

UNIQUENESS OF POSITIVE SOLUTIONS FOR SINGULAR PROBLEMS INVOLVING THE p -LAPLACIAN

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ABSTRACT. We study existence and uniqueness of positive eigenfunctions for the singular eigenvalue problem: $-\Delta_p u - \lambda \eta(x) \frac{u^{p-1}}{|x|^p} = \mu \frac{u^{p-1}}{|x|^p}$ on a bounded smooth domain $\Omega \subset \mathbb{R}^N$ with zero boundary condition. We also characterize all positive solutions of $-\Delta_p u = |\frac{N-p}{p}|^p \frac{u^{p-1}}{|x|^p}$ in $\mathbb{R}^N \setminus \{0\}$.

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1. INTRODUCTION

This note is concerned with solutions of the following nonlinear eigenvalue problem,

$$(1.1) \quad \begin{cases} -\Delta_p u - \lambda \eta(x) \frac{|u|^{p-2} u}{|x|^p} = \mu \frac{|u|^{p-2} u}{|x|^p} & \text{in } \Omega \setminus \{0\}, \\ u > 0 & \text{in } \Omega \setminus \{0\}, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here Ω is a bounded domain in \mathbb{R}^N of class C^2 with $0 \in \Omega$, $p \in (1, \infty) \setminus \{N\}$, $\eta \in C^\alpha(\overline{\Omega})$ ($\alpha \in (0, 1)$) such that $\eta \geq 0$, $\eta \not\equiv 0$ in $\overline{\Omega}$ and $\eta(0) = 0$, $\mu, \lambda \in \mathbb{R}$ are two parameters and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$. By the regularity results of Tolksdorf [11] and Di Benedetto [4] any weak solution $u \in \bigcap_{r>0} W^{1,p}(\Omega \setminus B_r(0))$ to (1.1) actually belongs to $C^1(\overline{\Omega} \setminus \{0\})$.

Naturally related to (1.1) is the following variational problem:

$$(1.2) \quad I[\lambda] := \inf_{0 \neq v \in W_0^{1,p}(\Omega \setminus \{0\})} \frac{\int_\Omega |\nabla v|^p - \lambda \int_\Omega \frac{\eta |v|^p}{|x|^p}}{\int_\Omega (|v|/|x|)^p}.$$

Recall that in the case $\lambda = 0$ it follows from the well known Hardy inequality,

$$(1.3) \quad \int_\Omega |\nabla u|^p \geq \left| \frac{N-p}{p} \right|^p \int_\Omega (|u|/|x|)^p, \quad \forall u \in W_0^{1,p}(\Omega \setminus \{0\}),$$

that $I[0] = c_{p,N}^* := \left| \frac{N-p}{p} \right|^p$. Moreover, by a simple construction of test functions approximating $|x|^{1-N/p}$, see [8], it can be shown that we have always, $I[\lambda] \leq c_{p,N}^*$. It was also proved in [8] (in the spirit of [3]) that there exists $\lambda^* > 0$ such that: $I[\lambda] = c_{p,N}^*$ for $\lambda \leq \lambda^*$ and $I[\lambda] < c_{p,N}^*$ for $\lambda > \lambda^*$. If $u \in W_0^{1,p}(\Omega \setminus \{0\})$ is a minimizer for (1.2) then u is clearly a solution to the equation in (1.1) with $\mu = I[\lambda]$. Since $|u|$ too is a minimizer, hence a solution to the same equation, it follows from Serrin's Harnack

principle [9] that $|u| > 0$ in $\Omega \setminus \{0\}$. Therefore, either u or $-u$ is a solution to (1.1). It turns out that solutions of (1.1) can be obtained as minimizers for (1.2) if and only if $I[\lambda] < c_{p,N}^*$. In this direction we have:

Theorem 1.1. *A solution u to (1.1) with $\mu = I[\lambda]$ which belongs to $W_0^{1,p}(\Omega \setminus \{0\})$ exists if and only if $I[\lambda] < c_{p,N}^*$. In this case u is a minimizer for (1.2) and it is the unique solution to (1.1) up to a multiplicative factor.*

