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ON A DISCRETE VARIATIONAL PROBLEM INVOLVING INTERACTING PARTICLES*

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Abstract. We consider the minimization problem

$$\min\ \mathcal{F}(\mathbf{z}) \equiv -\sum_{k<j} \log|z_k - z_j|, \qquad k,j = 1,2,\ldots,N,$$

for $\mathbf{z} = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N$ satisfying $\sum_{k=1}^N |z_k|^2 = 1$ and $z_k \neq z_j$ for $1 \leq k < j \leq N$.

This problem arises in different contexts in several physical models such as incompressible Euler equations and the Ginzburg–Landau model in superconductivity. We study the stability properties of some symmetric critical points such as regular polygons and configurations that consist of a regular polygon plus the origin. We also establish the existence of an infinite number of critical points enjoying different kinds of symmetry. We show that when the number of particles, N, exceeds a critical value, the global minimizer cannot be a regular polygon ($N \geq 6$), and if $N \geq 11$ it cannot be a star configuration (i.e., an $N-1$ sides regular polygon plus the origin).

Key words. interacting particles, symmetry breaking, Ginzburg–Landau model

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1. Introduction.

This paper deals with the following constrained minimization problem:

PROBLEM 1.

$$\mathcal{P}_N: \quad \min\ \mathcal{F}(\mathbf{z}) \equiv -\sum_{k<j} \log|z_k - z_j|, \qquad k,j = 1,2,\ldots,N, \tag{1.1}$$

for $\mathbf{z} = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N$ *satisfying the constraint* $\sum_{k=1}^N |z_k|^2 = 1$ *and* $z_k \neq z_j$ *for* $1 \leq k < j \leq N$.

This problem arises in different contexts in several physical models. Our motivation comes partly from the Ginzburg–Landau theory in superconductivity. Recently, Serfaty [10] showed that for certain values of an external applied magnetic field (which are close to the first critical value H_{c_1}), there exist local minimizers for the Ginzburg–Landau energy on a round cylinder, for which the limiting configuration of the vortices (after appropriate rescaling) is a minimizing configuration for the problem $\mathcal{P}_N$.

André and Shafrir [1] studied the effect of pinning due to variable thickness of the sample in a model problem involving the Ginzburg–Landau energy; see also [3, 4]. They showed that near nondegenerate minima points of the width, the vortices that appear have a limiting configuration which is a minimizer for problem $\mathcal{P}_N$.

Problem 1 is also relevant in fluid mechanics. The functional $\mathcal{F}(\mathbf{z})$ is the Hamiltonian for a system of point vortices in an ideal two-dimensional flow. The associated

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dynamics is considered with respect to the canonical symplectic form $\sum dz_i \wedge d\bar{z}_i$. The constraint $\sum_{k=1}^{N} |z_k|^2$ is the angular momentum associated with the system (see Batchelor [2, Chapter 7.3]). The minimizers of $\mathcal{F}(\mathbf{z})$ under the above constraint are stable configurations of point vortices which rotate in a uniform angular velocity corresponding to prescribed angular momentum.

Section 2 provides some preliminary details on the existence of minimizers, a scale invariant reformulation of the problem, and the solution in the real domain. Section 3 discusses the perfect polygon and the star configurations. Using stability analysis we prove that these critical points cannot be the global minimizers if N exceeds a critical value ($N \geq 6$ for the perfect polygon and $N \geq 11$ for the star configuration). Section 4 demonstrates some additional symmetric critical points enjoying different kinds of symmetry (e.g., invariant under rotations and under reflections). We conclude with some numerical optimization results in section 5 and present a few open problems in section 6.

2. Preliminaries. It should first be noted that Problem 1 always has a solution. This follows from the fact that $\mathcal{F}(\mathbf{z}) \to +\infty$ if two of the coordinates of $\mathbf{z}$ approach each other. Problem 1 is invariant under any permutation of the complex coordinates of $\mathbf{z}$. For brevity, we do not distinguish hereafter between a point (also referred to as "configuration") in $\mathbb{C}^N$ and the set of all of its permutations. It is easy to see that Problem 1 has several symmetry properties. Both $\mathcal{F}$ and the constraint are invariant under rotations by an arbitrary angle and under reflections with respect to any line that passes through the origin. Therefore, with no loss of generality, we may assume that one of the points z_j is real.

It is important to observe that Problem 1 can be formulated in the following equivalent way.

Problem 2.

$$\mathcal{P}_N : \quad \min \ \mathcal{G}(\mathbf{z}) \equiv -\sum_{k<j} \log |z_k - z_j| + \frac{1}{4} N(N-1) \log \left(\sum_{k=1}^{N} |z_k|^2 \right) \tag{2.1}$$

for $\mathbf{z} = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N$, *such that* $z_k \neq z_j$ *for* $1 \leq k < j \leq N$.

More precisely, every solution to Problem 1 is also a solution to Problem 2. On the other hand, since Problem 2 is invariant with respect to multiplication by a nonzero scalar, each solution to Problem 2 can be rescaled to a solution of Problem 1. We use both equivalent representations interchangeably, where the advantage of the latter formulation is in being constraint free and scale invariant.

It is insightful to consider the problem of minimizing $\mathcal{G}$ for real configurations, as follows.

Problem 3.

$$\mathcal{P}_N^* : \quad \min \ \mathcal{G}(\mathbf{z}) \equiv -\sum_{k<j} \log |z_k - z_j| + \frac{1}{4} N(N-1) \log \left(\sum_{k=1}^{N} |z_k|^2 \right) \tag{2.2}$$

for $\mathbf{z} = (z_1, z_2, \ldots, z_N) \in \mathbb{R}^N$, *such that* $z_k \neq z_j$ *for* $1 \leq k < j \leq N$.

The one-dimensional problem turns out to be a well-known one. It was completely solved by Schur [9] who showed that the unique minimizing configuration (modulo permutations) is given by the zeros of the Hermite polynomial of degree N. Details on this problem and similar ones (all motivated by problems from electrostatics) can be found in Szego [11]. The one-dimensional problem enjoys a convexity property: the functional $\mathcal{G}$ is convex on ordered real configurations. This convexity property implies uniqueness for critical points. On the other hand, in two dimensions convexity is lost, the structure of the set of critical points is richer, and the problem becomes much more complicated. Clearly, the minimizer for Problem 3 is a critical point for Problem 2. However, as the following proposition shows, this critical point is *not* a minimizer for Problem 2 for $N > 2$.

PROPOSITION 2.1. *For $N > 2$, the minimizing configuration for Problem 3 is unstable in $\mathbb{C}^N$.*

Proof. We deal with odd and even values of N separately.

1. Assume that $N = 2N_1 + 1$. We denote the minimizing configuration by

$$\mathbf{X} = (-x_{N_1}, -x_{N_1-1}, \dots, -x_2, -x_1, 0, x_1, x_2, \dots, x_{N_1-1}, x_{N_1}) \in \mathbb{R}^N. \tag{2.3}$$

We could assume $\mathbf{X}$ is symmetric since, as explained above, uniqueness holds for an ordered minimizing configuration. To prove the instability of $\mathbf{X}$, we shall show that $\mathcal{G}(\mathbf{X} + t\mathbf{v}) < \mathcal{G}(\mathbf{X})$ for small enough t with $\mathbf{v}$ given by

$$\mathbf{v} = (\underbrace{0, 0, \dots, 0}_{N_1 \ times}, i, \underbrace{0, 0, \dots, 0}_{N_1 \ times}) \tag{2.4}$$

(here $i = \sqrt{-1}$).

