MINIMIZATION OF A GINZBURG-LANDAU TYPE ENERGY WITH POTENTIAL HAVING A ZERO OF INFINITE ORDER

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1. Introduction

Let G be a bounded and smooth, simply connected domain in $\mathbb{R}^2$ and let $g : \partial G \to S^1$ be a boundary condition of degree $\deg(g, \partial G) = d \geq 0$ (as we may assume without loss of generality). Consider a C^2 functional $J : \mathbb{R} \to [0, \infty)$ satisfying the following conditions:

(H_1) $J(0) = 0$ and $J(t) > 0$ on $(0, \infty)$,
(H_2) $J'(t) > 0$ on $(0, 1]$,
(H_3) There exists $\eta_0 > 0$ such that $J''(t) > 0$ on $(0, \eta_0)$.
For $\varepsilon > 0$ consider the energy functional

$$E_\varepsilon(u) = \int_G |\nabla u|^2 dx + \frac{1}{\varepsilon^2} \int_G J(1 - |u|^2)\, dx \tag{1.1}$$

over

$$H^1_g(G, \mathbb{C}) := \{u \in H^1(G, \mathbb{C}) \text{ s.t. } u = g \text{ on } \partial G\}\,. \tag{1.2}$$

It is easy to see that $\min_{u \in H^1_g(G,\mathbb{C})} E_\varepsilon(u)$ is achieved by some smooth u_ε which satisfies:

$$\begin{cases} -\Delta u_\varepsilon = \dfrac{1}{\varepsilon^2} j(1 - |u_\varepsilon|^2) u_\varepsilon \ \text{ in } G, \\ \quad u_\varepsilon = g \ \text{ on } \partial G, \end{cases} \tag{1.3}$$

where $j(t) := J'(t)$. The case $J(u) = (1-|u|^2)^2$, corresponding to the Ginzburg-Landau (GL) energy, was studied by Bethuel, Brezis and Hélein [1, 2] (see also Struwe [5]), where it was shown that:

(i) For a subsequence $\varepsilon_n \to 0$ we have, $u_{\varepsilon_n} \to u_* = e^{i\phi} \prod_{j=1}^{d} \frac{z-a_j}{|z-a_j|}$ in $C^{1,\alpha}(\bar{G} \setminus \{a_1, ..., a_d\})$, where $a_1, ..., a_d$ are distinct points in G and ϕ is a smooth harmonic function determined by the requirement $u_* = g$ on ∂G.
(ii) $E_\varepsilon(u_\varepsilon) = 2\pi d|\log\varepsilon| + O(1)$ as $\varepsilon \to 0$.

The method of [1, 2, 5] can be adapted without difficulty to the case of J satisfying $(H_1)-(H_3)$ with a zero of *finite order* at $t = 0$. This applies for example to $J(t) = |t|^k$, $\forall k \geq 2$. The main objective of the current paper is to treat the case of J with zero of *infinite order* at $t = 0$, having in mind the examples

$$(1.4) \qquad J_k(t) = \begin{cases} \exp(-1/t^k) & \text{for } t > 0\,, \\ 0 & \text{for } t \leq 0\,, \end{cases}$$

for any $k > 0$. It turns out that a convergence result, as in (i) above, holds for such J's as well. The main difference with respect to the usual GL-energy is in the energy asymptotics. For J with a zero of infinite order the "energy cost" of a degree-one vortex may be much less than the cost of $2\pi\log\frac{1}{\varepsilon}$ for the GL-functional (see (ii) above). In fact, we shall see that this cost equals

$$2\pi\log\frac{1}{\varepsilon} - \bar{I}(\frac{1}{\varepsilon})\,,$$

where $\bar{I}(R)$ is a positive function satisfying $\bar{I}(R) = o(\log R)$ as $R \to \infty$, which is determined by the particular functional J. More precisely, the function $\bar{I}(R)$ satisfies

$$(1.5) \qquad \bar{I}(R) = \frac{1}{2}\int_{1/R^2}^{j(\eta_0)} j^{-1}(t)\frac{dt}{t} + O(1)\,, \quad \text{as } R \to \infty \quad \text{(see Lemma 2.4)}\,.$$

So for example, for J_1 in (1.4) we find $\bar{I}(R) = \log\log R + O(1)$ (see Proposition 4.1 in the Appendix), and the asymptotics for the energies in this case reads:

$$E_\varepsilon(u_\varepsilon) = 2\pi d\big(\log\frac{1}{\varepsilon} - \log\log\frac{1}{\varepsilon}\big) + O(1).$$

Somewhat surprisingly, it turns out that we may have $\bar{I}(R) = O(1)$ also for J with a zero of infinite order, as is the case for $k \in (0,1)$ in (1.4), see Proposition 4.1.

Our first main theorem describes the asymptotic behavior of the minimizers and their energies.

Theorem 1. *For each $\varepsilon > 0$, let u_ε be a minimizer for the energy E_ε over $H^1_g(G,\mathbb{C})$ with G, g (of degree $d \geq 0$) as above and J satisfying $(H_1) - (H_3)$. Then:*
(i) For a subsequence $\varepsilon_n \to 0$ we have

$$u_{\varepsilon_n} \to u_* = e^{i\phi} \prod_{j=1}^{d} \Big(\frac{z - a_j}{|z - a_j|}\Big) \quad \textit{in } C^{1,\alpha}(\bar{G} \setminus \{a_1, ..., a_d\}),$$

where $a_1, ..., a_d$ are distinct points in G and ϕ is a smooth harmonic function determined by the requirement $u_* = g$ on ∂G.
(ii) Setting

$$I_0(R) = \frac{1}{2} \int_{1/R^2}^{j(\eta_0)} j^{-1}(t) \frac{dt}{t}\,,$$

we have

$$E_\varepsilon(u_\varepsilon) = 2\pi d \Big(\log \frac{1}{\varepsilon} - I_0\big(\frac{1}{\varepsilon}\big) \Big) + O(1). \tag{1.6}$$

We show in Lemma 2.4 below that the function I_0 satisfies $I_0(R) = o(\log R)$. This implies that the leading term in the energy is always of the order $|\log \varepsilon|$. It is easy to see that $I_0(R)$ is a positive, monotone increasing, concave function of $\log R$ (for large R). It is natural to ask whether every function with these properties can appear in the second order term of the energy expansion, for some potential J. The answer to this "inverse problem" turns out to positive, as shown by our second theorem.

