n^\delta)'_\nu$ in the direction of $\nabla\Psi_1$, and $(\tilde{\gamma}_n^\delta)'_\sigma$ in the direction of $\nabla\tilde{s}_1$. We have

$$\Big(\int_0^1 \sqrt{W(\tilde{\gamma}_n^\delta)}\,|(\tilde{\gamma}_n^\delta)'|\Big)^2 = \int_0^1 W(\tilde{\gamma}_n^\delta)\,\Big(|(\tilde{\gamma}_n^\delta)'_\nu|^2 + |(\tilde{\gamma}_n^\delta)'_\sigma|^2\Big)\,. \tag{4.48}$$

Next, note that

$$\int_0^1 W(\tilde{\gamma}_n^\delta)\,|(\tilde{\gamma}_n^\delta)'_\nu|^2 \geq \Big(\int_0^1 \sqrt{W(\tilde{\gamma}_n^\delta)}\,|(\tilde{\gamma}_n^\delta)'_\nu|\Big)^2$$
$$= \Big(\int_0^1 |\nabla\Psi_1(\tilde{\gamma}_n^\delta)|\,|(\tilde{\gamma}_n^\delta)'_\nu|\Big)^2 \geq (\delta - \varepsilon_n^{1/2})^2\,. \tag{4.49}$$

Using (2.5) we find

$$\int_0^1 W(\tilde{\gamma}_n^\delta)\,|(\tilde{\gamma}_n^\delta)'_\sigma|^2 \geq c_1\varepsilon_n^{1/2}\Big(\int_0^1 |(\tilde{\gamma}_n^\delta)'_\sigma|\Big)^2 \geq c_2\varepsilon_n^{1/2}\,, \tag{4.50}$$

for some positive constants c_1 and c_2 which do not depend on ε_n or on δ (for δ small enough). Plugging (4.49)–(4.50) in (4.48) yields (provided δ is chosen small enough),

$$\int_0^1 \sqrt{W(\tilde{\gamma}_n^\delta)}\,|(\tilde{\gamma}_n^\delta)'| \geq \Big((\delta - \varepsilon_n^{1/2})^2 + c_2\varepsilon_n^{1/2}\Big)^{1/2} \geq \Big(\delta^2 + c_3\varepsilon_n^{1/2}\Big)^{1/2} \geq \delta + c_4\frac{\varepsilon_n^{1/2}}{\delta}\,. \tag{4.51}$$

From (4.51) we deduce that

$$d_W^{(\varepsilon_n^{1/2})}(p_1^{\varepsilon_n}, p_2^{\varepsilon_n}) \geq D + \varepsilon_n^{1/2} + c_4\frac{\varepsilon_n^{1/2}}{\delta}\,,$$

which contradicts (4.38), provided δ is chosen small enough. □

The next proposition provides the lower-bound estimate needed for assertion (ii) of Theorem 1.

Proposition 4.1. *We have*

$$\frac{1}{2\pi}E_\varepsilon(u_\varepsilon) \geq \cos\phi_0\Big(\frac{2D}{\varepsilon} + \sqrt{2\beta D}\frac{K}{\varepsilon^{1/2}}\Big) + o(\varepsilon^{-\frac{1}{2}})\,. \tag{4.52}$$

Proof. Plugging the estimates (4.39)–(4.40) in (4.21) and using the obvious estimate $d_W^{(\varepsilon^{1/2})}(p_1^\varepsilon, p_2^\varepsilon) \geq D$ yields

$$\frac{1}{2\pi}E_\varepsilon(u_\varepsilon) \geq \frac{2D\cos\phi_0}{\varepsilon} + \frac{\cos\phi_0}{\varepsilon^{1/2}}\sqrt{2\beta D}\left(\delta_1(p_1^\varepsilon, \mathcal{M}_1) + \delta_2(p_2^\varepsilon, \mathcal{M}_2)\right) + o(\varepsilon^{-1/2})\,. \tag{4.53}$$

Therefore, (4.52) is a direct consequence of (4.53), Lemma 4.6 and the definition (1.23) of K. □

5 A refined upper-bound and the convergence result

Next we prove the upper-bound part of the energy expansion (1.22).

Proposition 5.1. *We have*

$$\frac{1}{2\pi}E_\varepsilon(u_\varepsilon) \leq \cos\phi_0\Big(\frac{2D}{\varepsilon} + \sqrt{2\beta D}\frac{K}{\varepsilon^{1/2}}\Big) + o(\varepsilon^{-\frac{1}{2}})\,. \tag{5.1}$$

