On the distance between homotopy classes of maps between spheres

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Dedicated with great respect to Haim Brezis on the occasion of his 70th birthday

Abstract

Certain Sobolev spaces of maps between manifolds can be written as a disjoint union of homotopy classes. Rubinstein and Shafrir computed the distance between homotopy classes in the spaces $W^{1,p}(S^1,S^1)$, for different values of p, and in the space $W^{1,2}(\Omega,S^1)$, for certain multiply connected two dimensional domains Ω. We generalize some of these results to higher dimensions. Somewhat surprisingly we find that in $W^{1,p}(S^2,S^2)$, with $p>2$, the distance between any two distinct homotopy classes equals a universal positive constant $c(p)$. A similar result holds in $W^{1,p}(S^n,S^n)$, for any $n\geq 2$ and $p>n$.

1 Introduction

Sobolev spaces of maps from a domain or a manifold with values in spheres appear naturally in Geometry and Analysis, especially in the study of harmonic maps. Motivation to study such maps comes from several areas of Physics like Liquid Crystals and Superconductivity. In general, decomposition of the relevant space $W^{1,p}(D,S^n)$ (where D is either a domain or a manifold) into homotopy classes is a very subtle issue (see [2, 3, 6, 7, 8, 9]). In some cases a decomposition of the form

$$W^{1,p}(D,S^n)=\bigcup_d \mathcal{E}_d\,, \tag{1.1}$$

of the space into a disjoint union of homotopy classes, holds, where d is either an integer or a vector of integers. Such partitions for Sobolev spaces of mappings between two Riemannian manifolds were developed by B. White ([18]). The existence of such a partition for maps from a two dimensional disk to S^2 (for $p = 2$) was proved by Brezis and Coron who used it to establish existence of certain harmonic maps. Rubinstein and Sternberg [13] used such a partition for maps from the solid torus to S^1 to explain persistent currents in Superconductivity and in a later work, with Kim [10], to predict new structures in Liquid Crystals (here again $p = 2$). The partition (1.1) where D is a multiply connected domain, $m = 1$ and $p = 2$, arises naturally in the study by Bethuel, Brezis and Hélein [1] of minimizers of Ginzburg-Landau type energy (in this case d is a vector of length n, where n is the number of holes).

Usually partitions like (1.1) are used to prove existence of non-trivial p-harmonic maps, as minimizers of the p-energy within a specific homotopy classes. The study of the *distance* between two distinct homotopy classes seems to be initiated by Rubinstein and Shafrir in [12]. They considered two classes of maps. The first is

$$H^1(S^1, S^1) = W^{1,2}(S^1, S^1) = \bigcup_{d\in\mathbb{Z}} \mathcal{E}_d = \bigcup_{d\in\mathbb{Z}} \{u : \deg u = d\},$$

where for any two integers $d_1 \neq d_2$ the distance $\delta(d_1, d_2)$ between the homotopy classes $\mathcal{E}_{d_1}$ and $\mathcal{E}_{d_2}$ is defined by

$$\delta^2(d_1, d_2) = \inf\left\{\int_{S^1} |(u_1 - u_2)'|^2 : u_1 \in \mathcal{E}_{d_1}, u_2 \in \mathcal{E}_{d_2}\right\}. \tag{1.2}$$

Rubinstein and Shafrir found an explicit formula for $\delta(d_1, d_2)$, namely

$$\delta^2(d_1, d_2) = \frac{8(d_2 - d_1)^2}{\pi}. \tag{1.3}$$

They also proved analogous formula for different values of p. Here we study the distance between homotopy classes of self-maps of spheres in higher dimension. Since the case of maps from S^n to S^n turns out to be essentially the same for any $n \geq 2$, we restrict ourselves below to the case $n = 2$ (see Section 3 for details on the n-dimensional case) . A well-defined notion of degree for maps in $W^{1,p}(S^2, S^2)$ exists only for $p \geq 2$ (see Section 2 for details). We then have

$$W^{1,p}(S^2, S^2) = \bigcup_{d\in\mathbb{Z}} \mathcal{E}_d = \bigcup_{d\in\mathbb{Z}} \{u : \deg u = d\}.$$

The distance between $\mathcal{E}_{d_1}$ and $\mathcal{E}_{d_2}$ is defined by

$$\delta_p^p(d_1, d_2) = \inf\left\{\int_{S^2} |\nabla(u_1 - u_2)|^p : u_1 \in \mathcal{E}_{d_1}, u_2 \in \mathcal{E}_{d_2}\right\}.$$

We compute explicitly $\delta_p(d_1, d_2)$, and somewhat surprisingly, the results are quite different from those of [12] for the case $n = 1$. First, in the case $p = 2$ we found in Theorem 2 that $\delta_2(d_1, d_2) = 0$ for every $d_1, d_2 \in \mathbb{Z}$. Second, when $p > 2$, for every pair $d_1 \neq d_2$, the distance $\delta_p(d_1, d_2)$ equals a fixed positive value (independently of the degrees) that is given explicitly by $\frac{2}{C_p}$, where

$$C_p = (2\pi)^{-1/p} \left[\sqrt{\pi} \frac{\Gamma\left(\frac{p-2}{2p-2}\right)}{\Gamma\left(\frac{2p-3}{2p-2}\right)} \right]^{1-1/p} \quad \text{(see Theorem 3).}$$

The constant C_p arises as the best constant in a Sobolev type inequality on two dimensional spheres, which is due to Talenti [17].