The critical role played by the value $c_{p,N}^*$ for the existence of a minimizer is analogue to a phenomenon that was observed first by Marcus, Mizel and Pinchover [6] in the context of another version of Hardy inequality. The existence of a minimizer in the subcritical case $I[\lambda] < c_{p,N}^*$ is proved in the appendix, using a ‘‘concentration-compactness principle’’ [5], similar to the one used in [6, 7]. The uniqueness part in the subcritical case is then essentially known ([2, 1, 7]), see the appendix for details.

The next natural question is what happens in the critical case $I[\lambda] = c_{p,N}^*$. In that case the existence of a solution $u \in \bigcap_{q \in [1,p)} W_0^{1,q}(\Omega \setminus \{0\}) \setminus W_0^{1,p}(\Omega \setminus \{0\})$ was proved in [8]. We are left then with the problem of uniqueness. This is the subject of the main theorem of this note:

Theorem 1.2. *In the case $I[\lambda] = c_{p,N}^*$ the solution to (1.1) with $\mu = c_{p,N}^*$ is unique up to a multiplicative factor.*

Uniqueness in the linear case $p = 2$ is known and easy (for any value of μ , see Remark 3.1), but we could not find an easy proof for general p . Our argument, presented in Section 3, is based on Harnack inequality and on the following Liouville type theorem, which may be of independent interest.

Theorem 1.3. *Let u be a solution of,*

$$(1.4) \quad \begin{cases} -\Delta_p u = c_{p,N}^* \frac{u^{p-1}}{|x|^p} & \text{in } \mathbb{R}^N \setminus \{0\}, \\ u > 0 & \text{in } \mathbb{R}^N \setminus \{0\}. \end{cases}$$

Then $u(x) = C|x|^{1-N/p}$ for some positive constant C .

2. CHARACTERIZATION OF GLOBAL SOLUTIONS

In the proof of Theorem 1.3 we will make use of an extension of Picone’s identity for the p -Laplacian, due to Allegretto and Huang [1], that we now recall. Let Ω be a subdomain of \mathbb{R}^N . For u, v in $C^1(\Omega)$ such that $u \geq 0$ and $v > 0$ in Ω denote,

$$(2.1) \quad \begin{aligned} L(u, v) &= |\nabla u|^p + (p-1) \frac{u^p}{v^p} |\nabla v|^p - p \frac{u^{p-1}}{v^{p-1}} \nabla u |\nabla v|^{p-2} \nabla v, \\ R(u, v) &= |\nabla u|^p - \nabla \left(\frac{u^p}{v^{p-1}} \right) |\nabla v|^{p-2} \nabla v. \end{aligned}$$

Then $L(u, v) = R(u, v) \geq 0$ in Ω and $L(u, v) = R(u, v) = 0$ in Ω if and only if $u = Cv$ for some constant C , see [1].

Proof of Theorem 1.3. By the regularity results of Tolksdorf [11] and Di Benedetto [4] any weak solution to (1.4) belongs to $C^1(\mathbb{R}^N \setminus \{0\})$. Clearly $v(x) = |x|^{1-N/p}$ is a solution of (1.4). We must show that u is a multiple of v . For each $R > 0$ we define a cut-off function ψ_R as follows,

$$(2.2) \quad \psi_R(x) = \begin{cases} 0, & 0 \leq |x| \leq \frac{1}{R^2} \\ 2 + \frac{\log|x|}{\log R}, & \frac{1}{R^2} \leq |x| \leq \frac{1}{R} \\ 1, & \frac{1}{R} \leq |x| \leq R \\ 2 - \frac{\log|x|}{\log R}, & R \leq |x| \leq R^2 \\ 0, & |x| \geq R^2 \end{cases}$$