Theorem 2. *Let $h \in C^2[0, \infty)$ satisfy, for some $T > 0$,*

$$h'(t) > 0\,, \;\; h''(t) < 0\,, \quad \textit{for } t \geq T > 0\,, \tag{1.7}$$

and

$$\lim_{t \to \infty} h'(t) = 0\,. \tag{1.8}$$

Then, there exists a functional J satisfying $(H_1) - (H_3)$, such that the minimizers $\{u_\varepsilon\}$ over $H^1_g(G,\mathbb{C})$, for E_ε defined by (1.1) and g of degree d as above, satisfy

$$E_\varepsilon(u_\varepsilon) = 2\pi d \Big(\log \frac{1}{\varepsilon} - h\big(\log \frac{1}{\varepsilon}\big) \Big) + O(1)\,.$$

2. A study of an auxiliary optimization problem

Let us begin by explaining the main idea of the proof of Theorem 1 and by showing how it leads to a certain optimization problem which is the object of the current section. It is natural to estimate first the energy cost of a degree-one "vortex" in a disc, say the unit disc $B_1 = B_1(0)$. In the case of the Ginzburg-Landau energy, it is easy to guess the energy cost, by taking $v_\varepsilon(r^{i\theta}) = f_\varepsilon(r)e^{i\theta}$ with f_ε given by:

$$f_\varepsilon(r) = \begin{cases} \frac{r}{\varepsilon} & \text{for } 0 \le r < \varepsilon, \\ 1 & \text{for } \varepsilon \le r \le 1. \end{cases}$$

A simple computation gives

$$\int_{B_1} |\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1 - |v_\varepsilon|^2)^2 = 2\pi \log \frac{1}{\varepsilon} + O(1),$$

which turns out to be the optimal estimate, up to an additive constant, although the proof of this fact is far from trivial (see [2]). When looking for the right upper-bound for the energy in the general case, we keep the ansatz $v_\varepsilon(r) = f_\varepsilon(r)e^{i\theta}$, and try to optimize over the function f_ε (since we do not know a priori what form should it take, for our particular J). What we can assume a priori on that function is that it satisfies

$$E_\varepsilon(v_\varepsilon, B_\varepsilon) = O(1), \tag{2.1}$$

and

$$\frac{1}{\varepsilon^2} \int_{B_1} J(1 - |v_\varepsilon|^2) = O(1)\,. \tag{2.2}$$

Indeed, for a *minimizer* both (2.1) and (2.2) should hold, thanks to the estimates (3.1) and (3.2) that we shall verify below. Assuming then that f_ε is chosen in such a way that (2.1)–(2.2) are satisfied, we get for the energy of v_ε:

$$\begin{aligned} E_\varepsilon(v_\varepsilon) &= 2\pi \int_0^1 \big((f_\varepsilon')^2 + \frac{f_\varepsilon^2}{r^2} + \frac{1}{\varepsilon^2} J(1 - f_\varepsilon^2)\big)\, r dr \\ &= 2\pi \log \frac{1}{\varepsilon} - 2\pi \int_\varepsilon^1 \frac{1 - f_\varepsilon^2}{r}\, dr + \int_\varepsilon^1 (f_\varepsilon')^2\, r dr + O(1). \end{aligned} \tag{2.3}$$

In order to get minimal energy (up to an $O(1)$-term), we shall look for f_ε which *maximizes* the term $\int_\varepsilon^1 \frac{1-f_\varepsilon^2}{r}\, dr$ (representing the gain of energy w.r.t. the "usual" cost of $2\pi \log \frac{1}{\varepsilon}$) under the constraint $\int_\varepsilon^1 J(1 - f_\varepsilon^2)\, r dr \le C_0$. Here we did not take into account the contribution of the term $\int_\varepsilon^1 (f_\varepsilon')^2\, r dr$, but as we shall see below, this term is bounded for the solution of our optimization problem.

Rescaling by a factor of ε, we are led naturally to define the following quantity:

$$I(R, c) = \sup \Big\{ \int_1^R \frac{1 - f^2}{r} dr \,:\, \int_1^R J(1 - f^2) r\, dr \le c \Big\}, \tag{2.4}$$

for any $R > 1$ and $c > 0$.

Lemma 2.1. *For every $R > 1$ and $c > 0$, there exists a maximizer $f_0 = f_0^{(R)}$ in* (2.4) *satisfying $0 \leq f_0(r) \leq 1, \forall r$, such that $f_0(r)$ is nondecreasing. Moreover, if $r_0 = r_0(c)$ is defined by the equation*

$$c = J(1)\Big(\frac{r_0^2 - 1}{2}\Big)\,, \tag{2.5}$$

then there exists $\tilde{r}_0 = \tilde{r}_0(c, R) \in [1, r_0]$ such that

$$f_0(r) \begin{cases} = 0 & \text{if } r \in [1, R] \text{ and } r < \tilde{r}_0\,, \\ > 0 & \text{if } r > \tilde{r}_0\,. \end{cases} \tag{2.6}$$

Furthermore,

$$\int_1^R J(1 - f_0^2) r\, dr = c\,, \quad \text{for } R > r_0\,, \tag{2.7}$$

and

$$j(1 - f_0^2(r)) = \frac{1}{\lambda r^2}\,, \quad r > \tilde{r}_0\,, \tag{2.8}$$

for some $\lambda = \lambda(R, c) > 0$.

Proof. We may consider only admissible f satisfying $0 \leq f \leq 1$. Indeed, note that replacing f by $|f|$ does not change the value of the integrals appearing in the target functional and in the constraint functional. Furthermore, if the set $\{r : f(r) > 1\}$ has positive measure, then replacing $f(r)$ by 1 on this set would strictly increase the value of the target functional, $\int_1^R \frac{1-f^2}{r}\, dr$, without violating the constraint.

Moreover, we may assume that f is nondecreasing by applying Schwarz symmetrization to the function $f(x) = f(|x|)$ (w.r.t. the Lebesgue measure on $\mathbb{R}^2$, setting first $f \equiv 0$ on $B_1(0)$). Then, using Helly's selection principle we obtain the convergence a.e. of a subsequence of a maximizing sequence $\{f_n\}$ to a limit f_0 which is a maximizer. Clearly, for $R \leq r_0$ the maximizer is $f_0 \equiv 0$. On the other hand, for $r > r_0$ we must have $f_0(r) > 0$. This implies the existence of $\tilde{r}_0$ satisfying (2.6).

In order to prove (2.7), assume by negation that the inequality $\int_1^R J(1-f_0^2) r\, dr < c$ holds for some $R > r_0$. We can then choose a small enough interval where f_0 is positive, and redefine f_0 to be zero there, thus increasing the value of target functional, without violating the constraint $\int_1^R J(1 - f_0^2) r\, dr \leq c$. Contradiction. Finally, note that the maximizer f_0 satisfies the following Euler equation associated to (2.4):

$$f_0 j(1 - f_0^2(r)) = \frac{f_0}{\lambda r^2}\,, \quad r \in [1, R]\,, \tag{2.9}$$

where λ is a Lagrange multiplier. For $r > \tilde{r}_0$ we may divide both sides of (2.9) by $f_0(r)$ to obtain (2.8). □

Lemma 2.2. *There exist two constants* $0 < a(c) < b(c)$ *such that*

$$a(c) \leq \lambda \leq b(c), \ \ R \geq r_0 + 1.$$

Proof. First, applying (2.8) for $r = r_0$ yields

$$\frac{1}{\lambda} = r_0^2 j(1 - f_0^2(r_0)) \leq r_0^2 \max_{0\leq t\leq 1} j(t) \implies \lambda \geq a(c)\,. \tag{2.10}$$

In order to prove the upper-bound for λ we distinguish two cases:

(i) $1 - f_0^2(\tilde{r}_0) \geq \eta_0\,,$
(ii) $1 - f_0^2(\tilde{r}_0) < \eta_0\,,$
where we denoted $f_0(\tilde{r}_0) = \lim_{r\searrow \tilde{r}_0^+} f_0(r)$.

In case (i) we simply have: $j(1 - f_0^2(\tilde{r}_0)) \geq \min_{t\in[\eta_0,1]} j(t) := \alpha_0 > 0$ (by (H_2)). Using the last estimate in conjunction with (2.8) yields $\lambda \leq \frac{1}{\alpha_0 \tilde{r}_0^2} \leq \frac{1}{\alpha_0}$.