To study the effect of the perturbation $t\mathbf{v}$ we calculate the coefficient of t^2, denoted by $H_X(\mathbf{v})$, in the related Taylor expansion at the investigated critical point $\mathbf{X}$. The necessary formula is developed below in the more general context of Problem 2 (optimization in $\mathbb{C}^N$) and is given in (3.11). This yields

$$H_{\mathbf{X}}(\mathbf{v}) = -\sum_{j=1}^{N_1} \frac{1}{x_j^2} + \frac{N_1(2N_1+1)}{4\sum_{j=1}^{N_1} x_j^2} . \tag{2.5}$$

By the arithmetic-geometric mean inequality we have

$$\sum_{j=1}^{N_1} \frac{1}{x_j^2} \geq \frac{N_1^2}{\sum_{j=1}^{N_1} x_j^2} , \tag{2.6}$$

which leads to

$$H_{\mathbf{X}}(\mathbf{v}) \leq \frac{N_1/2 - N_1^2}{2\sum_{j=1}^{N_1} x_j^2} < 0. \tag{2.7}$$

2. Next, assume that $N = 2N_1$. We denote the minimizing configuration by

$$\mathbf{X} = (-x_{N_1}, -x_{N_1-1}, \dots, -x_2, -x_1, x_1, x_2, \dots, x_{N_1-1}, x_{N_1}) \in \mathbb{R}^N , \tag{2.8}$$

and claim that for $\mathbf{v}$ given by

$$\mathbf{v} = (\underbrace{0, 0, \dots, 0}_{N_1-1 \ times}, -i, i, \underbrace{0, 0, \dots, 0}_{N_1-1 \ times}) , \tag{2.9}$$

we have

$$H_{\mathbf{X}}(\mathbf{v}) < 0, \tag{2.10}$$

which implies the instability. To prove (2.10), note first that (see (3.11) below)

$$H_{\mathbf{X}}(\mathbf{v}) = -\sum_{j=2}^{N_1}\left(\frac{1}{(x_1-x_j)^2}+\frac{1}{(x_1+x_j)^2}\right)-\frac{1}{2x_1^2}+\frac{N_1(2N_1-1)}{2\sum_{j=1}^{N_1}x_j^2}. \tag{2.11}$$

By the arithmetic-geometric mean inequality we have

$$\frac{1}{2x_1^2}+\sum_{j=2}^{N_1}\frac{1}{(x_1-x_j)^2}+\frac{1}{(x_1+x_j)^2}\geq\frac{(2N_1-1)^2}{2x_1^2+\sum_{j=2}^{N_1}(x_1-x_j)^2+(x_1+x_j)^2}. \tag{2.12}$$

It follows that

$$H_{\mathbf{X}}(\mathbf{v}) \leq -\frac{(2N_1-1)^2}{2N_1x_1^2+2\sum_{j=2}^{N_1}x_j^2}+\frac{N_1(2N_1-1)}{2\sum_{j=1}^{N_1}x_j^2}. \tag{2.13}$$

Thus, in order to prove (2.10) it is sufficient to prove that

$$(2N_1-1)\sum_{j=1}^{N_1}x_j^2 > N_1^2x_1^2+N_1\sum_{j=2}^{N_1}x_j^2. \tag{2.14}$$

Since $x_1 < x_j$ for $j > 1$, (2.14) follows immediately. ∎

3. The symmetric critical points P_N and S_N. We denote the N sides regular polygon "perfect" configuration by

$$P_N = \left(\omega, \omega^2, \ldots, \omega^{N-1}, 1\right), \tag{3.1}$$

where $\omega \equiv e^{2\pi i/N}$, and the $N-1$ sides regular polygon plus the origin "star" configuration by

$$S_N = \left(\zeta, \zeta^2, \ldots, \zeta^{N-2}, 1, 0\right), \tag{3.2}$$

where $\zeta \equiv e^{2\pi i/(N-1)}$.

P_N and S_N are points in $\mathbb{C}^N$ that are invariant under rotation by $2\pi/N$ and $2\pi/(N-1)$, respectively. Due to the symmetry of the problem, P_N and S_N are natural candidates for being critical points of $\mathcal{G}(\mathbf{z})$. They also turn out to be the only critical points that we are able to give explicitly and not just prove the existence of or approximate numerically. To show that P_N and S_N are critical points, we compute the gradient of $\mathcal{G}$, denoted by $\mathcal{D}(\mathbf{z})$. The k_0th component of $\mathcal{D}(\mathbf{z})$ is

$$\frac{\partial}{\partial z_{k_0}}\mathcal{G}(\mathbf{z}) = -\sum_{j\neq k_0}\frac{z_{k_0}-z_j}{|z_{k_0}-z_j|^2}+\frac{1}{2}N(N-1)\ \frac{z_{k_0}}{\sum_j |z_j|^2}. \tag{3.3}$$

Next, we claim the following.

Lemma 3.1.

1. *P_N and S_N are critical points of $\mathcal{G}(\mathbf{z})$.*
2.

$$\mathcal{G}(P_N) = \frac{1}{4}N(N-3)\log(N), \tag{3.4}$$

$$\mathcal{G}(S_N) = \frac{1}{4}(N-1)(N-2)\log(N-1). \tag{3.5}$$

3. *P_N is not the global minimizer for $N \geq 6$.*

Proof. We use the identities

$$\sum_{j=1}^{N-1} \frac{1-\omega^j}{|1-\omega^j|^2} = \frac{1}{2}(N-1), \tag{3.6}$$

$$\prod_{j=1}^{N-1}(1-\omega^j) = N. \tag{3.7}$$

From (3.6) and (3.7) it follows that $\mathcal{D}(P_N) = 0$, which proves that P_N is a critical point. The proof for $\mathbf{z} = S_N$ is similar.

Using (3.7) we obtain the values of $\mathcal{G}(P_N)$ and $\mathcal{G}(S_N)$, as specified in (3.4) and (3.5). From (3.5) and (3.4) we see that

$$\mathcal{G}(S_N) > \mathcal{G}(P_N) \text{ for } N = 3, 4, 5, \tag{3.8}$$

$$\mathcal{G}(S_N) < \mathcal{G}(P_N) \text{ for } N \geq 6. \tag{3.9}$$

This implies immediately that P_N is not the global minimizer for $N \geq 6$. □

3.1. Stability analysis of P_N and S_N. After identifying P_N and S_N as critical points, the question of their stability arises naturally. This is an interesting question in its own right, and the answer also turns out to be important for further analysis of the problem. To investigate the stability of P_N and S_N we need a few results which are given in the following proposition.

PROPOSITION 3.2.

1. *Let $x \neq 0, v \in \mathbb{C}$, and $t \in \mathbb{R}$. Then*

$$\log(|x+tv|^2) = \log(|x|^2) + 2t\mathrm{Re}\left(\frac{x}{|x|^2}\bar{v}\right) + t^2\left(\frac{|v|^2}{|x|^2} - 2\left(\mathrm{Re}\left(\frac{x}{|x|^2}\bar{v}\right)\right)^2\right) + o(t^2). \tag{3.10}$$

2. *Let $\mathbf{v}, \mathbf{x} \in \mathbb{C}^N$ and $t \in \mathbb{R}$. Define $h(t) = \mathcal{G}(\mathbf{x} + t\mathbf{v})$. Then*

$$\begin{aligned} h(t) = &-\frac{1}{2}\sum_{1\leq k<j\leq N}\log(|x_k-x_j|^2) - t\sum_{1\leq k<j\leq N}\mathrm{Re}\left(\frac{x_k-x_j}{|x_k-x_j|^2}(\overline{v_k-v_j})\right) \\ &-\frac{1}{2}t^2\sum_{1\leq k<j\leq N}\frac{|v_k-v_j|^2}{|x_k-x_j|^2} + t^2\sum_{1\leq k<j\leq N}\left(\mathrm{Re}\left(\frac{x_k-x_j}{|x_k-x_j|^2}(\overline{v_k-v_j})\right)\right)^2 \\ &+\frac{N(N-1)}{4}\left\{\log\left(\sum_{k=1}^N|x_k|^2\right) + 2t\frac{\sum_{k=1}^N\mathrm{Re}(x_k\bar{v}_k)}{\sum_{k=1}^N|x_k|^2}\right. \\ &\left. + t^2\left(\frac{\sum_{k=1}^N|v_k|^2}{\sum_{k=1}^N|x_k|^2} - 2\left(\frac{\sum_{k=1}^N\mathrm{Re}(x_k\bar{v}_k)}{\sum_{k=1}^N|x_k|^2}\right)^2\right)\right\} + o(t^2). \end{aligned} \tag{3.11}$$