A brief explanation for the value $\frac{2}{C_p}$ goes as follows. It is not difficult to see that for any two maps, $u_1 \in \mathcal{E}_{d_1}$ and $u_2 \in \mathcal{E}_{d_2}$, the (scalar) function $v = |u_2 - u_1|$ must take both the value 0 and 2 somewhere (we assume here for simplicity that $d_2 \neq -d_1$). Then, Talenti's inequality applied to v yields,

$$\int_{S^2} |\nabla (u_1 - u_2)|^p \geq \int_{S^2} |\nabla v|^p \geq \frac{\max_{S^2} v - \min_{S^2} v}{C_p} = \frac{2}{C_p}\,.$$

This is essentially the proof of the lower bound of Theorem 3. The proof of the upper bound uses an explicit construction based on the profile of the optimal function in Talenti's inequality. The main difference from the case $n = 1$ is explained by the extra dimension, that allows us to construct maps possessing k-axial symmetry, i.e., maps of the form

$$u(\varphi, \theta) = (\sin \Phi(\varphi) \cos(k\theta), \sin \Phi(\varphi) \sin(k\theta), \cos \Phi(\varphi))\,.$$

The degrees of these maps are the result of rotations around the z-axis that do not affect the distance between the maps, see the proof of Theorem 3 for details. The rather straightforward generalization of the above results to maps between higher dimensional spheres is given in Section 3.

Remark 1. The M.Sc. thesis [11] also contains a generalization of the above results for the distance between homotopy classes of maps in $W^{1,p}\left(S^2, \Sigma\right)$, where $\Sigma = \partial K$ is a surface which is the boundary of a convex body $K \subset \mathbb{R}^2$ of class C^2_+ (see [14]). There it is proved that for $p > 2$ we have $\delta_p\left(d_1, d_2\right) = \frac{W}{C_p}$, where W is the *width* of the convex body K (i.e., the minimal distance between two parallel planes bounding K, see [14] for details). In [11] one can find also generalization of the result of [12] for $W^{1,p}(S^1, S^1)$ to the case of the space $W^{1,p}(S^1, C)$, where the closed curve C is the boundary of convex body in $\mathbb{R}^2$.

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2 Maps from S^2 to S^2

Let $S^2 = \left\{x \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\right\}$ denote the unit sphere in $\mathbb{R}^3$. Denote the north and south poles by $N = (0, 0, 1)$ and $S = (0, 0, -1)$. With a slight abuse of notation each $v : S^2 \to \mathbb{R}$ can be also viewed as a map from $[0, \pi] \times [0, 2\pi]$ to $\mathbb{R}$ such that $v\,(0, \theta)$ and $v\,(\pi, \theta)$ are independent of θ and also $v(\varphi, 0) = v(\varphi, 2\pi)$ for all $0 < \varphi < \pi$. Hence we can also write any $u : S^2 \to S^2$ as $u = (v_1, v_2, v_3)$ where $v_i : [0, \pi] \times [0, 2\pi] \to \mathbb{R}$. Here θ (longitude) and φ (colatitude) are geographical coordinates on S^2. Thus

$$\begin{aligned} x_1 &= \cos\theta \sin\varphi, \\ x_2 &= \sin\theta \sin\varphi, \\ x_3 &= \cos\varphi, \end{aligned}$$

where $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$.

Note that $\mathcal{H}^2(dx) = \sin\varphi \, d\varphi d\theta$ where x runs over S^2 and $\mathcal{H}^2$ denotes the Hausdorff 2-dimensional measure on S^2. We also have

$$\begin{aligned} |\nabla v_i| &= \sqrt{\left(\frac{\partial v_i}{\partial \varphi}\right)^2 + \left(\frac{1}{\sin\varphi} \frac{\partial v_i}{\partial \theta}\right)^2}, \quad i = 1, 2, 3\,, \\ |\nabla u| &= \sqrt{|\nabla v_1|^2 + |\nabla v_2|^2 + |\nabla v_3|^2}. \end{aligned}$$

Note that for $p > 2$, $W^{1,p}(S^2, S^2) \subset C^{1-2/p}(S^2, S^2)$, so that each $u \in W^{1,p}(S^2, S^2)$ has a well-defined degree. For $p = 2$ the degree is still well-defined thanks to the density of $C^1(S^2, S^2)$ in the Sobolev space $H^1(S^2, S^2)$ ([15]). This is a special case of the VMO degree that was developed by Brezis and Nirenberg in [4]. Thus, for $p \ge 2$ we may write

$$W^{1,p}(S^2, S^2) = \bigcup_{d \in \mathbb{Z}} \mathcal{E}_d = \bigcup_{d \in \mathbb{Z}} \{u \,:\, \deg u = d\}.$$

The distance between $\mathcal{E}_{d_1}$ and $\mathcal{E}_{d_2}$ is defined by

$$\delta_p^p(d_1, d_2) = \inf \left\{ \int_{S^2} |\nabla (u_1 - u_2)|^p \,:\, u_1 \in \mathcal{E}_{d_1}, u_2 \in \mathcal{E}_{d_2} \right\}. \tag{2.1}$$

In this Section we will compute $\delta_p(d_1, d_2)$ for any $p \ge 2$ and $d_1, d_2 \in \mathbb{Z}$. Interestingly, in contrast with the case of maps between S^1 to S^1 studied in [12], we will show that in dimension two (and higher) $\delta_p(d_1, d_2)$ is the same for any $d_1 \ne d_2$.