Clearly $\psi_R(x)$ is a Lipschitz function with compact support in \mathbb{R}^N . By Picone's identity (2.1) and (1.4),

$$(2.3) \quad \begin{aligned} 0 \leq \int_{\mathbb{R}^N} L(\psi_R v, u) &= \int_{\mathbb{R}^N} R(\psi_R v, u) = \int_{\mathbb{R}^N} |\nabla(\psi_R v)|^p - \int_{\mathbb{R}^N} \nabla \left(\frac{\psi_R^p v^p}{u^{p-1}} \right) |\nabla u|^{p-2} \nabla u \\ &= \int_{\mathbb{R}^N} \left(|\nabla(\psi_R v)|^p - \frac{C_{p,N}^*}{|x|^p} \psi_R^p v^p \right). \end{aligned}$$

On the other hand, testing (1.4) for v against $\psi_R^p v$ yields,

$$(2.4) \quad \int_{\mathbb{R}^N} \frac{C_{p,N}^*}{|x|^p} \psi_R^p v^p = \int_{\mathbb{R}^N} \psi_R^p |\nabla v|^p + p \psi_R^{p-1} v |\nabla v|^{p-2} \nabla v \nabla \psi_R.$$

Combining (2.3) with (2.4) we obtain,

$$(2.5) \quad 0 \leq I(R) := \int_{\mathbb{R}^N} L(\psi_R v, u) = \int_{\mathbb{R}^N} \left(|\psi_R \nabla v + v \nabla \psi_R|^p - \psi_R^p |\nabla v|^p - p \psi_R^{p-1} v |\nabla v|^{p-2} \nabla v \nabla \psi_R \right)$$

Next we shall prove that $\lim_{R \rightarrow \infty} I(R) = 0$. We recall the following elementary inequality which holds for all $z_1, z_2 \in \mathbb{R}^N$ (see [10, Lemma A.4]):

$$(2.6) \quad |z_1 + z_2|^p - |z_1|^p - p|z_1|^{p-2} z_1 \cdot z_2 \leq \begin{cases} \frac{p(p-1)}{2} (|z_1| + |z_2|)^{p-2} |z_2|^2, & \text{for } p > 2, \\ \gamma_p |z_2|^p & \text{for } p \leq 2, \end{cases}$$

for some constant $\gamma_p > 0$. Assume first that $p \leq 2$. Then from (2.2), (2.5) and (2.6) we get,

$$(2.7) \quad I(R) \leq C \left(\int_{\frac{1}{R^2}}^{\frac{1}{R}} r^{p-N} \left(\frac{1}{r \log R} \right)^p r^{N-1} dr + \int_R^{R^2} r^{p-N} \left(\frac{1}{r \log R} \right)^p r^{N-1} dr \right) = \frac{2C}{(\log R)^{p-1}},$$

hence $\lim_{r \rightarrow \infty} I(R) = 0$ in this case. Now consider the case $p > 2$. Thanks to (2.6) and (2.7) we need only to show that

$$(2.8) \quad \int_{A_R} |\nabla v|^{p-2} v^2 |\nabla \psi_R|^2 \rightarrow 0 \text{ as } R \rightarrow \infty,$$

where $A(R) = \{\frac{1}{R^2} < |x| < \frac{1}{R}\} \cup \{R < |x| < R^2\}$. Indeed, (2.8) follows from

$$\begin{aligned} \int_{A_R} |\nabla v|^{p-2} v^2 |\nabla \psi_R|^2 &\leq C \int_{\frac{1}{R^2}}^{\frac{1}{R}} r^{-\frac{N}{p}(p-2)} r^{2(1-\frac{N}{p})} \frac{r^{N-1} dr}{r^2 (\log R)^2} \\ &\quad + C \int_R^{R^2} r^{-\frac{N}{p}(p-2)} r^{2(1-\frac{N}{p})} \frac{r^{N-1} dr}{r^2 (\log R)^2} = \frac{2C}{\log R} \rightarrow 0 \text{ as } R \rightarrow \infty. \end{aligned}$$

Therefore, in all cases we have, $\lim_{R \rightarrow \infty} \int_{\mathbb{R}^N} L(\psi_R v, u) = 0$. By Fatou's Lemma we infer that $L(v, u) \in L^1(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} L(v, u) = 0$. Hence, $L(v, u) = 0$ in \mathbb{R}^N which implies that for some constant $C > 0$ we have: $u = Cv = C|x|^{1-N/p}$. \square

From Theorem 1.3 we next deduce the asymptotic radial symmetry at the origin of solutions on a bounded domain.