3. *The coefficient of t^2 in the Taylor expansion of $h(t)$ around 0, at the critical point $\mathbf{x} = P_N$, is given by*

$$\frac{1}{2}h''(0) \equiv H_{P_N}(\mathbf{v}) = -\frac{1}{2}\sum_{1\le k<j\le N}\left(\frac{|v_k-v_j|^2}{|\omega^k-\omega^j|^2} - 2\left(\mathrm{Re}\left(\frac{\omega^k-\omega^j}{|\omega^k-\omega^j|^2}\,\overline{(v_k-v_j)}\right)\right)^2\right)$$

$$+\frac{N-1}{4}\sum_{k=1}^{N}|v_k|^2 - \frac{N-1}{2N}\left(\sum_{k=1}^{N}\mathrm{Re}\left(\omega^k\bar{v}_k\right)\right)^2. \tag{3.12}$$

4. *The coefficient of t^2 in the Taylor expansion of $h(t)$ around 0, with $\mathbf{v}$ satisfying $v_N = 0$, at the critical point $\mathbf{x} = S_N$, is given by*

$$\frac{1}{2}h''(0) \equiv H_{S_N}(\mathbf{v}) = -\frac{1}{2}\sum_{1\le k<j\le N-1}\left(\frac{|v_k-v_j|^2}{|\zeta^k-\zeta^j|^2} - 2\left(\mathrm{Re}\left(\frac{\zeta^k-\zeta^j}{|\zeta^k-\zeta^j|^2}\,\overline{(v_k-v_j)}\right)\right)^2\right)$$

$$+\frac{N-2}{4}\sum_{k=1}^{N-1}|v_k|^2 - \frac{N}{2(N-1)}\left(\sum_{k=1}^{N-1}\mathrm{Re}\left(\zeta^k\bar{v}_k\right)\right)^2 + \sum_{k=1}^{N-1}\left(\mathrm{Re}\left(\zeta^k\bar{v}_k\right)\right)^2. \tag{3.13}$$

Proof. The formula (3.10) is proved easily by the expansion

$$\log(|x+tv|^2) = \log(|x|^2) + \log\left(1 + t\left(2\mathrm{Re}\left(\frac{x}{|x|^2}, v\right)\right) + t^2\frac{|v|^2}{|x|^2}\right). \tag{3.14}$$

Equation (3.11) follows from (3.10) and the definition of $\mathcal{G}$. Formulas (3.12) and (3.13) follow from (3.11), after the substitution $\mathbf{x} = P_N$ and $\mathbf{x} = S_N$, respectively. ∎

We now consider some special perturbations and their effect on the second-order differential of $\mathcal{G}$ at the investigated points P_N and S_N.

Lemma 3.3. *Consider the following vectors in $\mathbb{C}^N$: $\mathbf{v} = (0,0,\dots,1)$, $\tilde{\mathbf{v}} = (0,0,\dots,0,1,0)$, and $\mathbf{u}_m = (\omega^m, \omega^{2m}, \dots, \omega^{mN})$ for $m = 1,2,\dots,N$. Then*

1.

$$H_{P_N}(\mathbf{v}) = -\frac{1}{24}\frac{(N^2-11N+12)(N-1)}{N}; \tag{3.15}$$

2.

$$H_{S_N}(\tilde{\mathbf{v}}) = -\frac{1}{24}\frac{N(N-2)(N-13)}{N-1}; \tag{3.16}$$

3.

$$H_{P_N}(\mathbf{u}_1) = 0, \tag{3.17}$$

$$H_{P_N}(\mathbf{u}_m) = \frac{N(N-1)}{4} \quad \textit{for odd } N \textit{ and } m \ne 1, \tag{3.18}$$

$$H_{P_N}(\mathbf{u}_m) = \frac{N(N-1)}{4} \quad \textit{for even } N \textit{ and } m \ne 1, \frac{1}{2}N+1, \tag{3.19}$$

$$H_{P_N}(\mathbf{u}_m) = -\frac{1}{16}N^3 + \frac{1}{2}N^2 - \frac{1}{2}N \quad \textit{for even } N \textit{ and } m = \frac{1}{2}N+1. \tag{3.20}$$

Proof. For our calculations we use the following identity:

$$\sum_{k=1}^{N-1}\frac{1}{\sin^2(\frac{k\pi}{N})} = \frac{1}{3}(N^2-1). \tag{3.21}$$

For odd N, the identity (3.21) is proved by expressing $\sin(Nx)$ as a polynomial in $\sin(x)$, that is, defining the polynomial $p(y) = \sin(Nx)/\sin(x)$ where $\sin(x) = y$. The roots of this $N-1$ degree polynomial are $y_k = \arcsin(k\pi/N), k = 1, 2, \ldots, N-1$. Using Vietta's formulas we have

$$\sum_{k=1}^{N-1} \frac{1}{\sin^2(\frac{k\pi}{N})} = \lim_{x\to 0} \left(\left(\frac{p'(x)}{p(x)}\right)^2 - \frac{p''(x)}{p(x)} \right) = \frac{1}{3}(N^2-1) \tag{3.22}$$

which proves (3.21). For even N, the identity (3.21) is proved by repeating the above procedure with $p(y) = \sin(Nx)/\sin(2x)$. The proof of (3.15) and (3.16) follows from substituting $\mathbf{v} = (0, 0, \ldots, 1)$ and $\mathbf{v} = (0, 0, \ldots, 0, 1, 0)$ in (3.12) and (3.13), respectively, and using the identity (3.21).

When $m = 1$, (3.17) is obtained either by straightforward calculation, or alternatively, by using the scale invariance of $\mathcal{G}$. In order to prove (3.18)–(3.20) note first that by simple trigonometric identities we have for $\mathbf{v} = \mathbf{u}_m$ and $\theta = \frac{2\pi}{N}$

$$2\left(\mathrm{Re}\left(\frac{\omega^k-\omega^j}{|\omega^k-\omega^j|^2}\overline{(v_k-v_j)}\right)\right)^2 - \frac{|v_k-v_j|^2}{|\omega^k-\omega^j|^2} = \frac{\sin^2\left(\frac{m(k-j)\theta}{2}\right)}{\sin^2\left(\frac{(k-j)\theta}{2}\right)}\cos\left((m-1)(k+j)\theta\right). \tag{3.23}$$

Hence,

$$H_{P_N}(\mathbf{u}_m) = \frac{1}{4}\sum_{k\neq j} \frac{\sin^2\left(\frac{m(k-j)\theta}{2}\right)}{\sin^2\left(\frac{(k-j)\theta}{2}\right)}\cos\left((m-1)(k+j)\theta\right) + \frac{N(N-1)}{4} - \frac{N(N-1)}{2}\delta_{1m}. \tag{3.24}$$