In case (ii) we argue as follows. Since by (H_3) the function j is strictly monotone increasing on $[0, \eta_0]$, it has an inverse function that we denote by $\gamma = j^{-1}$. We may rewrite then (2.8) as

$$1 - f_0^2(r) = \gamma\Big(\frac{1}{\lambda r^2}\Big)\,, \quad r \in (\tilde{r}_0, R]\,. \tag{2.11}$$

Using the constraint (2.7) we obtain

$$c = \int_{\tilde{r}_0}^R J\Big(\gamma\big(\frac{1}{\lambda r^2}\big)\Big)\, r dr + \frac{\tilde{r}_0^2 - 1}{2} J(1)\,. \tag{2.12}$$

We use the r.h.s. of (2.12) to define a function of two variables

$$F(\lambda, t) = \int_t^R J\Big(\gamma\big(\frac{1}{\lambda r^2}\big)\Big)\, r dr + \frac{t^2 - 1}{2} J(1)\,.$$

A simple computation gives

$$\begin{aligned} F_\lambda(\lambda, t) &= -\int_t^R j\Big(\gamma\big(\frac{1}{\lambda r^2}\big)\Big)\gamma'\big(\frac{1}{\lambda r^2}\big)\,\frac{dr}{\lambda^2 r} \\ &= -\int_t^R \gamma'\big(\frac{1}{\lambda r^2}\big)\,\frac{dr}{\lambda^3 r^3} = -\frac{1}{2\lambda^2}\Big(\gamma\big(\frac{1}{\lambda t^2}\big) - \gamma\big(\frac{1}{\lambda R^2}\big)\Big) < 0\,, \end{aligned} \tag{2.13}$$

provided that $t < R$. In particular, λ is uniquely determined by (2.12) as a C^1-function $\lambda = l(t)$ in a neighborhood of $t = \tilde{r}_0$. Note also that

$$F_t(\lambda, t) = -J\Big(\gamma\big(\frac{1}{\lambda t^2}\big)\Big)t + tJ(1) = t\big(J(1) - J(1 - f_0^2(t))\big)\,. \tag{2.14}$$

Next, using (2.11) we get that

$$I(R,c)=\log\tilde r_0+\int_{\tilde r_0}^R\gamma\big(\frac{1}{l(\tilde r_0)r^2}\big)\,\frac{dr}{r}:=H(\tilde r_0)\,. \tag{2.15}$$

Since $t=\tilde r_0$ is a maximum point for the function $H(t)$, we must have

$$H'(\tilde r_0)=0\,. \tag{2.16}$$

Using (2.16) and (2.13)–(2.14) we obtain

$$\begin{aligned}0=H'(\tilde r_0)&=\frac{1}{\tilde r_0}-\gamma\Big(\frac{1}{l(\tilde r_0)\tilde r_0^2}\Big)\frac{1}{r_0}-\frac{l'(\tilde r_0)}{l^2(\tilde r_0)}\int_{\tilde r_0}^R\gamma'\Big(\frac{1}{l(\tilde r_0)r^2}\Big)\,\frac{dr}{r^3}\\ &\le\frac{1}{\tilde r_0}+l'(\tilde r_0)l(\tilde r_0)F_\lambda(l(\tilde r_0),\tilde r_0)=\frac{1}{\tilde r_0}-l(\tilde r_0)F_{\tilde r_0}(l(\tilde r_0),\tilde r_0)\\ &=\frac{1}{\tilde r_0}-l(\tilde r_0)\tilde r_0\big(J(1)-J(1-f_0^2(\tilde r_0))\big)\,.\end{aligned} \tag{2.17}$$

From (2.17) it follows (using (ii) and (H_2)) that

$$\lambda=l(\tilde r_0)\le\frac{1}{J(1)-J(\eta_0)}\,,$$

and the result follows in this case as well. $\square$

Remark 2.1. *The proof of Lemma 2.2 actually shows that the bounds for λ are uniform for c lying in a bounded interval.*

Lemma 2.3. *For every $c>0$ there exists a constant $C=C(c)$ such that for every $0<c_1,c_2\le c$ we have*

$$|I(R,c_1)-I(R,c_2)|\le C,\ \forall R\ge 1\,.$$

Proof. Let $f_1^{(R)}$ and $f_2^{(R)}$ denote, respectively, the maximizers for $I(R,c_1)$ and $I(R,c_2)$. By Lemma 2.1 we have

$$j(1-f_i^2(r))=\frac{1}{\lambda_i r^2}\,,\quad i=1,2,\ r\ge\max(r_0(c_1),r_0(c_2))\,,$$

for the Lagrange multipliers $\lambda_1=\lambda_1(R)=\lambda(R,c_1)$ and $\lambda_2=\lambda_2(R)=\lambda(R,c_2)$. Recall that by Lemma 2.2 we have

$$0<a\le\lambda_1^{(R)},\ \lambda_2^{(R)}\le b<\infty,\quad\forall R\ge\max(r_0(c_1),r_0(c_2))+1\,,$$

for some positive constants a and b. By Lemma 2.1, Lemma 2.2 and Remark 2.1 it follows that there exists $R_0=R_0(c)$ such that $1-f_i^2(r)\le\eta_0/2$, $i=1,2$, for $r\ge R_0$.

Therefore, for $i = 1, 2$ we have
(2.18)

$$I(R, c_i) = \int_1^R \frac{1 - f_i^2}{r}\, dr = \int_{R_0}^R j^{-1}\big(\frac{1}{\lambda_i r^2}\big)\, \frac{dr}{r} + O(1) = \frac{1}{2} \int_{\frac{1}{\lambda_i R^2}}^{\frac{1}{\lambda_i R_0^2}} j^{-1}(t)\, \frac{dt}{t} + O(1)\,.$$

Using $\max_{[0, j(\eta_0/2)]} j^{-1}(t) \leq C_1$, for some constant C_1, we obtain from (2.18),

$$|I(R, c_1) - I(R, c_2)| \leq | \int_{\frac{1}{\lambda_1 R^2}}^{\frac{1}{\lambda_2 R^2}} j^{-1}(t)\, \frac{dt}{t} | + O(1)$$

$$\leq C_1 | \log\big(\frac{\lambda_1}{\lambda_2}\big)| + O(1) \leq C_1 \log\big(\frac{b}{a}\big) + O(1)\,.$$

□

In view of Lemma 2.3 it is natural to set:

$$I(R) := I(R, 1)\,. \tag{2.19}$$

For any fixed $c_0 > 0$ we have then:

$$|I(R, c) - I(R)| \leq C(c_0)\,, \quad \forall c \leq c_0, \forall R \geq 1\,. \tag{2.20}$$

Next we prove, by the method of proof of Lemma 2.3, an explicit estimate for $I(R)$. In the sequel we shall denote by f_0 be a maximizer for $I(R) = I(R, 1)$ as given by Lemma 2.1.