It turns out that the computation of the distance between the homotopy classes is related to the best constant in a certain Sobolev type inequality on the sphere. We recall below the relevant result which is due to Talenti ([17]).

Theorem 1. *Let $p > 2$. If $v \in W^{1,p}(S^2, \mathbb{R})$, then*

$$\max_{S^2} v - \min_{S^2} v \le C_p \|\nabla v\|_{L^p(S^2)}\,, \tag{2.2}$$

where

$$C_p = (2\pi)^{-1/p} \left[\sqrt{\pi} \frac{\Gamma\left(\frac{p-2}{2p-2}\right)}{\Gamma\left(\frac{2p-3}{2p-2}\right)} \right]^{1-1/p}.$$

Inequality (2.2) *is sharp.*

Let v be a smooth map from S^2 to $\mathbb{R}$. Without loss of generality we assume $v \geq 0$. Let $V(t)$ be the level sets of v, $V(t) := \left\{x \in S^2 : v(x) > t\right\}$. The ***distribution function*** of v is given by $\mu(t) = \mathcal{H}^2(V(t))$. Let

$$v^*(s) = \int_0^\infty \chi_{[0,\mu(t)]}(s)\, dt,$$

denote the ***decreasing rearrangement*** of v in the sense of Hardy and Littlewood - i.e., the decreasing right-continuous map from $[0, 4\pi]$ into $[0, \infty)$ which is equidistributed with v. It can be shown that v^* is locally Lipschitz continuous.

The ***spherical symmetric rearrangement*** $v^\bigstar$ of v is a function from S^2 to $[0, \infty)$ which is equidistributed with v and whose level sets are concentric spherical caps. Hence, if φ is the colatitude of x and $B(\varphi)$ is the area of the cap which is intercepted on S^2 by a circular cone having its vertex in the center of S^2 and aperture 2φ, then

$$v^\bigstar(x) = v^*(B(\varphi)) = v^*\left(4\pi \sin^2 \frac{\varphi}{2}\right). \tag{2.3}$$

An important property of $v^\bigstar$ is that it does not increase the L^p- norm of the gradient. In fact, the following Lemma is a special case of a symmetrization theorem from [16]:

Lemma 1. *If* $p \geq 1$ *then*

$$\|\nabla v\|_{L^p(S^2)} \geq \left\|\nabla v^\bigstar\right\|_{L^p(S^2)} = \left[\int_0^{4\pi} [s(4\pi - s)]^{p/2} \left[\frac{dv^*}{ds}(s)\right]^p ds \right]^{1/p}.$$

We give below a sketch of proof for Theorem 1. For convenience, we assume $v \geq 0$. By the definition of v^* ,

$$\max_{S^2} v = v^*(0), \ \min_{S^2} v = v^*(4\pi-) \ \text{ and } \ \max_{S^2} v - \min_{S^2} v = \int_0^{4\pi} -\left[\frac{dv^*}{ds}(s)\right] ds.$$

Hence, Hölder inequality gives

$$\max_{S^2} v - \min_{S^2} v \leq \left[\int_0^{4\pi} [s(4\pi - s)]^{-p/2(p-1)} ds \right]^{1-1/p} \cdot \left[\int_0^{4\pi} [s(4\pi - s)]^{p/2} \left[\frac{dv^*}{ds}(s)\right]^p ds \right]^{1/p}. \tag{2.4}$$

Inequality (2.2) follows from Lemma 1 since the first term on the R.H.S of (2.4) equals C_p.

Remark 2. An inspection shows that equality holds in (2.2) if and only if v satisfies

$$v^*(s) = c_1 \int_{s/4\pi}^{1} [t(1-t)]^{-p/2(p-1)}\, dt + c_2, \tag{2.5}$$

for some constants c_1 and c_2. Using (2.2) we deduce the following corollary.

Corollary 1. *Equality holds in* (2.2) *for a radially symmetric function* $v : S^2 \to \mathbb{R}$ *if and only if* v *satisfies*

$$v(x) = c_1 \int_{\varphi}^{\pi} (\sin t)^{-1/(p-1)}\, dt + c_2, \tag{2.6}$$

where φ *is the colatitude of* x *and* c_1, c_2 *are constants.*

For $p > 2$ we define the function $f_0 = f_0^{(p)} : [0,\pi] \to [0,1]$ by

$$f_0(\varphi) = \Big(\int_0^{\pi} (\sin t)^{-1/(p-1)}\, dt \Big)^{-1} \cdot \int_0^{\varphi} (\sin t)^{-1/(p-1)}\, dt. \tag{2.7}$$

Let $v_0(x) := f_0(\varphi)$, where $x \in S^2$ and φ is the colatitude of x. The function f_0 will be useful later in the proof of the upper bound for δ_p. We have

$$v_0(N) = f_0(0) = 0 \quad \text{and} \quad v_0(S) = f_0(\pi) = 1.$$

From Corollary 1 equality holds in (2.2) for the function v_0. Thus,

$$\|\nabla v_0\|_{L^p(S^2)} = 2\pi \int_0^{\pi} \big(f_0'(\varphi) \big)^p \sin\varphi\, d\varphi = C_p^{-1}. \tag{2.8}$$

Lemma 2. *For* $d_1, d_2 \in \mathbb{Z}$, $d_1 \neq -d_2$, $p \geq 2$, *let* u_1 *and* u_2 *be two continuous maps in* $W^{1,p}(S^2, S^2)$ *with* $\deg u_i = d_i$, $i = 1, 2$. *Then, there is a point* $\tilde{x} \in S^2$ *such that* $u_2(\tilde{x}) = u_1(\tilde{x})$.