Corollary 2.1. *Let $u \in W_{loc}^{1,p}(B_R(0) \setminus \{0\})$ be a solution of*

$$(2.9) \quad \begin{cases} -\Delta_p u = \frac{u^{p-1}}{|x|^p} (c_{p,N}^* + \eta(x)) \text{ on } B_R(0) \setminus \{0\}, \\ u > 0 \text{ on } B_R(0) \setminus \{0\}, \end{cases}$$

where $\eta(x) \in C(B_R(0))$ satisfies $\eta(0) = 0$. Then,

$$(2.10) \quad \lim_{r \rightarrow 0^+} \frac{\bar{u}(r)}{\underline{u}(r)} = 1,$$

where for each $r \in (0, R)$ we denote,

$$(2.11) \quad \bar{u}(r) = \max_{\{|x|=r\}} u(x), \quad \underline{u}(r) = \min_{\{|x|=r\}} u(x).$$

Proof. Fix $n \in \mathbb{R}^N$ with $|n| = 1$ and for any $r \in (0, R/2)$ set,

$$(2.12) \quad v_r(y) = \frac{u(r y)}{u(r n)}, \quad \text{for } y \in B_{R/r}(0) \setminus \{0\}.$$

Then v_r satisfies

$$(2.13) \quad -\Delta_p v_r = \frac{v_r^{p-1}}{|y|^p} (c_{p,N}^* + \eta(r y)) \quad \text{on } B_{R/r}(0) \setminus \{0\},$$

and $v_r(n) = 1$. For any $D > 1$ denote $\mathcal{A}_D = \{1/D < |y| < D\}$. Then, by the Harnack inequality of Serrin [9] we infer that for each $D > 1$ there exist $K_D > 1$, such that for any $r < R/(2D)$ we have:

$$(2.14) \quad \frac{1}{K_D} \leq v_r(y) \leq K_D \quad \text{on } \mathcal{A}_D.$$

Then, from the regularity results of [11, 4] it follows that there exist $L_D > 0$ and $\alpha_D \in (0, 1)$, such that for each $r < R/(2D)$ we have,

$$(2.15) \quad \|v_r(y)\|_{C^{1,\alpha_D}(\mathcal{A}_{D/2})} \leq L_D,$$

where we denoted $\mathcal{A}_{D/2} = \{2/D < |y| < D/2\}$. Set $Q := \limsup_{r \rightarrow 0^+} \frac{\bar{u}(r)}{\underline{u}(r)}$ and choose a sequence $r_n \rightarrow 0$ such that $\lim_{n \rightarrow \infty} \frac{\bar{u}(r_n)}{\underline{u}(r_n)} = Q$. Using (2.15) we deduce from (2.13) that for a subsequence, still denoted by $\{r_n\}$, we have, $v_{r_n} \rightarrow w_D$ in $C^1(\mathcal{A}_{D/2})$, where w_D is a positive solution of,

$$-\Delta_p w_D = c_{p,N}^* \frac{w_D^{p-1}}{|y|^p} \quad \text{on } \mathcal{A}_{D/2}.$$

Choosing a sequence $D_m \nearrow \infty$, and passing to a diagonal subsequence we obtain a subsequence $r_{n_k} \rightarrow 0$ such that $v_{r_{n_k}} \rightarrow v_0$ in $C_{loc}^1(\mathbb{R}^N \setminus \{0\})$ where v_0 is a positive solution of (1.4). By Theorem 1.3, $v_0(x) = C|x|^{1-N/p}$, hence $Q = 1$. \square

3. UNIQUENESS OF THE EIGENFUNCTION IN THE CRITICAL CASE

This section is devoted to the proof of Theorem 1.2. We shall need the following comparison lemma.