Since the mapping $T(k,j) = (k-j, k+j) \pmod N$ is a bijection from $\mathbb{Z}_N$ to $\mathbb{Z}_N$ when N is odd, we have, denoting $r = k-j, s = k+j$,

$$H_{P_N}(\mathbf{u}_m) = \frac{1}{4}\sum_{r=1}^{N-1}\sum_{s=0}^{N-1} \frac{\sin^2\left(\frac{mr\theta}{2}\right)}{\sin^2\left(\frac{r\theta}{2}\right)}\cos\left((m-1)s\theta\right) + \frac{(N-1)N}{4} = \frac{(N-1)N}{4}. \tag{3.25}$$

To show (3.19) and (3.20), note that for even N the image of T consists only of pairs (r, s) in which the parity of r and s is equal. Moreover, each such pair is the image of exactly two preimages. For convenience we set $N_1 = \frac{1}{2}N$ and write

$$\begin{aligned} H_{P_N}(\mathbf{u}_m) &= \frac{1}{2}\sum_{r_1=1}^{N_1-1}\sum_{s_1=0}^{N_1-1} \frac{\sin^2(mr_1\theta)}{\sin^2(r_1\theta)}\cos\left(2(m-1)s_1\theta\right) \\ &+\frac{1}{2}\sum_{r_1=0}^{N_1-1}\sum_{s_1=0}^{N_1-1} \frac{\sin^2\left(\frac{m(2r_1+1)\theta}{2}\right)}{\sin^2\left(\frac{(2r_1+1)\theta}{2}\right)}\cos\left((m-1)(2s_1+1)\theta\right) + \frac{(N-1)N}{4} \\ &\equiv I_1 + I_2 + \frac{(N-1)N}{4}. \end{aligned} \tag{3.26}$$

If $m \neq 1, \frac{1}{2}N + 1$, then

$$\sum_{s_1=0}^{N_1-1} \cos\left(2(m-1)s_1\theta\right) = \sum_{s_1=0}^{N_1-1} \cos\left((m-1)(2s_1+1)\theta\right) = 0, \tag{3.27}$$

which proves (3.19). We are finally left with the case $m = N_1 + 1$. Clearly we have

$$I_1 = \frac{1}{2} N_1 \sum_{r_1=1}^{N_1-1} \frac{\sin^2\left(\frac{(N_1+1)r_1\pi}{N_1}\right)}{\sin^2\left(r_1\frac{\pi}{N_1}\right)} = \frac{1}{2} N_1(N_1-1). \tag{3.28}$$

Next, since

$$\sum_{s_1=0}^{N_1-1} \cos\left((2s_1+1)N_1\theta\right) = \sum_{s_1=0}^{N_1-1} \cos\left((2s_1+1)\pi\right) = -N_1, \tag{3.29}$$

we have

$$\begin{aligned} I_2 &= -\frac{1}{2} N_1 \sum_{r_1=0}^{N_1-1} \frac{\sin^2\left((N_1+1)r_1\theta + (N_1+1)\frac{\theta}{2}\right)}{\sin^2\left(r_1\theta + \frac{\theta}{2}\right)} \\ = -\frac{1}{2} N_1 \sum_{r_1=0}^{N_1-1} \frac{\sin^2\left(r_1\theta + \frac{\theta}{2} + \frac{\pi}{2}\right)}{\sin^2\left(r_1\theta + \frac{\theta}{2}\right)} &= -\frac{1}{2} N_1 \sum_{r_1=0}^{N_1-1} \frac{\cos^2\left(r_1\theta + \frac{\theta}{2}\right)}{\sin^2\left(r_1\theta + \frac{\theta}{2}\right)} \\ &= \frac{1}{2} N_1 \left(N_1 - \sum_{r_1=0}^{N_1-1} \frac{1}{\sin^2\left(\frac{(2r_1+1)\pi}{N}\right)} \right). \end{aligned} \tag{3.30}$$

Next, using the identity (3.21) we have

$$\begin{aligned} \sum_{r_1=0}^{N_1-1} \frac{1}{\sin^2\left(\frac{(2r_1+1)\pi}{N}\right)} &= \sum_{k=1}^{N-1} \frac{1}{\sin^2\left(\frac{k\pi}{N}\right)} - \sum_{k=1}^{N_1-1} \frac{1}{\sin^2\left(\frac{k\pi}{N_1}\right)} \\ &= \frac{1}{3}(N^2-1) - \frac{1}{3}(N_1^2-1) = N_1^2. \end{aligned} \tag{3.31}$$

Combining (3.26), (3.28)–(3.31) we finally obtain

$$\begin{aligned} H_{P_N}(\mathbf{u}_m) &= \frac{1}{2} N_1(N_1-1) - \frac{1}{2} N_1(N_1^2 - N_1) + \frac{1}{4} N(N-1) \\ &= -\frac{1}{16} N^3 + \frac{1}{2}(N^2 - N). \qquad \square \end{aligned} \tag{3.32}$$

We now have all the tools required for studying the stability of P_N and S_N for (almost) all N, and state the following theorem.

THEOREM 1.

1. *The critical points P_N are stable for $2 \leq N < 7$ and are unstable for $N \geq 8$.*
2. *The critical points S_N are stable for $4 \leq N < 10$ and are unstable for $N = 3$ and $N \geq 11$.*

Remark 1. Note that we are unable to decide on the stability of P_7 and S_{10}. The difficulty for these particular values is explained in the proof below.

Proof.

1. The critical points P_N: For $N \geq 10$, the instability of P_N follows directly from (3.15), and for $N = 8$ it follows from (3.20). To complete the proof of our statement concerning P_N it remains to investigate the cases $N = 3, 4, 5, 6$ (for stability) and $N = 9$ (for instability). This is done directly by computing the eigenvalues of the

Table 1

The eigenvalues of the Hessian matrix (represented as a $2N \times 2N$ real matrix) associated with $\mathcal{G}$, at the point P_N for $2 \leq N \leq 10$. The values are obtained by direct factorization of the characteristic polynomial.

N	Hessian matrix eigenvalues
2	0, 0, 1, 1
3	0, 0, 2, 2, 2, 2
4	0, 0, 2, 4, 3, 3, 3, 3
5	0, 0, 2, 2, 4, 4, 4, 4, 6, 6
6	0, 0, 1, 2, 2, 5, 5, 5, 5, 8, 8, 9
7	0, 0, 0, 0, 2, 2, 6, 6, 6, 6, 10, 10, 12, 1
8	-2, -1, -1, 0, 0, 2, 2, 7, 7, 7, 7, 12, 12, 15, 15, 16
9	-4, -4, -2, -2, 0, 0, 2, 2, 8, 8, 8, 8, 14, 14, 18, 18, 20, 20
10	-7, -6, -6, -3, -3, 0, 0, 2, 2, 9, 9, 9, 9, 16, 16, 21, 21, 24, 24, 25

Hessian matrix (represented as a $2N \times 2N$ real matrix) associated with $\mathcal{G}$, at the point P_N, for these specific values of N. The resulting characteristic polynomials have integer coefficients and integer roots. They can be factored analytically. Table 1 gives the list of the eigenvalues for $2 \leq N \leq 10$. Note that zero is always an eigenvalue of multiplicity 2 or 4. In fact, two of these directions correspond to the two obvious invariance properties of $\mathcal{G}$: with respect to rotations and rescaling ($v \to rv$). Thus, for the study of stability, these two zeros in each line of Table 1 can be ignored. By examining the signs of the nonzero eigenvalues we deduce the stability for $3 \leq N \leq 6$ and the instability for $N = 9$. For $N = 7$ the multiplicity of the eigenvalue zero is 4. Hence, there are two "nontrivial" eigenvectors and the stability question is not resolved by the present study of the second-order derivatives.