Proof. We claim that there exist $\tilde{x} \in S^2$ and $t_0 \in (0,1)$ such that

$$t_0 u_1(\tilde{x}) + (1-t_0)(-u_2(\tilde{x})) = 0. \tag{2.9}$$

Indeed, otherwise, the map $I : [0,1] \times S^2 \to S^2$ given by

$$I(t,x) = \frac{t u_1(x) + (1-t)(-u_2(x))}{\|t u_1(x) + (1-t)(-u_2(x))\|},$$

would be a homotopy between u_1 and $-u_2$. Since $\dim(S^2)$ is even, $\deg(-u_2) = -d_2$. Hence $d_1 =$

$-d_2$, contradicting our initial assumption. From (2.9) we get $\|t_0 u_1(\tilde{x})\| = \|(1-t_0)(-u_2(\tilde{x}))\|$. Therefore, $t_0 = \frac{1}{2}$ and the result follows from (2.9). Note that the continuity assumption is needed only for $p = 2$ since for $p > 2$ every $u \in W^{1,p}\left(S^2, S^2\right)$ has a continuous representative. □

Lemma 3. *If $d_1 \neq -d_2$, $p \geq 2$, then $\delta_p(d_1+k, d_2+k) \leq \delta_p(d_1, d_2)$, $\forall k \in \mathbb{Z}$.*

Proof. Take any $u_1 \in \mathcal{E}_{d_1}$, $u_2 \in \mathcal{E}_{d_2}$. We may assume without loss of generality that u_1, u_2 are smooth maps. Since $d_1 \neq -d_2$, by Lemma 2 there is a point $\tilde{x} \in S^2$ such that $u_1(\tilde{x}) = u_2(\tilde{x})$. We may choose the coordinates axes in the domain and in the range of the maps u_i such that $\tilde{x} = S$ and $u_1(\tilde{x}) = u_2(\tilde{x}) = S$. Thus, $u_1(S) = u_2(S) = S$.

For a small $\varepsilon > 0$ define the maps $\tilde{u}_i = \tilde{u}_i^{(\varepsilon)}$, $i = 1, 2$, on S^2 by

$$\tilde{u}_i(\varphi,\theta) = \begin{cases} u_i\left(\frac{\pi}{\pi-\varepsilon}\varphi, \theta\right) & \varphi \in [0, \pi-\varepsilon], \\ \left(\sin\left[\frac{\pi}{\varepsilon}(\pi-\varphi)\right]\cos(-k\theta), \sin\left[\frac{\pi}{\varepsilon}(\pi-\varphi)\right]\sin(-k\theta), \cos\left[\frac{\pi}{\varepsilon}(\pi-\varphi)\right]\right) & \varphi \in (\pi-\varepsilon, \pi]. \end{cases} \tag{2.10}$$

The maps $\tilde{u}_i$ belong to $\mathcal{E}_{d_i+k}$, $i = 1, 2$, and satisfy

$$\begin{aligned} \int_{S^2} |\nabla(\tilde{u}_2 - \tilde{u}_1)|^p &= \int_0^{2\pi} d\theta \int_0^{\pi-\varepsilon} \left|\nabla\left(u_2\left(\frac{\pi}{\pi-\varepsilon}\varphi,\theta\right) - u_1\left(\frac{\pi}{\pi-\varepsilon}\varphi,\theta\right)\right)\right|^p \sin\varphi \, d\varphi \\ &= \frac{\pi-\varepsilon}{\pi} \int_0^{2\pi} d\theta \int_0^{\pi} |\nabla(u_2(\tilde{\varphi},\theta) - u_1(\tilde{\varphi},\theta))|^p \sin\left(\frac{\pi-\varepsilon}{\pi}\tilde{\varphi}\right) d\tilde{\varphi}. \end{aligned}$$

Since the maps u_1, u_2 are smooth and $\sin\left(\frac{\pi-\varepsilon}{\pi}\tilde{\varphi}\right) \leq \sin(\tilde{\varphi}) + \varepsilon$ for $0 \leq \tilde{\varphi} \leq \pi$, we get

$$\int_{S^2} |\nabla(\tilde{u}_2 - \tilde{u}_1)|^p \leq 2\pi^2 M^p \varepsilon + \int_{S^2} |\nabla(u_2 - u_1)|^p,$$

where $M := \|\nabla(u_2 - u_1)\|_{L^\infty(S^2)} < \infty$. Hence

$$\lim_{\varepsilon \to 0} \int_{S^2} |\nabla(\tilde{u}_2 - \tilde{u}_1)|^p \leq \int_{S^2} |\nabla(u_2 - u_1)|^p.$$

The result follows since u_i can be chosen arbitrarily in $\mathcal{E}_{d_i}$. □

Now we treat the case $p = 2$.