Proof. We are going to refine the construction used in the proof of Proposition 2.1, using the insight we got from the lower-bound estimates we established so far. We first fix $x^{(1)} \in \mathcal{M}_1, x^{(2)} \in \mathcal{M}_2$ and $\underline{\gamma}^{(i_0)} \in \mathcal{G}$ that realizes the minimum for K in (1.23) and denote by $p_1 = \underline{\zeta}_1^{(i_0)}$ and $p_2 = \underline{\zeta}_2^{(i_0)}$ the end points of $\underline{\gamma}^{(i_0)}$. For these values of $x^{(1)}, x^{(2)}, p_1$ and p_2 we shall slightly modify the construction of a test function v_ε, that was described in the proof of Proposition 2.1 on 7 intervals, from (i) to (vii). Actually, we shall modify the construction only on the intervals (i)–(ii) and (vi)–(vii). From Lemma 1.1 it follows that there exists a minimizer w_1 that realizes $\delta_1(p_1, \mathcal{M}_1)$ with $\lim_{t\to\infty} w_1(t) = x^{(1)}$. We then define

$$\widetilde{w}_1(t) = w_1(\sqrt{2\beta D}t)\,,\ t \in [0,\infty)\,,$$

so that

$$\int_0^\infty \left(|\widetilde{w}_1'|^2 + 2\beta D(|\widetilde{w}_1| - m_1)\right)dt = \sqrt{2\beta D}\,\delta_1(p_1, \mathcal{M}_1)\,.$$

Let $\bar{\phi}$ be a number whose distance to ϕ_0 is of the order $O(\varepsilon^{1/2})$, that will be determined later. We first set

$$v_\varepsilon(\phi) = \widetilde{w}_1(\frac{\phi - \bar{\phi}}{\varepsilon^{1/2}}) \quad \text{for } \phi \in [\bar{\phi}, \frac{\phi_0 + \pi/2}{2}]\,. \tag{5.2}$$

Then, on the interval $[\frac{\phi_0+\pi/2}{2},\pi/2]$ define v_ε in such a way that it follows the curve Γ_1 from the point $v_\varepsilon(\frac{\phi_0+\pi/2}{2})$ to the point $x^{(1)}$ in a constant velocity. From (1.20) and (5.2) it easily follows that

$$\frac{1}{2\pi}E_\varepsilon(v_\varepsilon,A_{\bar\phi,\pi/2})=\cos\bar\phi\,\sqrt{\beta D/2}\,\delta_1(p_1,\mathcal{M}_1)\varepsilon^{-1/2}+o(\varepsilon^{-1/2})\,. \tag{5.3}$$

The above construction replaces the construction on the intervals (i) and (ii) in the proof of Proposition 2.1. From here we follow exactly that construction on the intervals (iii)–(v). Finally, the construction on the intervals (vi)–(vii) is modified in a similar manner to the above, and yields the analogous estimate to (5.3), namely

$$\frac{1}{2\pi}E_\varepsilon(v_\varepsilon,A_{-\pi/2,\bar\phi-2\varepsilon-(\frac{1}{c_1}+\frac{1}{c_2})\varepsilon\ln 1/\varepsilon})=\cos\bar\phi\,\sqrt{\beta D/2}\,\delta_2(p_2,\mathcal{M}_2)\varepsilon^{-1/2}+o(\varepsilon^{-1/2})\,. \tag{5.4}$$

Combining (5.3) and (5.4) with the estimates from the proof of Proposition 2.1 yields i.e.,

$$\frac{1}{2\pi}E_\varepsilon(v_\varepsilon)\le\cos\bar\phi\Big(\frac{2D}{\varepsilon}+\sqrt{2\beta D}\,\big(\delta_1(p_1,\mathcal{M}_1)+\delta_2(p_2,\mathcal{M}_2)\big)\varepsilon^{-1/2}\Big)+o(\varepsilon^{-1/2})\,. \tag{5.5}$$

It remains to fix the value of $\bar\phi$ in such a way that the constraint (2.10) is satisfied. Put

$$\mu_1=\frac{\int_{\bar\phi}^{\pi/2}|v_\varepsilon|\cos\phi\,d\phi}{1-\sin\bar\phi}\quad\text{and }\mu_2=\frac{\int_{-\pi/2}^{\bar\phi}|v_\varepsilon|\cos\phi\,d\phi}{1+\sin\bar\phi}\,.$$

By our construction of v_ε and (1.20) we have

$$(\mu_1-m_1)(1-\sin\bar\phi)=\int_{\bar\phi}^{\pi/2}(|v_\varepsilon|-m_1)\cos\phi=\frac{\delta_1(p_1,\mathcal{M}_1)}{2\sqrt{2\beta D}}(\cos\bar\phi)\varepsilon^{1/2}+o(\varepsilon^{1/2}) \tag{5.6}$$

and

$$(\mu_2-m_2)(1+\sin\bar\phi)=\int_{-\pi/2}^{\bar\phi}(|v_\varepsilon|-m_2)\cos\phi=\frac{\delta_2(p_2,\mathcal{M}_2)}{2\sqrt{2\beta D}}(\cos\bar\phi)\varepsilon^{1/2}+o(\varepsilon^{1/2})\,. \tag{5.7}$$

We have also a third equation coming from the constraint,

$$\mu_1(1-\sin\bar\phi)+\mu_2(1+\sin\bar\phi)=m_1(1-\sin\phi_0)+m_2(1+\sin\phi_0)\,. \tag{5.8}$$