2. The critical points S_N: For $N \geq 14$, the instability of S_N follows directly from (3.16). To complete the proof of our statement concerning S_N it remains to investigate the cases $4 \leq N \leq 9$ (for stability) and $N = 3, 11, 12, 13$ (for instability). Again, this is done directly by computing the eigenvalues of the Hessian matrix (represented as a $2N \times 2N$ real matrix) associated with $\mathcal{G}$, at the point S_N for these specific values of N. The resulting characteristic polynomials have integer coefficients. Their roots are integers for $N = 3, 4$ and, for $5 \leq N \leq 13$, they are either integers or of the form $N \pm 2\sqrt{N}$ (each with multiplicity 2). These polynomials can be factored analytically and the roots are given in Table 2. As above, for each N the multiplicity of the eigenvalue zero is 2 or 4. Two of the eigenvectors of zero can be explained by the invariance properties of $\mathcal{G}$. So for $N \neq 10$ the result follows by examining the signs of the nonzero eigenvalues. For $N = 10$ the multiplicity of zero is 4 and we cannot resolve the question of stability by the present study of the second-order derivatives. ∎

4. Critical points enjoying different kinds of symmetry. So far we have met three kinds of critical points of $\mathcal{G}$, namely, P_N, S_N, and the minimizing configuration for Problem 3 (i.e., minimization over $\mathbb{R}^N$). All three possess certain symmetries. P_N is symmetric with respect to rotation by $2\pi/N$ and S_N is symmetric with respect to rotation by $2\pi/N-1$. The solution of Problem 3 enjoys symmetry with respect to multiplication by -1, in addition to being (trivially) invariant under complex conjugation. This observation leads us to look for critical points of $\mathcal{G}$ (or $\mathcal{F}$) in (real) linear subspaces which are the fixed points set for a certain group of isometries of $\mathbb{C}^N$. This is established by the following lemma.

Lemma 4.1. *Let G be a finite group of (real) linear isometries from $\mathbb{C}^N$ to $\mathbb{C}^N$ and let V denote the (real) linear subspace of fixed points of G which is assumed to*

Table 2

The eigenvalues of the Hessian matrix (represented as a $2N\times 2N$ real matrix) associated with $\mathcal{G}$, at the point S_N for $3 \le N \le 13$. The values are obtained by direct factorization of the characteristic polynomial. The integer eigenvalues are listed first, followed by the irrational eigenvalues when appropriate.

N	Hessian matrix eigenvalues
3	-3, 0, 0, 3, 3, 9
4	0, 0, 0, 0, 4, 4, 8, 8
5	0, 0, 4, 5, 5, 6, 5 $\pm\sqrt{20}$, 5 $\pm\sqrt{20}$
6	0, 0, 6, 6, 6, 6, 6, 6, 6 $\pm\sqrt{24}$, 6 $\pm\sqrt{24}$
7	0, 0, 5, 6, 6, 7, 7, 8, 8, 9, 12, $\pm\sqrt{48}$, 12 $\pm\sqrt{48}$
8	0, 0, 4, 4, 6, 6, 8, 8, 10, 10, 12, 12, 8 $\pm\sqrt{32}$, 8 $\pm\sqrt{32}$
9	0, 0, 2, 3, 3, 3, 3, 6, 6, 9, 9, 12, 12, 15, 15, 15, 15, 16
10	0, 0, 0, 0, 2, 2, 6, 6, 10, 10, 14, 14, 18, 18, 20, 20, 10 $\pm\sqrt{40}$, 10 $\pm\sqrt{40}$
11	-3, -2, -2, 0, 0, 1, 1, 6, 6, 11, 11, 16, 16, 21, 21, 24, 24, 25, 11 $\pm\sqrt{44}$, 11 $\pm\sqrt{44}$
12	-6, -6, -4, -4, 0, 0, 0, 0, 6, 6, 12, 12, 18, 18, 24, 24, 28, 28, 30, 30, 12 $\pm\sqrt{48}$, 12 $\pm\sqrt{48}$
13	-10, -9, -9, -6, -6, -1, -1, 0, 0, 6, 6, 13, 13, 20, 20, 27, 27, 32, 32, 34, 35, 36, 13 $\pm\sqrt{52}$, 13 $\pm\sqrt{52}$

be nontrivial. Let $\tilde{\mathcal{G}}$ denote the restriction of $\mathcal{G}$ to V. Then any critical point $x_0 \in V$ of $\tilde{\mathcal{G}}$ is also a critical point of $\mathcal{G}$.

Proof. This lemma is a special case of a more general principle of symmetric criticality (see Palais [6]). For the sake of completeness, we give a short proof for our case.

We denote the elements of the group by $G = \{g_1, g_2, \ldots, g_M\}$. For any $v \in \mathbb{C}^N$ and $g_i \in G$, we may use the invariance of $\mathcal{G}$ under isometries of $\mathbb{C}^N$ and write

$$\mathcal{G}\big(x_0 + \epsilon v\big) = \mathcal{G}\big(g_i(x_0 + \epsilon v)\big) = \mathcal{G}\big(g_i(x_0) + \epsilon g_i(v)\big). \tag{4.1}$$

Therefore,

$$\langle \mathcal{G}'(x_0), v\rangle = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \mathcal{G}(x_0 + \epsilon v)$$

$$= \frac{1}{M}\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \sum_{i=1}^{M} \mathcal{G}(x_0 + \epsilon g_i(v)) = \left\langle \mathcal{G}'(x_0), \frac{1}{M}\sum_{i=1}^{M} g_i(v) \right\rangle, \tag{4.2}$$

but we clearly have $\tilde{v} \equiv \frac{1}{M}\sum_{i=1}^{M} g_i(v) \in V$ and hence

$$0 = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \tilde{\mathcal{G}}(x_0 + \epsilon\tilde{v}) = \langle \tilde{\mathcal{G}}'(x_0), \tilde{v}\rangle. \tag{4.3}$$

Combining (4.2) and (4.3) we see that $\langle \mathcal{G}'(x_0), v\rangle = 0$ for all $v \in \mathbb{C}^N$, which proves the desired result. □

This lemma provides an alternative proof for the fact that P_N and S_N are critical points of $\mathcal{G}$. For example, to show that S_N is a critical point, it is enough to minimize $\mathcal{G}$ on the invariant subspace of the group G generated by the isometry g of $\mathbb{C}^N$ defined by

$$g(z_1, z_2, \ldots, z_{N-1}, z_N) = (\zeta z_2, \zeta z_3, \ldots, \zeta z_1, \zeta z_N) \tag{4.4}$$

(as above, $\zeta \equiv e^{2\pi i/(N-1)}$). We give several other applications for this lemma below.

4.1. Critical points, other than P_N and S_N, that are invariant under rotations. We now investigate the existence of critical points which are invariant under rotation by $2\pi/N$ and which are different from P_N and S_{N+1}.