Theorem 2. *For every $d_1, d_2 \in \mathbb{Z}$ we have $\delta_2(d_1, d_2) = 0$.*

Proof. We begin with a brief description of our strategy. It would be enough to deal with the case where one of the degrees is zero, and then use Lemma 3 to deduce the general case. We shall construct two maps, both with $m-$axial symmetry, one of degree zero and the second one of degree $m = d_2 - d_1 \neq 0$. On a small sphere of order ε on S^2, centered at the north pole, the two maps are identical, each covering the upper hemisphere. On the remaining (much larger) part of S^2 one of the maps "goes back" from the equator to the north pole, so its degree is zero. On the other hand, the values taken by the second map on that part of S^2 are just the reflection w.r.t. the $xy-$plane of the values taken by the first map. The degree of the second map equals therefore to m and the difference between the two maps has a nonzero component only in the z-direction. Using the fact that a point has zero $2-$capacity in dimension two, we can arrange to have arbitrarily small energy contribution from that component. The detailed construction is given below.

For any small $\varepsilon > 0$ define the maps $\Phi_i^{(\varepsilon)} : [0,\pi] \to [0,\pi]$ by

$$\Phi_i(\varphi) = \Phi_i^{(\varepsilon)}(\varphi) = \begin{cases} \frac{\pi}{2\varepsilon}\varphi & \varphi \in [0,\varepsilon], \\ \frac{\pi}{2}\left(1 - (-1)^i \frac{\log\varphi - \log\varepsilon}{\log\pi - \log\varepsilon}\right) & \varphi \in (\varepsilon,\pi], \end{cases} \tag{2.11}$$

where $i = 1, 2$. A direct computation yields,

$$\lim_{\varepsilon\to 0} \int_{\varepsilon}^{\pi} \left(\Phi_i'(\varphi)\right)^2 \sin\varphi \, d\varphi = 0. \tag{2.12}$$

We define the maps $u_i = u_i^{(\varepsilon)}$ from S^2 to S^2 by

$$u_i(\varphi,\theta) = (\sin\Phi_i(\varphi)\sin(m\theta), \sin\Phi_i(\varphi)\cos(m\theta), \cos\Phi_i(\varphi)), \ i = 1, 2 \ ,$$

where $0 \le \varphi \le \pi, 0 \le \theta \le 2\pi$ and $m = d_2 - d_1 \ne 0$. Since $|\nabla u_i| < \frac{c}{\varepsilon}$, $i = 1, 2$, (c is a positive constant, independent of ε) the maps belong to $W^{1,\infty}(S^2, S^2)$ and satisfy $u_1 \in \mathcal{E}_{d_2-d_1}$ and $u_2 \in \mathcal{E}_0$. We have

$$u_2 - u_1 = \begin{cases} (0,0,0) & \varphi \in [0,\varepsilon], \\ (0,0,2\cos\Phi_2(\varphi)) & \varphi \in (\varepsilon,\pi]. \end{cases}$$

Thus,

$$\int_{S^2} |\nabla(u_2 - u_1)|^2 \le 2^3\pi \int_{\varepsilon}^{\pi} \left(\Phi_2'(\varphi)\right)^2 \sin\varphi \, d\varphi. \tag{2.13}$$

From (2.12), (2.13) and Lemma 3 (applied to 0 and $d_2 - d_1$) we get

$$\delta_2(d_1, d_2) \le \delta_2(d_2 - d_1, 0) = 0.$$

□

Remark 3. The above theorem is analogous to a result from [4](see Lemma 6 and Remark 6 there) which states that the distance between homotopy classes in $H^{1/2}(S^1, S^1)$ is always zero.

Next we turn to the case $p > 2$. The lower-bound is given by the following lemma.

Lemma 4. *If $d_1 \neq d_2$ then for $p > 2$ we have:*

$$\delta_p(d_1, d_2) \geq \frac{2}{C_p}.$$

Proof. Take any $u_1 \in \mathcal{E}_{d_1}$, $u_2 \in \mathcal{E}_{d_2}$. Since $d_1 \neq d_2$ Lemma 2 applied to u_1 and $-u_2$ implies that there is a point $\tilde{x}_1 \in S^2$ such that $u_2(\tilde{x}_1) = -u_1(\tilde{x}_1)$. We may assume W.l.o.g. that $d_2 \neq 0$. We choose the coordinates axes in the range so that $u_2(\tilde{x}_1) = N$. Since $d_2 \neq 0$, it follows that there is $\tilde{x}_2 \in S^2$ such that $u_2(\tilde{x}_2) = S$. Let $v_3 : S^2 \to \mathbb{R}$ be the third component of $u_2 - u_1$. The function v_3 belongs to $W^{1,p}(S^2, \mathbb{R})$ and satisfies $v_3(\tilde{x}_1) = 2,\ v_3(\tilde{x}_2) \leq 0$. From Theorem 1 we get

$$\|\nabla(u_2 - u_1)\|_{L^p(S^2)} \geq \|\nabla v_3\|_{L^p(S^2)} \geq \frac{\max_{S^2} v_3 - \min_{S^2} v_3}{C_p} \geq \frac{2}{C_p}. \tag{2.14}$$

□

Next we prove the main result of this Section.