Lemma 2.4. *We have*

$$I(R) = \frac{1}{2} \int_{\frac{1}{R^2}}^{j(\eta_0)} j^{-1}(t)\, \frac{dt}{t} + O(1), \quad \forall R \geq 1\,. \tag{2.21}$$

In particular,

$$\lim_{R \to \infty} \frac{I(R)}{\log R} = 0\,. \tag{2.22}$$

Proof. By Lemma 2.1 we have $j(1 - f_0^2(r)) = \frac{1}{\lambda r^2}$ for $r > r_0(1)$ and by Lemma 2.2 we have

$$\lambda = \lambda(R) \in [a, b], \quad \text{for } R \geq r_0(1) + 1\,, \tag{2.23}$$

for some two positive constants a and b. Using hypothesis (H_3) we conclude that

$$1 - f_0^2(r) = j^{-1}\Big(\frac{1}{\lambda r^2}\Big)\,, \quad \text{for } R \geq r \geq \mu_0 := \max\big(r_0(1), \frac{1}{\sqrt{a j(\eta_0)}}\big)\,. \tag{2.24}$$

It follows that

$$I(R) = \int_{\mu_0}^{R} j^{-1}\Big(\frac{1}{\lambda r^2}\Big)\frac{dr}{r} + O(1) = \frac{1}{2}\int_{\frac{1}{\lambda R^2}}^{j(\eta_0)} j^{-1}(t)\frac{dt}{t} + O(1)\,.$$

In order to get (2.21) it suffices to notice that

$$\begin{aligned}\left|\int_{\frac{1}{\lambda R^2}}^{j(\eta_0)} j^{-1}(t)\frac{dt}{t} - \int_{\frac{1}{R^2}}^{j(\eta_0)} j^{-1}(t)\frac{dt}{t}\right| &\le \left|\int_{\frac{1}{R^2}}^{\frac{1}{\lambda R^2}} j^{-1}(t)\frac{dt}{t}\right| \\ &\le C|\log\lambda| \le C\max(|\log b|, |\log a|) = O(1)\,.\end{aligned}$$

Finally we note that (2.22) follows easily from (2.21) since $j^{-1}(0) = 0$. □

The next lemma provides an estimate that we shall use in the proof of the upper-bound for the energy in subsection 3.2.

Lemma 2.5. *We have*

$$\int_{\mu_0}^{R} (f_0')^2\, r dr \le C, \quad \forall R > \mu_0\,, \tag{2.25}$$

for a as in (2.23) *and μ_0 as defined in* (2.24).

Proof. Differentiating the equality (2.24) yields for $r \ge \mu_0$,

$$-2f_0 f_0' = (j^{-1})'\Big(\frac{1}{\lambda r^2}\Big)\cdot\Big(-\frac{2}{\lambda r^3}\Big)\,,$$

which implies

$$f_0'(r) \le C(j^{-1})'\Big(\frac{1}{br^2}\Big)\cdot\frac{1}{r^3}\,,$$

with b given by (2.23). Therefore, denoting by C different positive constants, we get

$$\begin{aligned}\int_{\mu_0}^{R} (f_0')^2\, r dr &\le C\int_{\mu_0}^{R}\Big[(j^{-1})'\Big(\frac{1}{br^2}\Big)\Big]^2\frac{dr}{r^5} = C\int_{\frac{1}{bR^2}}^{\frac{1}{b\mu_0^2}}\big[(j^{-1})'(\alpha)\big]^2\alpha\, d\alpha \\ &= C\int_{\frac{1}{bR^2}}^{\frac{1}{b\mu_0^2}}\frac{\alpha d\alpha}{\big(j'(j^{-1}(\alpha))\big)^2} = C\int_{j^{-1}(\frac{1}{bR^2})}^{j^{-1}(\frac{1}{b\mu_0^2})}\frac{j(\beta)}{j'(\beta)}\,d\beta\,.\end{aligned} \tag{2.26}$$

It is elementary to verify that

$$\lim_{\beta\to 0^+}\frac{j(\beta)}{j'(\beta)} = 0\,. \tag{2.27}$$

Indeed, if $J''(0) = j'(0) > 0$ then

$$\lim_{\beta\to 0^+}\frac{j(\beta)}{j'(\beta)} = \lim_{\beta\to 0^+}\frac{J'(\beta)}{J''(\beta)} = 0\,,$$

since $J'(0)=0$ by (H_1), while if $J''(0)=0$ then by L'hôpital rule

$$\lim_{\beta\to 0^+}\frac{j(\beta)}{j'(\beta)}=\lim_{\beta\to 0^+}\frac{J'(\beta)}{J''(\beta)}=\lim_{\beta\to 0^+}\frac{J(\beta)}{J'(\beta)}=0\,,$$

since by convexity $J(\beta)=\int_0^\beta J'(s)\,ds\le \beta J'(\beta)$ for $\beta\le\eta_0$. Therefore, (2.25) follows from (2.26) and (2.27). $\square$

We next study a similar functional to that of (2.4). It will serve in the proof of the lower-bound of the energy in subsection 3.3. For any $R>1$ and $c>0$ set

$$\widetilde{I}(R,c)=\sup\left\{\int_1^R\left(\frac{1-f^2}{r}+4\frac{(1-f^2)^2}{r}\right)dr\ :\ \int_1^R J(1-f^2)r\,dr=c\right\}. \tag{2.28}$$

Lemma 2.6. *There exists a constant $C=C(c)$ such that*

$$|\widetilde{I}(R,c)-I(R,c)|\le C\,,\quad \forall R\ge 1\,. \tag{2.29}$$

Proof. The existence of a maximizer $\tilde f_0=\tilde f_0^{(R)}$ which is a nondecreasing function on $[1,R]$ satisfying $0\le\tilde f_0\le 1$, follows as in the proof of Lemma 2.1. Moreover, there exists $\tilde r_0'\in[1,r_0]$ (with r_0 given by (2.5)) such that $\tilde f_0(r)>0$ for $r>\tilde r_0'$ and $\tilde f_0(r)=0$ for $r<\tilde r_0'$. The Euler-Lagrange equation for $\tilde f_0$ reads

$$-\frac{2\tilde f_0}{r}-\frac{16\tilde f_0(1-\tilde f_0^2)}{r}=\lambda r j(1-\tilde f_0^2)(-2\tilde f_0)\,,\quad 1\le r\le R\,.$$

Thus,

$$\frac{j(1-\tilde f_0^2)}{9-8\tilde f_0^2}=\frac{1}{\lambda r^2}\,,\quad \tilde r_0'<r\le R\,. \tag{2.30}$$

Setting $\tilde j(t)=\frac{j(t)}{8t+1}$, we may rewrite (2.30) as

$$\tilde j(1-\tilde f_0^2)=\frac{1}{\lambda r^2}\,,\quad \tilde r_0'<r\le R\,.$$

We claim that there exists $\tilde\eta_0>0$ such that $\tilde j$ is strictly monotone increasing on $[0,\tilde\eta_0]$. Indeed, this is a direct consequence of

$$\Big(\frac{j(t)}{8t+1}\Big)'=\frac{(8t+1)j'(t)-8j(t)}{(8t+1)^2}$$

and (2.27). We can then repeat the argument of Lemma 2.2, with $\tilde j$ and $\tilde\eta_0$ taking the place of j and η_0, respectively, to conclude the existence of $\tilde a(c),\tilde b(c)$ such that $0<\tilde a(c)\le\lambda\le\tilde b(c)<\infty$. We can then apply the argument of Lemma 2.3 to deduce (2.29). Indeed, we only need to notice that $\tilde j(t)=j(t)+o(t)$ implies that $\tilde j^{-1}(t)=j^{-1}(t)+o(t)$. $\square$

By using the above arguments we also obtain the following result.