Writing $\bar\phi=\phi_0-\tau\varepsilon^{1/2}$ and using first order Taylor expansion yields from (5.6)–(5.8) a perturbed linear system of 3 equations in the 3 unknowns μ_1,μ_2 and τ which has a solution

$$\tau=\sqrt{\frac{\beta}{8D}}\big(\delta_1(p_1,\mathcal{M}_1)+\delta_2(p_2,\mathcal{M}_2)\big)\cot\phi_0+o(1)\,. \tag{5.9}$$

For the value of τ given by (5.9) we find by the Taylor expansion for the cosine function that

$$\cos\bar{\phi} = \cos\phi_0 + \tau\varepsilon^{1/2}\sin\phi_0 + o(\varepsilon^{1/2}) = \cos\phi_0\Big(1 + \sqrt{\frac{\beta}{8D}}(\delta_1+\delta_2)\varepsilon^{1/2}\Big) + o(\varepsilon^{1/2})\,. \quad (5.10)$$

Finally, plugging (5.10) in (5.5) gives (5.1). □

Proof of Theorem 1 completed. Since the energy estimate (1.22) follows from Proposition 4.1 and Proposition 5.1, it remains to prove the convergence result, i.e., assertion (i). Passing to a subsequence, we may deduce from Lemma 4.6 that $\lim p_1^{\varepsilon_n} = p_1 = \zeta_1^{(i_0)}$ and $\lim p_2^{\varepsilon_n} = p_2 = \zeta_2^{(i_0)}$ for some geodesic $\underline{\gamma}^{(i_0)} \in \mathcal{G}$. Moreover, from (4.53) and (5.1) we deduce that there is a pair of corresponding points, $x^{(j)} \in \mathcal{M}_j$, $j = 1, 2$, that together with $\underline{\gamma}^{(i_0)}$, realize the minimum K in (1.23). Next we shall show that $u_{\varepsilon_n} \to x^{(1)}$ uniformly on $[\phi_0+\delta, \frac{\pi}{2}-\delta]$ for any $\delta > 0$ (the uniform convergence $u_{\varepsilon_n} \to x^{(2)}$ on $[-\frac{\pi}{2}+\delta, \phi_0-\delta]$ is proved in the same manner). A direct consequence of (3.34) is that

$$d(u_\varepsilon, \Gamma_1) \to 0 \quad \text{uniformly on } [\phi_0+\delta, \frac{\pi}{2}-\delta]\,. \quad (5.11)$$

Therefore, whenever a sequence $\{\phi_n\} \subset [\phi_0+\delta, \frac{\pi}{2}-\delta]$ satisfies $u_{\varepsilon_n}(\phi_n) \to x_1$, then necessarily $x_1 \in \Gamma_1$. Combining (4.21) with the upper-bound (5.1) (or (2.9)) we get

$$\int_{\gamma_1}^{\pi/2-\varepsilon^{1/3}} \Big(|v_{1,\varepsilon}'|^2 + \frac{2\beta d_W^{(\varepsilon^{1/2})}(p_1^\varepsilon, p_2^\varepsilon)}{\varepsilon}(|v_{1,\varepsilon}(\phi)| - m_1)\Big)\cos\phi \le C\varepsilon^{1/2}\,, \quad (5.12)$$

where $v_{1,\varepsilon} = \tilde{s}_1(u_\varepsilon)$. A direct consequence of (5.12) is that for each $\eta > 0$ we have

$$\text{meas}\Big(\{\phi \in [\phi_0+\delta, \frac{\pi}{2}-\delta]\} \,:\, |v_{1,\varepsilon}(\phi)| - m_1 > \eta\Big) \le C(\eta)\varepsilon^{1/2}\,. \quad (5.13)$$

Since (5.13) implies that $\lim_{\varepsilon\to 0}|v_{1,\varepsilon}(\phi)| = m_1$ in measure, for a subsequence we have $\lim_{\varepsilon_n\to 0}|v_{1,\varepsilon_n}(\phi)| = m_1$ a.e. on $(\phi_0, \frac{\pi}{2})$. Consider then a point $\tilde{\phi} \in [\phi_0+\delta, \frac{\pi}{2}-\delta]$ such that (possibly for a further subsequence) $\lim_{\varepsilon_n\to 0} u_{\varepsilon_n}(\tilde{\phi}) = \lim_{\varepsilon_n\to 0} v_{1,\varepsilon_n}(\tilde{\phi}) = x_1 \in \mathcal{M}_1$. From the proof of Lemma 4.5 it follows that x_1 must coincide with $x^{(1)}$; otherwise we would get a contradiction to the upper bound (5.1). In the last conclusion we used hypothesis (H_4), which implies that the "distance" according to the expression in (1.19) between two distinct points of Γ_1 is positive. The above argument implies that if $\lim_{\varepsilon_n\to 0} u_{\varepsilon_n}(\phi_n) = \lim_{\varepsilon_n\to 0} v_{1,\varepsilon_n}(\phi_n) = x$, with $\{\phi_n\} \subset [\phi_0+\delta, \frac{\pi}{2}-\delta]$, then $x = x^{(1)}$. The claimed uniform convergence on $[\phi_0+\delta, \frac{\pi}{2}-\delta]$ follows. □

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