Theorem 3. *If $d_1 \neq d_2$ then for $p > 2$ we have:*

$$\delta_p(d_1, d_2) = \frac{2}{C_p}. \tag{2.15}$$

Proof. Thanks to the lower bound of Lemma 4, it is enough to prove that the following upper bound holds,

$$\delta_p(d_1, d_2) \leq \frac{2}{C_p}, \tag{2.16}$$

for all $d_1 \neq d_2$. The construction of pairs of maps that realize (2.16) in the limit shares some similarities with the construction used in the proof of Theorem 2. Indeed, once again it is enough to consider the case where one of the degrees is zero and both maps are taken to be equal on a small sphere, whose image by each of the maps is the upper hemisphere. On the remaining part of S^2 one map (the one of zero degree) covers again the upper hemisphere while the second map is just the reflection w.r.t. the xy-plane of the first one. This time however we arrange so that the difference between the two maps (which is in the direction of the z-axis) is equal approximately to f_0 (see (2.7)-(2.8)) which is the profile of the minimizer in Theorem 1.

For any small $\varepsilon > 0$ consider the following approximation $F = F^{(\varepsilon)} : [\varepsilon, \pi] \to [0, 1]$ of the function f_0 (defined in (2.7)):

$$F^{(\varepsilon)}(\varphi) = J_\epsilon(f_0(\varphi)), \tag{2.17}$$

where the map $J_\varepsilon : [f_0(\varepsilon), 1] \to [0,1]$ is a C^{m+1}-map satisfying the following properties:

$$\begin{array}{cc} J_\varepsilon(1) = 1, & J'_\varepsilon(1) = \cdots = J_\varepsilon^{(m)}(1) = 0, \\ J_\varepsilon(f_0(\varepsilon)) = 0, & J'_\varepsilon(f_0(\varepsilon)) = \cdots = J_\varepsilon^{(m)}(f_0(\varepsilon)) = 0, \\ J_\varepsilon(s) = s, & 2f_0(\varepsilon) \le s \le 1-\varepsilon, \\ 0 \le J'_\varepsilon(s) < c_0, & f_0(\varepsilon) \le s \le 1, \\ \left|J_\varepsilon^{(m+1)}(s)\right| < \frac{c_1}{\varepsilon^m}, & f_0(\varepsilon) \le s \le 1, \\ \frac{c_2}{\varepsilon^m} < \left|J_\varepsilon^{(m+1)}(s)\right| < \frac{c_1}{\varepsilon^m}, & 1-\frac{\varepsilon}{2} < s \le 1, \end{array} \tag{2.18}$$

for some constants c_0, c_1, c_2 (independent of ε) and m an integer that satisfies $m \ge 1 + \frac{2}{p-2}$.
In fact, we need to construct a function J_ϵ on the two intervals $[f_0(\varepsilon), 2f_0(\varepsilon)]$ and $[1-\varepsilon, 1]$ "connecting" the values $J_\varepsilon(f_0(\varepsilon)) = 0$ and $J_\varepsilon(1-\varepsilon) = 1-\varepsilon$ to the values $J_\varepsilon(s) = s$ on the interval $[2f_0(\varepsilon), 1-\varepsilon]$, that satisfies the estimates in (2.18). This requires a change of order $f_0(\varepsilon)$ for J_ε on the interval $[f_0(\varepsilon), 2f_0(\varepsilon)]$ and of order ε on the interval $[1-\varepsilon, 1]$. An appropriate J_ε will then have a derivative of order $O(1)$. But now the change of order 1 between $J'_\varepsilon(f_0(\varepsilon))$ and $J'_\varepsilon(2f_0(\varepsilon))$ and between $J'_\varepsilon(1-\varepsilon) = 1$ and $J'_\varepsilon(1) = 0$ requires J''_ε of order $\max(\frac{1}{f_0(\varepsilon)}, \frac{1}{\varepsilon}) = \frac{1}{\varepsilon}$ (since $f_0(\varepsilon) \sim \varepsilon^{\frac{p-2}{p-1}} \gg \varepsilon$). Similarly, for higher order derivatives we will get $|J_\varepsilon^{(k)}(s)| \le \frac{c}{\varepsilon^{k-1}}$. Since what we are requiring is just interpolation between certain given values of the function and some of its derivatives at two pairs of points, it is clear that we can even take a polynomial for J_ε.

Using Taylor formula around $s = f_0(\varepsilon)$ and $s = 1$ in conjunction with (2.18) yields for $s \in [f_0(\varepsilon), 1]$,

$$|J'_\varepsilon(s)| \le \frac{c}{\varepsilon^m}(s - f_0(\varepsilon))^m, \tag{2.19}$$

$$|J'_\varepsilon(s)| \le \frac{c}{\varepsilon^m}(1-s)^m, \tag{2.20}$$

where c is a constant independent of ε. For $s \in \left[1-\frac{\varepsilon}{2}, 1\right]$ we use again Taylor formula around 1 to obtain,

$$J_\varepsilon(s) = 1 + \frac{J_\varepsilon^{(m+1)}(\theta)}{(m+1)!}(s-1)^{m+1},\ \theta \in (s,1),$$

implying

$$\left|\frac{J'_\varepsilon(s)}{(1-J_\varepsilon(s))^{1/2}}\right| \le \frac{c}{\varepsilon^{m/2}}(1-s)^{(m-1)/2},\ s \in [1-\frac{\varepsilon}{2}, 1]. \tag{2.21}$$