Lemma 3.1. *Let Ω be a bounded domain in \mathbb{R}^N of class C^2 and let $K \subset\subset \Omega$ be a compact subset. Suppose that $u_1, u_2 \in C(\overline{\Omega \setminus K}) \cap W^{1,p}(\Omega \setminus K)$ are such that $u_2 > 0$ on $(\Omega \setminus K) \cup \partial K$ and $u_1 = 0$ on $\partial\Omega$. Suppose further that:*

$$\begin{aligned} -\Delta_p u_1 - a(x)|u_1|^{p-2}u_1 &\leq 0 & \text{in } \Omega \setminus K, \\ -\Delta_p u_2 - a(x)|u_2|^{p-2}u_2 &\geq 0 & \text{in } \Omega \setminus K, \end{aligned}$$

for some $a(x) \in L_{loc}^\infty(\Omega \setminus K)$, and that

$$u_2 \geq u_1 \quad \text{on } \partial K.$$

Then,

$$u_2(x) \geq u_1(x) \quad \text{on } \Omega \setminus K.$$

Proof. The result follows directly from the argument of Proposition A.1 and Corollary A.1 in [10]. The results in [10] deal with the case where $K = \{x \in \Omega, \text{dist}(x, \partial\Omega) \geq \beta\}$ for some $\beta > 0$, but it is clear that the argument there works for general K . \square

Proof of Theorem 1.2. Let u_1, u_2 be two solutions to (1.1). Again by regularity theory we have $u_1, u_2 \in C^1(\overline{\Omega} \setminus \{0\})$. For all sufficiently small $r > 0$ set,

$$(3.1) \quad f(r) = \min_{\{|x|=r\}} \frac{u_2(x)}{u_1(x)}, \quad F(r) = \max_{\{|x|=r\}} \frac{u_2(x)}{u_1(x)}.$$

Thanks to Corollary 2.1 we have $\lim_{r \rightarrow 0^+} F(r)/f(r) = 1$ and we can define,

$$m := \liminf_{r \rightarrow 0^+} f(r) = \liminf_{r \rightarrow 0^+} F(r) \in [0, \infty],$$

$$M := \limsup_{r \rightarrow 0^+} f(r) = \limsup_{r \rightarrow 0^+} F(r) \in [0, \infty].$$

We can assume without loss of generality that $m < \infty$ (otherwise we interchange between u_1 and u_2). For every $\varepsilon > 0$ there exists a sequence $r_n \rightarrow 0$ such that,

$$u_2(x) \leq (m + \varepsilon)u_1(x) \quad \text{on } \{|x| = r_n\}.$$

By Lemma 3.1 it follows that,

$$(3.2) \quad u_2(x) \leq (m + \varepsilon)u_1(x) \quad \text{on } \Omega \setminus B_{r_n}(0).$$

Letting n tend to ∞ in (3.2) and then sending ε to 0 we obtain that,

$$u_2(x) \leq mu_1(x) \quad \text{on } \Omega \setminus \{0\}.$$

It follows that $M = m > 0$. Interchanging between u_1 and u_2 and repeating the above argument we get that also $u_2(x) \geq mu_1(x)$ and we conclude that $u_2 = mu_1$. \square

Remark 3.1. In the *linear case* $p = 2$ the uniqueness (up to a multiplicative factor) of a solution to (1.1) is easy for any μ and λ . Indeed, if u and v are two solutions we have by (4.12) below that $v \geq c_1 u$ in Ω for some $c_1 > 0$. Defining $\bar{c}_1 := \sup\{c_1 > 0; v \geq c_1 u\}$ we have $v - \bar{c}_1 u \geq 0$ in Ω . If $v - \bar{c}_1 u = 0$ we are done. Otherwise, since $v - \bar{c}_1 u$ is also a solution of the equation in (1.1) (recall that we are in the linear case), we have by the strong maximum principle $v - \bar{c}_1 u > 0$ in Ω . Hence by the above argument there exists $c_2 > 0$ such that $v - \bar{c}_1 u > c_2 u$ in Ω . This is a contradiction to the maximality of \bar{c}_1 .