Lemma 2.7. *For every $c_0, \alpha > 0$ there exists a constant $C_1(c_0, \alpha)$ such that*

$$\begin{cases} |I(\alpha R, c) - I(R)| \leq C_1(c, \alpha) \\ |\widetilde{I}(\alpha R, c) - I(R)| \leq C_1(c, \alpha) \end{cases} \quad \text{for } R > \max(1, \frac{1}{\alpha}) \text{ and } c \in (0, c_0]\,. \tag{2.31}$$

3. Proof of the main results

In this section we shall give the proof of our main results Theorem 1 and Theorem 2. We begin with some basic estimates for the minimizer u_ε, which follow as in the case of the GL-energy (see [1, 2]).

3.1. Some basic estimates for u_ε.

The next lemma provides L^∞-estimates for u_ε and its gradient.

Lemma 3.1. *Any solution u_ε of (1.3) satisfies:*

$$\|u_\varepsilon\|_{L^\infty(G)} \leq 1 \quad \text{and} \quad \|\nabla u_\varepsilon\|_{L^\infty(G)} \leq \frac{C}{\varepsilon}. \tag{3.1}$$

Proof. The first estimate follows easily form the observation that replacing $u_\varepsilon(x)$ by $u_\varepsilon(x)/|u_\varepsilon(x)|$ on the set $\{x \in G : |u_\varepsilon(x)| > 1\}$ strictly decreases the energy if the latter set has a positive measure. The second estimate in (3.1) follows from a simple rescaling argument and standard elliptic estimates as in [1, 5]. □

In the case of a starshaped G the following Pohozaev identity holds for u_ε (actually it is valid for any solution of problem (1.3)). The proof is identical to the one for the GL-energy in [2], so we omit it.

Lemma 3.2. *If G is starshaped then*

$$\frac{1}{\varepsilon^2}\int_G J(1 - |u_\varepsilon|^2) \leq C_0,\ \forall \varepsilon > 0\,. \tag{3.2}$$

We shall show later that the assumption of starshapeness of the domain can be dropped, by applying an argument of del Pino and Felmer [4].

3.2. The upper-bound for the energy.

This subsection is devoted to the proof of the following proposition which provides the upper-bound assertion of Theorem 1. Recall that u_ε is a minimizer for E_ε over $H^1_g(G, \mathbb{C})$. We assume without loss of generality that $d \geq 0$.

Proposition 3.1. *We have*

$$E_\varepsilon(u_\varepsilon) \leq 2\pi d\Big(\log\left(\frac{1}{\varepsilon}\right) - I\Big(\frac{1}{\varepsilon}\Big)\Big) + O(1),\ \forall \varepsilon > 0\,. \tag{3.3}$$

Proof. We treat only the case $d > 0$ since the case $d = 0$ is trivial. We shall define $U_\varepsilon \in H^1_g(G,\mathbb{C})$ for which $E_\varepsilon(U_\varepsilon)$ satisfies the bound (3.3). Fix d distinct points $b_1,\dots,b_d \in G$, and let ρ satisfy

$$0 < \rho < \frac{1}{4}\min\big(\min_{i\neq j}|b_i - b_j|, \min_i \operatorname{dist}(b_i,\partial G)\big)\,.$$

Fix any $k \in \{1,\dots,d\}$. Let $f_0(r)$ be a maximizer for $I(\frac{\rho}{\varepsilon})$ as given by Lemma 2.1, see (2.19), and let θ_k denote a polar coordinate around b_k. Next we define U_ε on $B_{2\rho}(b_k)$ as follows:

$$U_\varepsilon(x) = \begin{cases} \frac{|x-b_k|}{\mu_0\varepsilon} f_0(\mu_0)e^{i\theta_k} & \text{on } B_{\mu_0\varepsilon}(b_k)\,,\\ f_0(\frac{|x-b_k|}{\varepsilon})e^{i\theta_k} & \text{on } B_\rho(b_k)\setminus B_{\mu_0\varepsilon}(b_k)\,,\\ \Big(f_0(\frac{\rho}{\varepsilon}) + \big(\frac{|x-b_k|-\rho}{\rho}\big)\big(1-f_0(\frac{\rho}{\varepsilon})\big)\Big)e^{i\theta_k} & \text{on } B_{2\rho}(b_k)\setminus B_\rho(b_k)\,. \end{cases}$$

We claim that

$$E_\varepsilon(U_\varepsilon, B_{2\rho}(b_k)) = 2\pi\Big(\log\Big(\frac{1}{\varepsilon}\Big) - I\Big(\frac{1}{\varepsilon}\Big)\Big) + O(1)\,. \tag{3.4}$$

Clearly,

$$E_\varepsilon(U_\varepsilon, B_{\mu_0\varepsilon}(b_k)) = O(1)\,. \tag{3.5}$$

Next, by the same computation as in (2.3) we obtain

$$\begin{aligned} E_\varepsilon(U_\varepsilon, B_\rho(b_k)\setminus B_{\mu_0\varepsilon}(b_k)) &= 2\pi\int_{\mu_0}^{\rho/\varepsilon}\Big((f_0')^2 + \frac{f_0^2}{r^2} + J(1-f_0^2)\Big)\,rdr \\ &= 2\pi\log\Big(\frac{\rho}{\mu_0\varepsilon}\Big) - 2\pi\int_{\mu_0}^{\rho/\varepsilon}\frac{1-f_0^2}{r}\,dr + 2\pi\int_{\mu_0}^{\rho/\varepsilon}(f_0')^2\,rdr + O(1) \\ &:= A_1 - A_2 + A_3 + O(1)\,, \end{aligned} \tag{3.6}$$

where we used the constraint satisfied by f_0. Clearly, $A_1 = 2\pi\log(1/\varepsilon) + O(1)$. By the definition of f_0 and Lemma 2.7 we get that $A_2 = 2\pi I(1/\varepsilon) + O(1)$. Moreover, $A_3 = O(1)$ thanks to (2.25). Therefore, (3.4) will follow from (3.5) and (3.6) once we show that

$$E_\varepsilon\big(U_\varepsilon, B_{2\rho}(b_k)\setminus B_\rho(b_k)\big) = O(1)\,. \tag{3.7}$$

In order to verify (3.7) we write on $B_{2\rho}(b_k)\setminus B_\rho(b_k)$,

$$U_\varepsilon(b_k + re^{i\theta_k}) = z(r)e^{i\theta_k} \quad\text{with}\quad z(r) = f_0(\frac{\rho}{\varepsilon}) + \big(\frac{r-\rho}{\rho}\big)\big(1 - f_0(\frac{\rho}{\varepsilon})\big)$$

and compute

$$\int_{B_{2\rho}(b_k)\setminus B_\rho(b_k)} |\nabla U_\varepsilon|^2\,dx = \int_{B_{2\rho}(b_k)\setminus B_\rho(b_k)} z^2|\nabla\theta_k|^2 + 2\pi\int_\rho^{2\rho}(z')^2\,rdr \tag{3.8}$$
$$= O(1) + 2\pi\Big(\frac{1-f_0(\rho/\varepsilon)}{\rho}\Big)^2\int_\rho^{2\rho} r\,dr = O(1)\,.$$

As for the second term of the energy, we obtain, using again the inequality $J(t) \le tj(t)$, (2.8) and Lemma 2.2

$$\frac{1}{\varepsilon^2}\int_{B_{2\rho}(b_k)\setminus B_\rho(b_k)} J(1-|U_\varepsilon|^2) \le \frac{C}{\varepsilon^2}\int_{B_{2\rho}(b_k)\setminus B_\rho(b_k)} j(1-|U_\varepsilon|^2) \tag{3.9}$$

$$\le \frac{C}{\varepsilon^2} j\big(1-f_0^2(\frac{\rho}{\varepsilon})\big) = \frac{C}{\varepsilon^2}\frac{1}{\lambda\cdot(\frac{\rho}{\varepsilon})^2} \le C\,. \tag{3.10}$$

Therefore, (3.7) follows from (3.8)–(3.9).