Define the functions $\widetilde{\Phi}_i = \widetilde{\Phi}_i^{(\varepsilon)} : [0,\pi] \to [0,\pi]$, $i = 1,2$, by

$$\widetilde{\Phi}_1(\varphi) = \begin{cases} \frac{\pi}{2\varepsilon}\varphi & [0,\varepsilon], \\ \pi - \arccos F(\varphi) & (\varepsilon,\pi], \end{cases},\quad \widetilde{\Phi}_2(\varphi) = \begin{cases} \frac{\pi}{2\varepsilon}\varphi & [0,\varepsilon], \\ \arccos F(\varphi) & (\varepsilon,\pi], \end{cases}. \tag{2.22}$$

Then, define the maps $u_i = u_i^{(\varepsilon)} : S^2 \to S^2$, for $i = 1, 2$, by

$$u_i(\varphi, \theta) = \left(\sin \widetilde{\Phi}_i(\varphi) \cos(k\theta), \sin \widetilde{\Phi}_i(\varphi) \sin(k\theta), \cos \widetilde{\Phi}_i(\varphi)\right), \tag{2.23}$$

where $0 \le \varphi \le \pi, 0 \le \theta \le 2\pi$ and $k = d_2 - d_1 \ne 0$.

Next we prove that $u_1 \in \mathcal{E}_{d_2-d_1}$ and $u_2 \in \mathcal{E}_0$. Computation of the degrees of u_1 and u_2 gives

$$\begin{aligned} \deg u_i &= \frac{1}{4\pi} \int_{S^2} u_i \cdot u_{i\varphi} \wedge u_{i\theta} = \frac{2\pi k}{4\pi} \int_0^\pi \widetilde{\Phi}_i'(\varphi) \sin \widetilde{\Phi}_i(\varphi)\, d\varphi \\ &= \frac{k}{2} \left[\cos \widetilde{\Phi}_i(0) - \cos \widetilde{\Phi}_i(\pi)\right] = \begin{cases} k & i = 1, \\ 0 & i = 2. \end{cases} \end{aligned}$$

We will prove that $u_i \in W^{1,\infty}(S^2, S^2) \subset W^{1,p}(S^2, S^2)$ by showing that the derivatives of F and $\sqrt{1-F^2}$ are bounded. Obviously it is enough to consider the intervals $[\varepsilon, \varepsilon + \tilde{\varepsilon}]$ and $[\pi - \tilde{\varepsilon}, \pi]$

for some $\tilde{\varepsilon} > 0$. Let $\tilde{\varepsilon}$ be such that $f_0(\pi - \tilde{\varepsilon}) > 1 - \frac{\varepsilon}{2}$. On the interval $[\varepsilon, \varepsilon + \tilde{\varepsilon}]$ we have

$$\begin{aligned} f_0'(\varphi) &= (\sin \varphi)^{-1/(p-1)} \le c\varphi^{-1/(p-1)}, \\ f_0(\varphi) &= c \int_0^\varphi (\sin t)^{-1/(p-1)}\, dt \le c\varphi^{(p-2)/(p-1)}. \end{aligned}$$

From (2.19) we get

$$\begin{aligned} F'(\varphi) &= J_\epsilon'(f_0(\varphi)) f_0'(\varphi) \le \frac{c}{\varepsilon^m} (f_0(\varphi) - f_0(\varepsilon))^m f_0'(\varphi) \\ &\le \frac{c}{\varepsilon^m} (f_0(\varphi))^m f_0'(\varphi) \le \frac{c}{\varepsilon^m} \varphi^{m((p-2)/(p-1))-1/(p-1)} \le \frac{c}{\varepsilon^m}. \end{aligned}$$

In the last inequality we used that

$$m\frac{p-2}{p-1} - \frac{1}{p-1} = \frac{p-2}{p-1}\left[m - \frac{1}{p-2}\right] > 0.$$

For the function $\sqrt{1-F^2}$ we simply have

$$\left|\left(\sqrt{1-F^2}\right)'(\varphi)\right| = \frac{FF'}{\sqrt{1-F^2}} \le cF' \le \frac{c}{\varepsilon^m}.$$

On the interval $[\pi - \tilde{\varepsilon}, \pi]$ the functions f_0 and f_0' satisfy

$$\begin{aligned} f_0'(\varphi) &= (\sin(\pi - \varphi))^{-1/(p-1)} \le c(\pi - \varphi)^{-1/(p-1)}, \\ 1 - f_0(\varphi) &= c \int_{\pi-\varphi}^{\pi} (\sin t)^{-1/(p-1)}\, dt \le c(\pi - \varphi)^{(p-2)/(p-1)}. \end{aligned}$$

Hence, using (2.20)

$$\begin{aligned} F'(\varphi) &= J_\epsilon'(f_0(\varphi)) f_0'(\varphi) \le \frac{c}{\varepsilon^m}(1-f_0(\varphi))^m f_0'(\varphi) \\ &\le \frac{c}{\varepsilon^m}(\pi-\varphi)^{m[(p-2)/(p-1)]-1/(p-1)} \le \frac{c}{\varepsilon^m}. \end{aligned}$$