4. APPENDIX: EXISTENCE AND UNIQUENESS IN THE SUBCRITICAL CASE

This section is devoted to the subcritical case $I[\lambda] < c_{p,N}^*$. Existence of a minimizer in (1.2) is established in the next proposition.

Proposition 4.1. *Assume that $I[\lambda] < c_{p,N}^*$. Let $\{u_n\} \subset W_0^{1,p}(\Omega)$ be a sequence of nonnegative functions satisfying,*

$$(4.1) \quad \int_{\Omega} \frac{u_n^p}{|x|^p} = 1, \quad \forall n,$$

and

$$(4.2) \quad \lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n|^p - \lambda \int_{\Omega} \frac{\eta |u_n|^p}{|x|^p} = I[\lambda].$$

Then $u_n \rightarrow u$ strongly in $W_0^{1,p}(\Omega \setminus \{0\})$ where u is a minimizer for (1.2).

Proof. We shall first prove the convergence of a subsequence. It is clear from (4.1)–(4.2) that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega \setminus \{0\})$, and passing to a subsequence we may assume that,

$$(4.3) \quad u_n \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega \setminus \{0\}) \quad \text{and} \quad u_n \rightarrow u \text{ in } L^p(\Omega),$$

for some nonnegative $u \in W_0^{1,p}(\Omega \setminus \{0\})$. We denote $v_n = u_n - u$. Fix any $r > 0$ such that $B_{2r}(0) \subset \Omega$. Then

$$(4.4) \quad \left| \int_{B_{2r}(0)} |\nabla u_n|^p - |\nabla v_n|^p \right| \leq C \|\nabla(u_n - v_n)\|_{L^p(B_{2r}(0))} = C \|\nabla u\|_{L^p(B_{2r}(0))} := \varepsilon_r,$$

with $\varepsilon_r \rightarrow 0$ as $r \rightarrow 0$. Let $\varphi_r \in C_c^\infty(\mathbb{R}^N)$ denote a cut-off function satisfying:

- (i) $\varphi_r(x) = 1$ for $|x| \leq r$,
- (ii) $0 \leq \varphi_r(x) \leq 1$ for $r < |x| < 2r$,
- (iii) $\varphi_r(x) = 0$ for $|x| \geq 2r$,
- (iv) $\|\nabla \varphi_r\|_{L^\infty} \leq 4/r$.

By Hölder inequality,

$$\begin{aligned} \int_{B_{2r}(0)} |\nabla(\varphi_r v_n)|^p &\leq \int_{B_{2r}(0)} |\varphi_r|^p |\nabla v_n|^p + p \int_{B_{2r}(0)} |v_n \nabla \varphi_r| (|v_n \nabla \varphi_r| + |\varphi_r \nabla v_n|)^{p-1} \\ &\leq \int_{B_{2r}(0)} |\nabla v_n|^p + \frac{4p}{r} \left(\int_{B_{2r}(0)} (|v_n \nabla \varphi_r| + |\varphi_r \nabla v_n|)^p \right)^{\frac{p-1}{p}} \left(\int_{B_{2r}(0)} |v_n|^p \right)^{\frac{1}{p}}. \end{aligned}$$

Therefore, using (4.3) we infer that,

$$(4.5) \quad \int_{B_{2r}(0)} |\nabla(\varphi_r v_n)|^p \leq \int_{B_{2r}(0)} |\nabla v_n|^p + D_r \alpha_n,$$

with $D_r > 0$ and $\alpha_n \rightarrow 0$. By Hardy inequality (1.3) we get,

$$\int_{B_{2r}(0)} |\nabla(\varphi_r v_n)|^p \geq c_{p,N}^* \int_{B_{2r}(0)} \frac{|\varphi_r|^p |v_n|^p}{|x|^p} \geq c_{p,N}^* \int_{B_r(0)} \frac{|v_n|^p}{|x|^p},$$

which together with (4.4) and (4.5) yields,

$$(4.6) \quad \int_{B_{2r}(0)} |\nabla u_n|^p \geq c_{p,N}^* \int_{B_r(0)} \frac{|v_n|^p}{|x|^p} - D_r \alpha_n - \varepsilon_r.$$