LEMMA 4.2. *Let $N = n + k$, with $n > k$. Denote*

$$Q^{\beta}_{n\,k} = (\xi_1, \xi_1^2, \ldots, \xi_1^n, \beta\xi_2, \beta\xi_2^2, \ldots, \beta\xi_2^k) \in \mathbb{C}^N, \tag{4.5}$$

where $0 < \beta < 1$ and

$$\xi_1 = e^{2\pi i/n}, \qquad \xi_2 = e^{2\pi i/k}. \tag{4.6}$$

Then there exists $n_0(k)$ such that if $n > n_0$ and $\beta^2 < (k-1)/k$ we have

$$\mathcal{G}(Q^{\beta}_{n\,k}) < \mathcal{G}(S_N) < \mathcal{G}(P_N). \tag{4.7}$$

Proof. From (3.5) we have

$$\mathcal{G}(S_N) = \frac{1}{4}(N-1)(N-2)\log(N-1). \tag{4.8}$$

The value of $\mathcal{G}(Q^{\beta}_{n\,k})$ depends only on β, the ratio of the radii of the two n sides and k sides regular polygons of which $Q^{\beta}_{n\,k}$ is composed. We denote $d = \gcd(n,k)$ and claim that the value of $\mathcal{G}(Q^{\beta}_{n\,k})$ is

$$\mathcal{G}(Q^{\beta}_{n\,k}) = \frac{1}{4}\left((n+k)^2 - n - k\right)\log(n + k\beta^2) - \log\left(n^{\frac{1}{2}n}k^{\frac{1}{2}k}\beta^{\frac{1}{2}k(k-1)}(1-\beta^{nk/d})^d\right). \tag{4.9}$$

To see that we first use (3.7) to obtain

$$\prod_{1\le j<l\le n}|\xi_1^j - \xi_1^l| = n^{n/2} \quad\text{and}\quad \prod_{1\le j<l\le k}|\xi_2^j - \xi_2^l| = k^{k/2}. \tag{4.10}$$

Next, we have

$$I \equiv \prod_{j=1}^{n}\prod_{l=1}^{k}|\xi_1^j - \beta\xi_2^l| = \left|\prod_{l=1}^{k}(\beta^n\xi_2^{ln} - 1)\right| = \beta^{kn}\left|\prod_{l=1}^{k}\left(\xi_2^{ln} - \frac{1}{\beta^n}\right)\right|. \tag{4.11}$$

We note that the identity (4.11) follows by substituting $z = \beta\xi_2^l$ in the identity $z^n - 1 = \prod_{j=1}^{n}(z - \xi_1^j)$. We now write $n = n_1 d$ and $k = k_1 d$. Since ξ_2^d is a primitive root of unity of order k_1, we have

$$r(z) \equiv z^{k_1} - 1 = \prod_{l=1}^{k_1}\left(z - \xi_2^{nl}\right). \tag{4.12}$$

Therefore, we can conclude

$$I = \beta^{kn}\left(r(1/\beta^n)\right)^d \tag{4.13}$$

and the expression for the value of $\mathcal{G}(Q^{\beta}_{n\,k})$, as in (4.9), follows.

Asymptotic expansions for $\mathcal{G}(Q^{\beta}_{n\,k})$ and $\mathcal{G}(S_N)$ give

$$\mathcal{G}(Q^{\beta}_{n\,k}) = \frac{1}{4}n^2\log(n) + \frac{1}{4}(2k-3)n\log(n) + \frac{1}{4}k\beta^2 n + O(\log(n)) \tag{4.14}$$

and

$$\mathcal{G}(S_N) = \frac{1}{4}n^2\log(n) + \frac{1}{4}(2k-3)n\log(n) + \frac{1}{4}(k-1)n + O(\log(n)). \tag{4.15}$$

Therefore, the result follows directly from (4.14) and (4.15). □

THEOREM 2. *Let $k > 1$ be a given positive integer. Then, there exists a number m_0 such that for all $m \geq m_0$*

1. *there exists a critical point for $\mathcal{P}_{mk}$, different from P_{mk}, which is invariant under rotation by $2\pi/k$;*

2. *there exists a critical point for $\mathcal{P}_{mk+1}$, different from S_{mk+1}, which is invariant under rotation by $2\pi/k$.*

Proof.

1. We take β satisfying $\beta^2 < (k-1)/k$ and $N = k + mk$, where m is chosen large enough so that

$$\mathcal{G}(Q^{\beta}_{mk\,k}) < \mathcal{G}(S_N) < \mathcal{G}(P_N). \tag{4.16}$$

The existence of such m is guaranteed by Lemma 4.2.

Consider the set $A_{N,k}$ of all points $\mathbf{z} \in \mathbb{C}^N$ that are invariant under rotation by $2\pi/k$. The functional $\mathcal{G}$ attains its minimum on $A_{N,k}$ at some point $\mathbf{a} \in A_{N,k}$, which is a critical point on $A_{N,k}$. From Lemma 4.1 it follows that $\mathbf{a}$ is also a critical point of $\mathcal{F}$ on $\mathbb{C}^N$. From (4.16) we conclude that $\mathbf{a} \neq P_N$.

2. The proof for this case is similar to the previous one. We take β such that $\beta^2 < (k-1)/k$ and $N = 1 + k + mk$, where m is chosen large enough so that $\mathcal{G}(Q^{\beta}_{mk+1\,k}) < \mathcal{G}(S_N) < \mathcal{G}(P_N)$, and we consider a minimizer for $\mathcal{G}$ over the set $B_{N,k}$ of all points $\mathbf{z} \in \mathbb{C}^N$ that are invariant under rotation by $2\pi/k$. □

4.2. A class of critical points invariant under reflections. From Theorem 1 it follows that for $N \geq 11$ neither P_N nor S_N is the minimizer of $\mathcal{P}_N$. This implies that the minimizers have fewer symmetry properties or none. On the other hand, Lemma 4.2 guarantees the existence of an infinite number of critical points with "partial" symmetry. For example, for any $N = 4l + 2$, we study the set $A_{N,4l,2}$ of the points in $\mathbb{C}^N$ that are composed of l rectangles with sides that are parallel to the axes, and center at the origin (called canonic rectangles hereafter), plus two points on the x-axis. In fact, the subspace $A_{N,4l,2}$ is the fixed point set of the group of isometries G generated by g_1 and g_2 defined by

$$\begin{aligned} &g_1(z_1, z_2, z_3, z_4, \ldots, z_{4l-3}, z_{4l-2}, z_{4l-1}, z_{4l}, z_{4l+1}, z_{4l+2}) \\ &\quad = \big(R_1(z_2), R_1(z_1), R_1(z_4), R_1(z_3), \ldots, \\ &R_1(z_{4l-2}), R_1(z_{4l-3}), R_1(z_{4l-1}), R_1(z_{4l}), R_1(z_{4l+2}), R_1(z_{4l+1})\big) \end{aligned} \tag{4.17}$$

where R_1 denotes reflection with respect to the y-axis, and

$$\begin{aligned} &g_2(z_1, z_2, z_3, z_4, \ldots, z_{4l-3}, z_{4l-2}, z_{4l-1}, z_{4l}, z_{4l+1}, z_{4l+2}) \\ &\quad = \big(R_2(z_3), R_2(z_4), R_2(z_1), R_2(z_2), \ldots, \\ &R_2(z_{4l-1}), R_2(z_{4l}), R_2(z_{4l-3}), R_2(z_{4l-2}), R_2(z_{4l+1}), R_2(z_{4l+2})\big), \end{aligned} \tag{4.18}$$

where R_2 denotes reflection with respect to the x-axis.

By the same argument given in section 2 we can conclude that $\mathcal{G}$ attains its minimum on $A_{N,4l,2}$ at a certain point denoted $\mathbf{b}_{N,4l,2}$ which, by means of Lemma 4.1, is also a critical point of $\mathcal{G}$ in $\mathbb{C}^N$. Since we know that $P_N \in A_{N,4l,2}$ it is not clear a priori whether or not $\mathbf{b}_{N,4l,2} \neq P_N$. This problem is treated by the following theorem.