We use the same construction on each the annuli $B_{2\rho}(b_k)$, $k = 1,\dots,d$, and finally on $G\setminus\cup_{k=1}^d B_{2\rho}(b_k)$ we set $U_\varepsilon = w$ where w is a fixed smooth S^1-valued map which equals to $e^{i\theta_k}$ on each $\partial B_{2\rho}(b_k)$ and to g on ∂G. The result follows since clearly $E_\varepsilon(u_\varepsilon) \le E_\varepsilon(U_\varepsilon)$. □

3.3. **The lower-bound of the energy and the convergence result.** The following simple lemma, which is a variant of [3, Theorem 3], is a basic tool in the proof of the lower bound.

Lemma 3.3. *Let $u \in C^1(S^1,\mathbb{C})$ satisfy $|u(x)| \ge \frac12$ on S^1 and deg$(u/|u|) = d$. Then,*

$$\int_{S^1}\Big(|u'|^2 + d^2(1-|u|^2) + 4d^2(1-|u|^2)^2\Big)\,d\tau \ge 2\pi d^2\,. \tag{3.11}$$

Proof. We may write $u = \rho e^{i(d\theta+\psi)}$ with $\psi \in C^1(S^1,\mathbb{R})$. We have

$$|u'|^2 = (\rho')^2 + \rho^2\big(d^2 + 2d\psi' + (\psi')^2\big)\,.$$

Integrating over S^1, using the equality $\int_{S^1}\rho^2\psi' = \int_{S^1}(\rho^2-1)\psi'$ and the Cauchy-Schwarz inequality gives

$$\int_{S^1}|u'|^2 + d^2(1-\rho^2) \ge 2\pi d^2 + \int_{S^1} 2d(\rho^2-1)\psi' + \rho^2(\psi')^2$$
$$\ge 2\pi d^2 - \int_{S^1}\Big(4d^2(\rho^2-1)^2 + \frac14(\psi')^2 - \rho^2(\psi')^2\Big) \ge 2\pi d^2 - \int_{S^1} 4d^2(\rho^2-1)^2\,,$$

and (3.11) follows. □

An immediate consequence is a lower bound on an annulus.

Proposition 3.2. *Let A_{R_1,R_2} denote the annulus $\{R_1 < |x| < R_2\}$ and let $u \in C^1(A_{R_1,R_2},\mathbb{C}) \cap C(\overline{A_{R_1,R_2}},\mathbb{C})$ satisfy:*

$$deg(u,\partial B_{R_j}(0)) = d,\ j = 1,2\,, \tag{3.12}$$

$$\frac{1}{2} \le |u| \le 1 \quad on\ A_{R_1,R_2}\,, \tag{3.13}$$

and

$$\frac{1}{R_1^2}\int_{A_{R_1,R_2}} J(1-|u|^2) \le c_0\,, \tag{3.14}$$

for some constant c_0. Then, there is a constant c_1, depending only on c_0 such that

$$\int_{A_{R_1,R_2}} |\nabla u|^2 \ge 2\pi d^2\Big(\log\big(\frac{R_2}{R_1}\big) - I\big(\frac{R_2}{R_1}\big)\Big) - d^2 c_1\,. \tag{3.15}$$

Proof. The rescaled version of (3.11) reads (with $u' := \frac{\partial u}{\partial \tau}$)

$$\int_{\partial B_r(0)} \Big(|u'|^2 + \frac{d^2}{r^2}(1-|u|^2) + 4\frac{d^2}{r^2}(1-|u|^2)^2\Big)\,d\tau \ge 2\pi\frac{d^2}{r}\,. \tag{3.16}$$

Integrating (3.16) over $r \in (R_1,R_2)$ yields

$$\int_{A_{R_1,R_2}} |\nabla u|^2 \ge 2\pi d^2 \log\big(\frac{R_2}{R_1}\big) - d^2\int_{A_{R_1,R_2}} \big(\frac{1-|u|^2}{|x|^2} + \frac{4(1-|u|^2)^2}{|x|^2}\big)\,dx\,. \tag{3.17}$$

Consider the rescaled map $\tilde{u}(x) = u(R_1 x)$ on the annulus $A_{1,R_2/R_1} = \{1 < |x| < R_2/R_1\}$. Let $\tilde{f}$ denote the symmetric nondecreasing rearrangement of $\tilde{u}$ with respect to the Lebesgue measure of $\mathbb{R}^2$. We have

(3.18)
$$\begin{aligned}\int_{A_{R_1,R_2}} \big(\frac{1-|u|^2}{|x|^2} + \frac{4(1-|u|^2)^2}{|x|^2}\big)\,dx &= \int_{A_{1,R_2/R_1}} \big(\frac{1-|\tilde{u}|^2}{|x|^2} + \frac{4(1-|\tilde{u}|^2)^2}{|x|^2}\big)\,dx \\ &\le 2\pi\int_1^{R_2/R_1} \big(\frac{1-|\tilde{f}|^2}{r} + \frac{4(1-|\tilde{f}|^2)^2}{r}\big)\,dr\,.\end{aligned}$$

We have also

$$\int_{A_{1,R_2/R_1}} J(1-|\tilde{f}|^2)\,dx = \int_{A_{1,R_2/R_1}} J(1-|\tilde{u}|^2)\,dx = \frac{1}{R_1^2}\int_{A_{R_1,R_2}} J(1-|u|^2)\,dx \le c_0\,, \tag{3.19}$$

by (3.14). Therefore, from (3.18)–(3.19) and Lemma 2.7 we conclude that

$$\int_{A_{R_1,R_2}} \big(\frac{1-|u|^2}{|x|^2} + \frac{4(1-|u|^2)^2}{|x|^2}\big)\,dx \le \tilde{I}(R_2/R_1,c_0) \le I(R_2/R_1) + c_1\,,$$

with c_1 depending only on c_0, which combined with (3.17) clearly implies (3.15). $\square$

We next use Proposition 3.2 in order to establish a lower bound on a more general perforated domain. The proof uses a variant of an argument of Struwe [5].

Proposition 3.3. *Let $x_1, \dots, x_m$ be m points in $B_\sigma(0)$ satisfying*

$$|x_i - x_j| \geq 4\delta,\ \forall i \neq j \quad \text{and} \quad |x_i| < \frac{\sigma}{4},\ \forall i\,,$$

with $\delta \leq \sigma/32$. Set $\Omega = B_\sigma(0) \setminus \bigcup_{j=1}^m B_\delta(x_j)$ and let u be a C^1-map from Ω into $\mathbb{C}$, which is continuous on $\partial\Omega$, satisfying

$$\frac{1}{2} \leq |u| \leq 1 \text{ in } \Omega \quad \text{and} \quad \deg(u, \partial B_\sigma(x_j)) = d_j,\ \forall j,$$

and

$$\frac{1}{\delta^2}\int_\Omega J(1-|u|^2) \leq K\,.$$

Then, denoting $d = \sum_{j=1}^m d_j$, we have

$$\int_\Omega |\nabla u|^2 \geq 2\pi|d|\Big(\log\frac{\sigma}{\delta} - I\big(\frac{\sigma}{\delta}\big)\Big) - C, \tag{3.20}$$

with $C = C(K, m, \sum_{j=1}^m |d_j|)$.