On the other hand, as in (4.4) we get,

$$(4.7) \quad \left| \int_{B_r(0)} \frac{u_n^p}{|x|^p} - \int_{B_r(0)} \frac{|v_n|^p}{|x|^p} \right| \leq C \left(\int_{B_r(0)} \left(\frac{u}{|x|} \right)^p \right)^{1/p} := \delta_r,$$

with $\delta_r \rightarrow 0$ as $r \rightarrow 0$ (by (1.3)). Moreover, by (4.3) and (1.3) we easily deduce that,

$$(4.8) \quad \lim_{n \rightarrow \infty} \int_{\Omega} \frac{\eta u_n^p}{|x|^p} - \int_{\Omega} \frac{\eta u^p}{|x|^p} = 0.$$

By (4.1), (4.6) and (4.7) we obtain,

$$(4.9) \quad \int_{\Omega} |\nabla u_n|^p - \lambda \int_{\Omega} \frac{\eta u_n^p}{|x|^p} \geq \int_{\Omega \setminus B_{2r}(0)} |\nabla u_n|^p + c_{p,N}^* \left(1 - \int_{\Omega \setminus B_r(0)} \frac{u_n^p}{|x|^p}\right) - \lambda \int_{\Omega} \frac{\eta u_n^p}{|x|^p} \\ - c_{p,N}^* \delta_r - D_r \alpha_n - \varepsilon_r.$$

Passing to the limit $n \rightarrow \infty$ in (4.9), using (4.2), (4.3) and (4.8), yields

$$(4.10) \quad I[\lambda] \geq \int_{\Omega \setminus B_{2r}(0)} |\nabla u|^p + c_{p,N}^* \left(1 - \int_{\Omega \setminus B_r(0)} \frac{u^p}{|x|^p}\right) - \lambda \int_{\Omega} \frac{\eta u^p}{|x|^p} - c_{p,N}^* \delta_r - \varepsilon_r.$$

Passing to the limit $r \rightarrow 0$ in (4.10) gives, using the definition of $I[\lambda]$,

$$(4.11) \quad I[\lambda] \geq \int_{\Omega} |\nabla u|^p + c_{p,N}^* \left(1 - \int_{\Omega} \frac{u^p}{|x|^p}\right) - \lambda \int_{\Omega} \frac{\eta u^p}{|x|^p} \\ \geq c_{p,N}^* \left(1 - \int_{\Omega} \frac{u^p}{|x|^p}\right) + I[\lambda] \int_{\Omega} \frac{u^p}{|x|^p}.$$

By (4.11) and our assumption $I[\lambda] < c_{p,N}^*$ it follows that $\int_{\Omega} \frac{u^p}{|x|^p} = 1$ and then by (4.11) we infer that u is a minimizer. Since $\int_{\Omega} \frac{u_n^p}{|x|^p} \rightarrow \int_{\Omega} \frac{u^p}{|x|^p} = 1$ we get also that $\int_{\Omega} |\nabla u_n|^p \rightarrow \int_{\Omega} |\nabla u|^p$ and the strong convergence $u_n \rightarrow u$ in $W_0^{1,p}(\Omega \setminus \{0\})$ follows. So far we proved the convergence of a subsequence of $\{u_n\}$. But since it is known that a solution to (1.1) in $W_0^{1,p}(\Omega \setminus \{0\})$ is unique up to a multiplicative factor, see [1, Theorem 2.1] and [7, Proposition 3.2], the full convergence $u_n \rightarrow u$ follows as well. \square

Proof of Theorem 1.1. Existence of a positive minimizer u for (1.2), which is then a solution of (1.1), follows directly from Proposition 4.1. Next we turn to the proof of the uniqueness part. Let v be another solution to (1.1) with $\mu = I[\lambda]$. We claim that there exists $c_1 > 1$ such that

$$(4.12) \quad \frac{u}{c_1} \leq v \leq c_1 u \quad \text{on } \Omega \setminus \{0\}.$$

Indeed, to see for example why u/v is bounded assume by negation that $\sup_{\Omega} u/v = \infty$.