THEOREM 3. *Let* $N = 4l + 2$. *Then*

1. *there exist critical points of* $\mathcal{G}$ *in* $A_{N,4l,2}$, *different from* P_N *if* N *is sufficiently large;*
2. $N = 10$ *is the smallest* N *with this property.*

Proof. The existence of critical points in $A_{N,4l,2}$ which are different from P_N is guaranteed by Lemmas 4.2 and 4.1. To prove the second statement, we first show that P_6 is the only critical point in $A_{6,4,2}$. In general, when $N = 4l + 2$, solving the problem $\mathcal{P}_N$ is equivalent to minimizing a real polynomial function of $2l+1$ variables with one quadratic constraint. For $N = 6$ this can be reduced to *maximizing* the function $f(a,\alpha)$ in the region $a \geq 0$, $\alpha \geq 0$, $4a^2 + 2\alpha^2 \leq 1$ where

$$f(a,\alpha) = \frac{1}{32}\alpha\, a^2 \left(1 - 4\,a^2 - 2\,\alpha^2\right)\left(1 - 2\,\alpha^2\right)\left(2\,\alpha^2 + 1 - 8\,a\alpha\right)^2 \left(2\,\alpha^2 + 1 + 8\,a\alpha\right)^2. \tag{4.19}$$

A point in $A_{6,4,2}$ can be geometrically described as one canonic rectangle plus two symmetric points on the x-axis. With this interpretation, $2a$ is one of the sides of the rectangle, and $\pm\alpha$ is the x-coordinate of the two real points. The maximum of $f(a,\alpha)$ is $\sqrt{6}/7776$ and is attained at $(a,\alpha) = (\sqrt{6}/12, \sqrt{6}/6)$, and the value of $\mathcal{G}$ at this point is $-\log(\sqrt{6}/7776)$. This maximum point corresponds exactly to P_6. There are no other nondegenerate critical points in $A_{6,4,2}$.

To complete the proof we show that there exists a critical point in $A_{10,8,2}$, other than P_{10}. It is enough to find a point $\mathbf{a}_{10,8,2} \in A_{10,8,2} \subset \mathbb{C}^{10}$, such that $\mathbf{a}_{10,8,2} \neq P_{10}$ and for which $\mathcal{G}(\mathbf{a}_{10,8,2}) < \mathcal{G}(P_{10})$. Such a point is shown in Figure 1. Observing Figure 1, one might suspect that $\mathbf{a}_{10,8,2}$ is a configuration composed of two points on the x-axis that lie inside a regular octagon. However, this is not the case because it can easily be proved (using (3.3)) that no configuration consisting of two symmetric points on the x-axis lying inside a regular polygon is a critical point of $\mathcal{G}$. ∎

5. Numerical optimization results. Numerical optimization techniques are a helpful tool for acquiring insights and approximating the minimizers. We briefly discuss here the techniques we used, and we report on some numerical results. Although Problems 1 and 2 are equivalent, and the paper is written mainly in terms of Problem 1, we prefer dealing with Problem 2 for numerical investigation because the equality constraint $\sum_{k=1}^{N} |z_k|^2 = 1$ limits the numerical search to finite ranges. For any given value of N, solving Problem 1 is equivalent to a $2N$ variables real optimization problem. We used the sequential quadratic programming method for constrained optimization problems [5]. Powell's method (see, e.g., [7] for implementation) was tested as well and yielded satisfactory results. In all tests, we chose the initial conditions randomly and repeated each run several times to increase the reliability of the numerical results.

Numerical optimization of Problem 1 indicates that the minimizing configurations are P_3, P_4, P_5 for $N = 3,4,5$, respectively, and S_6, S_7, S_8, S_9 for $N = 6,7,8,9$, respectively. For $N = 10$, the point (configuration) $\mathbf{a}_{10,8,2}$ that is plotted in Figure 1 is an approximation of $\mathbf{b}_{N,4l,2}$, the critical point in $A_{10,8,2}$. In fact, to the extent that

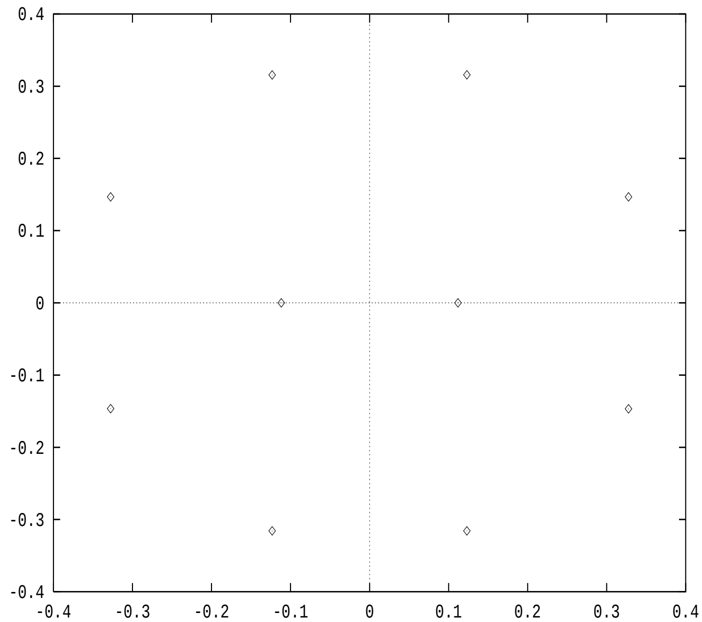


FIG. 1. *A point* $\mathbf{a}_{10,8,2} \in A_{10,8,2} \subset \mathbb{C}^{10}$, *different from* P_{10} *that satisfies* $\mathcal{G}(\mathbf{a}_{10,8,2}) < \mathcal{G}(P_{10})$. *This geometric configuration is composed of two canonic rectangles with dimensions* $2a \times 2b$ *and* $2c \times 2d$, *where* $a = 0.32765$, $b = 0.14669$, $c = 0.12323$, $d = 0.31572$, *and two points on the x-axis* $(0.11183, 0)$ *and* $(-0.11183, 0)$. $\mathcal{G}(\mathbf{a}_{10,8,2}) < 39.465 < \mathcal{G}(P_{10}) = 17.5 \log(10) \approx 40.2952$. *We point out that numerical investigation indicates that the point* $\mathbf{a}_{10,8,2}$ *is an approximation to the critical point* $\mathbf{b}_{N,4l,2}$ *of* $\mathcal{G}$ *which is probably also a global minimizer for* $\mathcal{P}_{10}$. *Note: the eight points which are the vertices of the two rectangles in this configuration* do not *form a regular octagon.*

numerical investigation is an indication this point is an approximation of the unique global minimizer of $\mathcal{P}_{10}$. Calculations show that $\mathcal{G}(\mathbf{a}_{10,8,2}) < 39.465 < \mathcal{G}(S_{10}) \approx 39.55 < \mathcal{G}(P_{10}) = 17.5 \log(10) \approx 40.2952$. Note again that the eight points which are the vertices of the two rectangles in this configuration *do not* form a regular octagon although they are close to such an octagon.

Figures 2 and 3 show a few numerical results for various values of N. They indicate that for large N, the minimizing configuration is composed of layers, each one a regular polygon "approximately," with increasing radii.