Proof. We use induction on m. The case $m = 1$ follows from Proposition 3.2. Suppose we are given m such points and that the assertion holds for any fewer number of points satisfying the above conditions. Put

$$d_j^{(1)} = d_j, x_j^{(1)} = x_j,\ \forall j \quad \text{and} \quad R^{(1)} = \delta, m^{(1)} = m \text{ and } J^{(1)} = \{1, \dots, m\}.$$

Set $r^{(1)} = \frac{1}{2}\min_{i\neq j}|x_i^{(1)} - x_j^{(1)}|$ and $A_j^{(1)} = B(x_j^{(1)}, r^{(1)}) \setminus B(x_j^{(1)}, R^{(1)})$, $\forall j$. By Proposition 3.2 we have

$$\begin{aligned}\int_{\bigcup_{j=1}^m A_j^{(1)}} |\nabla u|^2 &\geq 2\pi\sum_{j=1}^m \big(d_j^{(1)}\big)^2\Big(\log\big(\frac{r^{(1)}}{R^{(1)}}\big) - I\big(\frac{r^{(1)}}{R^{(1)}}\big)\Big) - C \\ &\geq 2\pi d\Big(\log\big(\frac{r^{(1)}}{R^{(1)}}\big) - I\big(\frac{r^{(1)}}{R^{(1)}}\big)\Big) - C\,.\end{aligned} \tag{3.21}$$

Next, define $R^{(2)}$ as the minimal number $R \in [r^{(1)}, (3/4)\sigma]$ for which there exists a subset $J^{(2)} \subset J^{(1)}$ such that:

$$\bigcup_{j=1}^m B_{R^{(1)}}(x_j^{(1)}) \subset \bigcup_{i\in J^{(2)}} B_{R/4}(x_i^{(1)}) \quad \text{and} \quad |x_{i_1}^{(1)} - x_{i_2}^{(1)}| \geq 4R,\ i_1 \neq i_2 \text{ in } J^{(2)}. \tag{3.22}$$

If no such $R^{(2)}$ exists, then necessarily $r^{(1)} \geq \alpha\sigma$ for some $\alpha = \alpha(m) > 0$ and the result follows from (3.21). So assume in the sequel that $R^{(2)}$ does exist. In that case we have

$$R^{(2)} \leq \beta r^{(1)}, \quad \text{for some constant } \beta = \beta(m)\,.$$

We relabel the points $\{x_i^{(1)}\}_{i\in J^{(2)}}$ by $\{x_j^{(2)}\}_{j=1}^{m(2)}$ with $m^{(2)} = |J^{(2)}|$. If $R^{(2)} \geq \sigma/8$ then we stop the construction with $k = 2$. Otherwise, we continue and define

$$r^{(2)} = \begin{cases} \frac{1}{2}\min_{i\neq j}|x_i^{(2)} - x_j^{(2)}| & \text{if } |J^{(2)}| \geq 2\,, \\ \frac{2\sigma}{3} & \text{if } |J^{(2)}| = 1\,. \end{cases} \tag{3.23}$$

Continuing the above construction yields

$$R^{(1)} < r^{(1)} < R^{(2)} < r^{(2)} < \cdots < R^{(k-1)} < r^{(k-1)} < R^{(k)} \leq \infty\,,$$

where k is the first index for which, either $R^{(k)}$ exists and satisfies $R^{(k)} \geq \sigma/8$, or $R^{(k)}$ does not exist, and then we set $R^{(k)} = \infty$. At each stage $R^{(l)}$ is chosen to satisfy an analogous condition to (3.22), and similarly, $r^{(l)}$ is chosen according to an analogous condition to (3.23). Note that we have

$$r^{(k-1)} \geq \alpha\sigma\,, \quad \text{for some } \alpha = \alpha(m) > 0\,. \tag{3.24}$$

For each l denote the resulting set of points by $\{x_j^{(l)}\}_{j=1}^{m^{(l)}}$. The corresponding degrees are

$$d_j^{(l)} = \deg\big(u, \partial B_{R^{(l)}}(x_j^{(l)})\big), \quad j = 1,\ldots,m^{(l)},\ l = 1,\ldots,k-1\,.$$

Note that

$$\sum_{j=1}^{m^{(l)}} d_j^{(l)} = d\,,\ l = 1,\ldots,k-1\,. \tag{3.25}$$

Using the induction hypothesis together with (3.24), (3.25) and (2.31) yields

$$\begin{aligned} \int_\Omega |\nabla u|^2 &\geq \sum_{j=1}^{m^{(k-1)}} \int_{B_{r^{(k-1)}}(x_j^{(k-1)})} |\nabla u|^2 \\ &\geq 2\pi \sum_{j=1}^{m^{(k-1)}} |d_j^{(k-1)}|\Big(\log\big(\frac{r^{(k-1)}}{\delta}\big) - I\big(\frac{r^{(k-1)}}{\delta}\big)\Big) - C \\ &\geq 2\pi|d|\Big(\log\big(\frac{\sigma}{\delta}\big) - I\big(\frac{\sigma}{\delta}\big)\Big) - C\,. \end{aligned} \tag{3.26}$$

□

3.4. **Proof of Theorem 1.** Let $R > 0$ be large enough so that $G \subset B_R(0)$. Fix any smooth map $U : \overline{B_R(0)} \setminus G \to S^1$ such that $U|_{\partial G} = g$ and let $\tilde{g} = U|_{\partial B_R(0)}$ (which has necessarily degree d). Denote for each ε by $\tilde{u}_\varepsilon$ a minimizer for E_ε over $H^1_{\tilde{g}}(B_R(0), \mathbb{C})$. Clearly,

(3.27)

$$\int_{B_R(0)} |\nabla \tilde{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2} J(1-|\tilde{u}_\varepsilon|^2) \le \int_G |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2} J(1-|u_\varepsilon|^2) + \int_{B_R(0)\setminus G} |\nabla U|^2 = E_\varepsilon(u_\varepsilon) + C\,.$$

Since $B_R(0)$ is star-shaped, we know from Lemma 3.2 that the estimate (3.2) holds for $\tilde{u}_\varepsilon$. Since (3.1) also holds for $\tilde{u}_\varepsilon$ we may apply the argument of [2] to obtain the existence of an integer N and of a real $\lambda > 0$, such that for each ε there exists a collection of discs $\{B_{\lambda\varepsilon}(x_i^\varepsilon)\}_{i=1}^{N_\varepsilon}$ such that

$$\widetilde{S}_\varepsilon := \{x \in B_R \,:\, |u_\varepsilon(x)| < \frac{1}{2}\} \subset \bigcup_{i=1}^{N_\varepsilon} B_{\lambda\varepsilon}(x_i^\varepsilon)\,,$$

and

$$|x_i^\varepsilon - x_j^\varepsilon| \ge 8\lambda\varepsilon\,, \quad \forall i, j, \text{ such that } i \ne j\,.$$

We fix $R_1 > 4R$ and consider a S^1-valued extension U_1 of U to $\overline{B_{R_1}(0)} \setminus B_R(0)$, of class C^1. This induces an extension of each $\tilde{u}_\varepsilon$ to $\overline{B_{R_1}(0)}$ (only a constant is added to its energy). Applying Proposition 3.3 yields

$$E_\varepsilon(\tilde{u}_\varepsilon, B_{R_1}(0)) \ge 2\pi d\Big(\log\frac{R_1}{\lambda\varepsilon} - I\big(\frac{R_1}{\lambda\varepsilon}\big)\Big) - C\,.$$

It follows that

(3.28)

$$E_\varepsilon(u_\varepsilon, G) \ge E_\varepsilon(\tilde{u}_\varepsilon, B_R(0)) - C \ge E_\varepsilon(\tilde{u}_\varepsilon, B_{R_1}(0)) - C \ge 2\pi d\Big(\log\frac{1}{\varepsilon} - I\big(\frac{1}{\varepsilon}\big)\Big) - C\,.$$

The energy estimate (1.6) is an immediate consequence of (3.28) and the upper-bound (3.3).