By Hopf boundary lemma [12] we must have $\lim_{r \rightarrow 0} \sup_{B_r(0) \setminus \{0\}} u/v = \infty$. Using Harnack inequality [9] we obtain for some sequence $r_n \rightarrow 0$ that $u \geq n v$ on $\partial B_{r_n}(0)$. But then we get by Lemma 3.1 that $u \geq n v$ on $\Omega \setminus B_{r_n}(0)$. This is clearly a contradiction for n large enough, and (4.12) follows. On the other hand, by a simple rescaling argument, Harnack inequality ([9]) and C^1 regularity estimates ([11, 4]) it follows (see [10, Lemma A.3]) that, for some $c_2 > 0$ we have,

$$(4.13) \quad \left| \left(\frac{\nabla v}{v} \right) (x) \right| \leq \frac{c_2}{|x|} \quad \text{on } B_d(0) \setminus \{0\}, \quad \text{with } d := \frac{1}{2} \min\{|y|; y \in \partial\Omega\}.$$

Combining (4.12) and (4.13) we get that,

$$(4.14) \quad |\nabla v(x)| \leq c_1 c_2 \frac{u(x)}{|x|} \quad \text{on } B_d(0) \setminus \{0\}.$$

By (4.14) and (1.3) we obtain that $v \in W_0^{1,p}(\Omega \setminus \{0\})$ and therefore v is a minimizer for (1.2). The fact that v is a multiple of u follows from the argument cited at the end of the proof of Proposition 4.1. To complete the proof we remark that in [8] it was proved that there is no solution in $W_0^{1,p}(\Omega \setminus \{0\})$ to (1.1) with $\mu = c_{p,N}^*$. \square

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REFERENCES

- [1] W. Allegretto and Y. X. Huang, *A Picone's identity for the p -Laplacian and applications*, Nonlinear Analysis TMA **32** (1998), 819–830.
- [2] A. Anane, *Simplicité et isolation de la première valeur propre du p -laplacien avec poids*, C. R. Acad. Sci. Paris **305**, Série I (1987), 725–728.
- [3] H. Brezis and M. Marcus, *Hardy's inequality revisited*, Ann. Sc. Norm. Pisa. **25** (1997), 217–237.
- [4] E. Di Benedetto, *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. TMA **7** (1983), 827–850.
- [5] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case. I.*, Rev. Mat. Iberoamericana **1** (1985), 145–201.
- [6] M. Marcus, V. J. Mizel and Y. Pinchover, *On the best constant for Hardy's inequality in \mathbb{R}^n* , Trans. A.M.S. **350** (1998), 3237–3255.
- [7] M. Marcus and I. Shafrir, *An eigenvalue problem related to Hardy's L^p inequality*, Ann. Sc. Norm. Pisa. **29** (2000), 581–604.
- [8] A. Poliakovsky, *On minimization problems which approximate Hardy's L^p inequality*, preprint.
- [9] J. Serrin, *Local behavior of solutions of quasi-linear equations*, Acta Math. **111** (1964), 247–302.
- [10] I. Shafrir, *Asymptotic behavior of minimizing sequences for Hardy's inequality*, Commun. Contemp. Math., **2** (2000), 151–189.
- [11] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Diff. Eqns. **51** (1984), 121–150.
- [12] J. L. Vázquez, *A strong Maximum Principle for Some Quasilinear Elliptic Equations*, Appl. Math. Optim. **12** (1984), no. 3, 191–202.

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