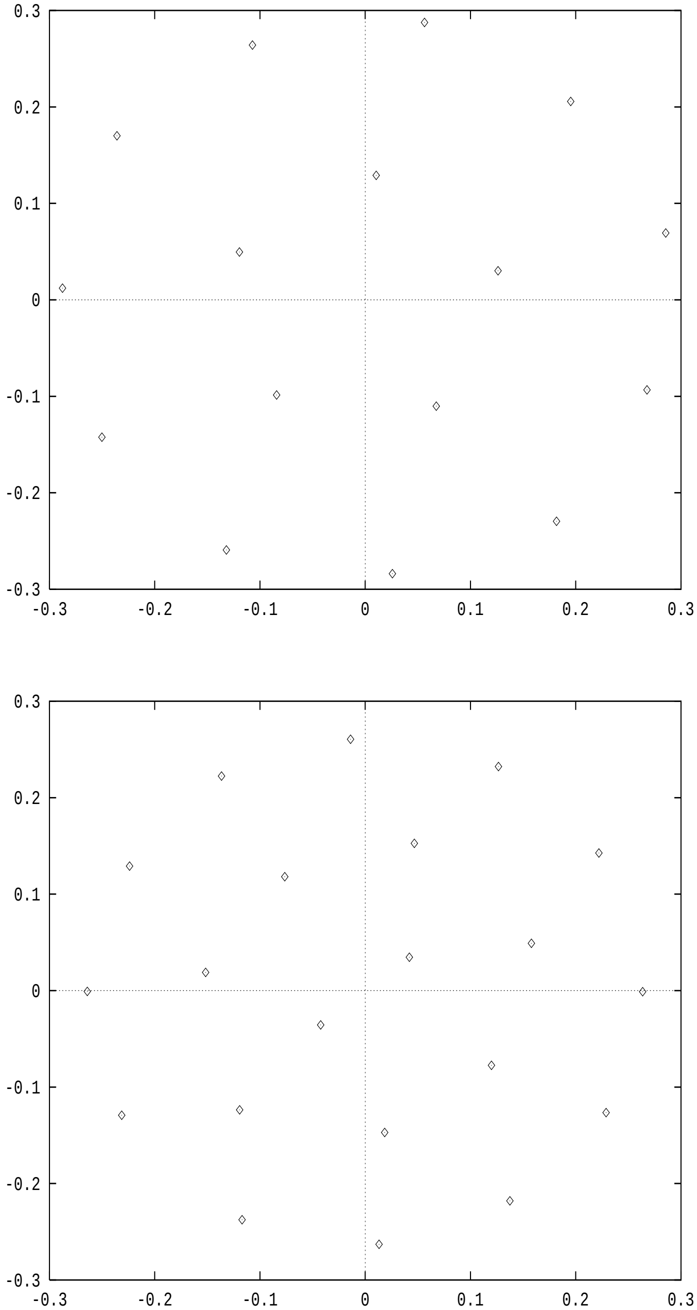
6. Some open problems.

1. *Can one give explicitly a minimizer for* $\mathcal{P}_N$ *for, say,* $N > 10$*?*

Numerical optimization can be applied. Unfortunately, it is not clear that such techniques indeed approximate the minimizers, to what degree of accuracy, and particularly, how they tackle the question of uniqueness of the minimizers. Also, as N increases, the complexity of the problem increases, which makes numerical methods less reliable.

2. *For which values of* N *is there a unique (up to permutations and rotations) minimizer for* $\mathcal{P}_N$*?*

For small N ($N \leq 15$) numerical investigation suggests that uniqueness holds. Also, for small N (e.g., $N = 2, 3$) the problem is trivial: For $N = 2$, the unique minimizer consists of two symmetric points on the x-axis, while for $N = 3$ the minimizer is an equilateral triangle.



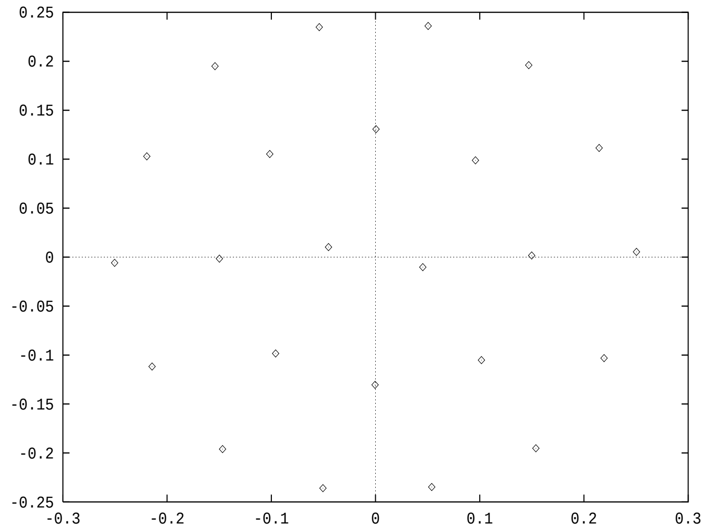
(a)

(b)

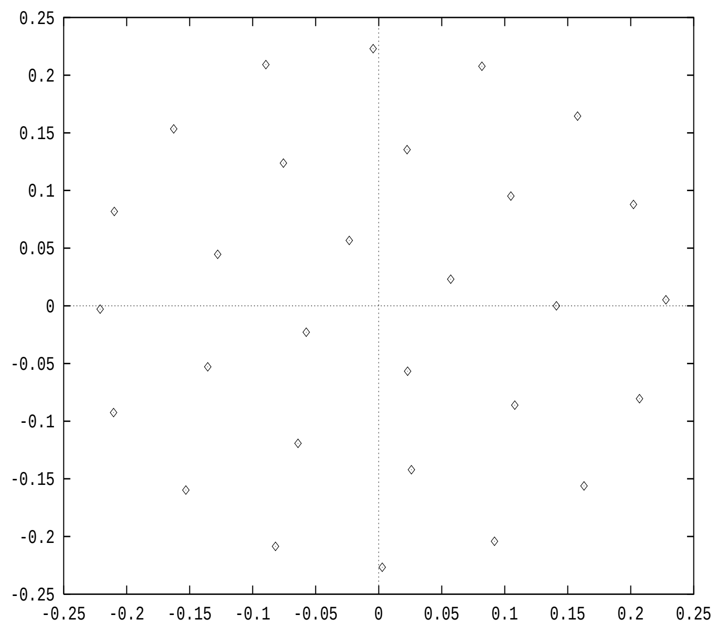
FIG. 2. *The result of numerical optimization of* $\mathcal{P}_N$ *for* $N = 16$, 21. *Multiple layers appear.*

3. *Is there a minimizer, or more generally, a critical point, for* $\mathcal{P}_N$ *which is not invariant under any nontrivial group of isometries of* $\mathbb{C}^N$*?*

We believe that the answer to this question is positive. In fact, already for $N = 11$



(a)



(b)

FIG. 3. *The result of numerical optimization of $\mathcal{P}_N$ for $N = 24, 28$. Multiple layers appear.*

we found numerical evidence for such an example.

4. *Can one describe the behavior of the minimizers for $N \gg 1$? Can one at least estimate the value of $\mathcal{G}$ at the minimizers, for $N \gg 1$?*

To obtain some intuition, see Figures 2 and 3 that indicate the formation of layers, each one an "approximately" regular polygon, with increasing radii.

In this context we mention a related discrete minimization problem which is also motivated by the Ginzburg–Landau theory ([3, Problem 12]). This problem is to minimize

$$\mathcal{J}(\mathbf{z}) = -\sum_{k\neq j} \log\left(|z_k - z_j|\right) - \sum_{k,j} \log(|1 - z_k \bar{z}_j|), \quad k, j = 1, 2, \ldots, N,$$

for $\mathbf{z} = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N$ satisfying $|z_k| \leq 1$ for all k and $z_k \neq z_j$ for $1 \leq k < j \leq N$.

Sandier and Soret [8] showed that a minimizing configuration approaches the unit circle as $N \to \infty$. More precisely, they showed that if $(z_1^{(N)}, z_2^{(N)}, \ldots, z_N^{(N)})$ is a minimizing configuration for $\mathcal{J}$, then the sequence of measures

$$\left\{ \frac{1}{N} \sum_{j=1}^{N} \delta_{z_j^{(N)}} \right\}_{N\geq 1}$$

converges in the sense of measures to the uniform measure of the circle $\{|z| = 1\}$. We conjecture that if the minimizing configurations $(z_1^{(N)}, z_2^{(N)}, \ldots, z_N^{(N)})$ for $\mathcal{G}$ are normalized so that $\sum_{j=1}^{N} |z_j^{(N)}|^2 = N$, then $\frac{1}{N}\sum_{j=1}^{N} \delta_{z_j^{(N)}}$ converges to a uniform measure on a *disc* of radius R_0 centered at the origin. Indication for this conjecture is given by our numerical results for large N (see Figure 3).

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