An argument of del Pino and Felmer [4] can now be used to show that (3.2) holds without the assumption on the starshapeness of G. In fact, applying (3.28) for 2ε instead of ε yields

$$\int_G |\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2} J(1-|u_\varepsilon|^2) \ge \int_G |\nabla u_{2\varepsilon}|^2 + \frac{1}{4\varepsilon^2} J(1-|u_{2\varepsilon}|^2) \ge 2\pi|d|\log\frac{1}{2\varepsilon} - C\,. \tag{3.29}$$

On the other hand, by the upper bound (3.3),

$$\int_G |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2} J(1-|u_\varepsilon|^2)^2 \le 2\pi|d|\log\frac{1}{\varepsilon} + C\,. \tag{3.30}$$

Subtracting (3.29) from (3.30) yields the result (3.2).

Having the estimate (3.2) on our hands, we can now follow the bad-discs construction of [2] and complete the convergence assertion of Theorem 1. Since the arguments are identical to those of [2], we omit the details.

3.5. **The inverse problem (proof of Theorem 2).** In view of Theorem 1, it is enough to find a C^2 functional J, which is strictly convex on some interval $(0,\eta_0)$, such that $j = J'$ satisfies

$$\frac{1}{2}\int_{\frac{1}{R^2}}^{\eta_0} j^{-1}(t)\frac{dt}{t} = h(\log R)\,.$$

Setting $\delta = \frac{1}{R^2}$, we obtain the equivalent condition

$$\frac{1}{2}\int_{\delta}^{\eta_0} j^{-1}(t)\frac{dt}{t} = h(\frac{1}{2}\log\frac{1}{\delta})\,.$$

Differentiating with respect to δ gives

$$-\frac{1}{2\delta}j^{-1}(\delta) = h'(\frac{1}{2}\log\frac{1}{\delta})\cdot(-\frac{1}{2\delta})\,.$$

Therefore,

$$j^{-1}(\delta) = h'\big(\frac{1}{2}\log\frac{1}{\delta}\big)\,. \tag{3.31}$$

In order to be able to recover j, and then J, from (3.31), we shall verify that the r.h.s. of (3.31) is a strictly increasing function, for δ small enough. Indeed,

$$\frac{d}{d\delta}\Big(h'\big(\frac{1}{2}\log\frac{1}{\delta}\big)\Big) = -\big(\frac{1}{2\delta}\big)h''\big(\frac{1}{2}\log\frac{1}{\delta}\big) > 0 \quad \text{for } \delta \le e^{-2T}\,,$$

since $h''(t) < 0$ for $t \ge T$ by (1.7). Note also that $j(0) = j^{-1}(0) = 0$ by (1.8). Therefore, we have constructed J such that $j = J'$ satisfies (3.31) on an interval $[0,\eta_0]$, with $\eta_0 = e^{-2T}$, so that (H_3) is satisfied. Finally, it is easy to see that J can be extended to all of $\mathbb{R}$ as a C^2-functional satisfying $(H_1) - (H_2)$.

4. Appendix

In this Appendix we compute the energy cost of a degree one vortex for the functionals J_k, $k > 0$, that were defined in (1.4). In view of Theorem 1 it suffices to compute for each $k > 0$:

$$I_{0,k}(R) := \frac{1}{2}\int_{1/R^2}^{j_k(\eta_k)} j_k^{-1}(t)\frac{dt}{t}\,, \tag{4.1}$$

with $j_j = J_k'$ and $\eta_k = \big(\frac{k}{k+1}\big)^{1/k}$ (a simple computation shows that $J_k'' > 0$ on $(0,\eta_k)$).

Proposition 4.1. *As R goes to the infinity, we have:*

(4.2)
$$I_{0,k}(R) = \begin{cases} O(1), & 0 < k < 1\,, \\ \frac{1}{2}\log\log R + O(1), & k = 1\,, \\ 2^{-\frac{1}{k}}\frac{k}{k-1}\big(\log(R)\big)^{\frac{k-1}{k}} + O(1), & k > 1\,. \end{cases}$$

Proof. The change of variable $s = j_k^{-1}(t)$ gives

(4.3)
$$I_{0,k}(R) = \frac{1}{2}\int_{j_k^{-1}(1/R^2)}^{\eta_k} s\frac{j_k'(s)}{j_k(s)}\,ds = \frac{1}{2}\int_{j_k^{-1}(1/R^2)}^{\eta_k}\Big(\frac{k}{s^k} - (k+1)\Big)\,ds\,.$$

If $k < 1$ then it follows immediately from (4.3) that $I_{0,k}(R) = O(1)$.

For $k > 1$ we obtain from (4.3) that

(4.4)
$$I_{0,k}(R) = \frac{k}{2(k-1)}\Big((j_k)^{-1}\big(\frac{1}{R^2}\big)\Big)^{1-k} + O(1)\,.$$

Set $\alpha = \alpha(R) = j_k^{-1}\big(\frac{1}{R^2}\big)$. Since $j_k(\alpha) = \big(\frac{k}{\alpha^{k+1}}\big)\exp(-1/\alpha^k)$, we have

$$\frac{1}{R^2} = \big(\frac{k}{\alpha^{k+1}}\big)\exp(-1/\alpha^k)\,.$$

Taking the logarithm of both sides gives

(4.5)
$$-2\log R = \log k - (k+1)\log\alpha - \frac{1}{\alpha^k}\,,\quad \text{for } k > 0\,.$$

By (4.5) we have $\lim_{R\to\infty} 2\alpha^k\log R = 1$, which we plug in (4.4) to obtain the case $k > 1$ in (4.2).

Finally, if $k = 1$ then by (4.3) we have

(4.6)
$$I_{0,1}(R) = \frac{1}{2}\int_{j_1^{-1}(1/R^2)}^{\eta_1}\big(\frac{1}{s} - 2\big)\,ds = -\frac{1}{2}\log\Big(j_1^{-1}\big(\frac{1}{R^2}\big)\Big) + O(1) = \frac{1}{2}\log\alpha + O(1)\,,$$

with $\alpha = j_1^{-1}\big(\frac{1}{R^2}\big)$, as above. In our case (4.5) gives $\lim_{R\to\infty} 2\alpha\log R = 1$, which implies that $\log\alpha = \log\Big(\frac{1}{2\log R}\Big) + o(1)$. Plugging it in (4.6) gives the result (4.2) for $k = 1$. □

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