Since $f_0(\pi-\tilde{\varepsilon}) > 1-\frac{\varepsilon}{2}$, from (2.21) we get

$$\begin{aligned} \left(\sqrt{1-F^2}\right)'(\varphi) &\le c\left|\frac{J_\varepsilon' f_0'}{(1-J_\varepsilon)^{1/2}}\right| \le \frac{c}{\varepsilon^{m/2}}(1-f_0(\varphi))^{(m-1)/2} f_0'(\varphi) \\ &\le \frac{c}{\varepsilon^{m/2}}(\pi-\varphi)^{[(m-1)/2]\cdot[(p-2)/(p-1)]-1/(p-1)} \le \frac{c}{\varepsilon^{m/2}}. \end{aligned}$$

In the last inequality we used that

$$\frac{m-1}{2}\cdot\frac{p-2}{p-1}-\frac{1}{p-1} = \frac{p-2}{2(p-1)}\left[m-1-\frac{2}{p-2}\right] \ge 0.$$

Our next step will be to prove that

$$\lim_{\varepsilon\to 0}\int_{S^2}|\nabla(u_2-u_1)|^p \le \left(\frac{2}{C_p}\right)^p. \tag{2.24}$$

Note that

$$u_2-u_1 = \begin{cases}(0,0,0) & \varphi\in[0,\varepsilon], \\ (0,0,2F(\varphi)) & \varphi\in(\varepsilon,\pi].\end{cases}$$

Therefore,

$$\int_{S^2}|\nabla(u_2-u_1)|^p \le 2^p\cdot 2\pi\int_\varepsilon^\pi \left(F'(\varphi)\right)^p \sin\varphi\, d\varphi. \tag{2.25}$$

Set $\tilde{\varepsilon} = 2f_0(\varepsilon)$. Note that

$$\begin{gathered} \tilde{\varepsilon} \le c\varepsilon^{1-1/(p-1)}, \\ f_0([\tilde{\varepsilon},\pi-\tilde{\varepsilon}]) \subset [2f_0(\varepsilon), 1-\varepsilon]. \end{gathered} \tag{2.26}$$

Since

$$\begin{aligned} \int_\varepsilon^{\varepsilon+\tilde{\varepsilon}} \left(F'(\varphi)\right)^p \sin\varphi\, d\varphi &\le c\int_\varepsilon^{\varepsilon+\tilde{\varepsilon}} \left(f_0'(\varphi)\right)^p \sin\varphi\, d\varphi \le c\int_\varepsilon^{\varepsilon+\tilde{\varepsilon}} \varphi^{-p/(p-1)}\varphi\, d\varphi \\ &\le c\tilde{\varepsilon}^{1-1/(p-1)} \le c\varepsilon^{[1-1/(p-1)]^2}, \end{aligned}$$

we have

$$\lim_{\varepsilon\to 0}\int\limits_{\varepsilon}^{\varepsilon+\tilde{\varepsilon}}\left(F'\left(\varphi\right)\right)^{p}\sin\varphi\,d\varphi=0. \tag{2.27}$$

Moreover,

$$\begin{aligned}\int\limits_{\pi-\tilde{\varepsilon}}^{\pi}\left(F'\left(\varphi\right)\right)^{p}\sin\varphi\,d\varphi &\leq c\int\limits_{\pi-\tilde{\varepsilon}}^{\pi}\left(f_0'\left(\varphi\right)\right)^{p}\sin\varphi\,d\varphi\leq c\int\limits_{\pi-\tilde{\varepsilon}}^{\pi}\left(\pi-\varphi\right)^{-p/(p-1)}\left(\pi-\varphi\right)\,d\varphi\\ &= c\tilde{\varepsilon}^{1-1/(p-1)}\leq c\varepsilon^{[1-1/(p-1)]^2},\end{aligned}$$

implying that

$$\lim_{\varepsilon\to 0}\int\limits_{\pi-\tilde{\varepsilon}}^{\pi}\left(F'\left(\varphi\right)\right)^{p}\sin\varphi\,d\varphi=0. \tag{2.28}$$

Finally, on $[\varepsilon+\tilde{\varepsilon},\pi-\tilde{\varepsilon}]$, we have by (2.26) and (2.18) that $F(\varphi)=f_0(\varphi)$. From (2.8) we obtain

$$\int\limits_{\varepsilon+\tilde{\varepsilon}}^{\pi-\tilde{\varepsilon}}\left(F'\left(\varphi\right)\right)^{p}\sin\varphi\,d\varphi\leq\int\limits_{0}^{\pi}\left(f_0'\left(\varphi\right)\right)^{p}\sin\varphi\,d\varphi=\frac{1}{2\pi}(C_p)^{-p}. \tag{2.29}$$

From (2.25) and (2.27)-(2.29) we deduce (2.24). Hence $\delta_p^p\left(d_2-d_1,0\right)\leq(\frac{2}{C_p})^p$. Lemma 3 (applied to 0 and d_2-d_1) yields (2.16). □

Next we turn to the question of attainability of $\delta_p(d_1,d_2)$.

Theorem 4. *For $p>2$, $\delta_p(d_1,d_2)$ is not attained for all $d_1\neq d_2$.*

Proof. Assume by negation that there exist maps $u_1\in\mathcal{E}_{d_1}$ and $u_2\in\mathcal{E}_{d_2}$ such that

$$\left\|\nabla\left(u_2-u_1\right)\right\|_{L^p(S^2)}=\delta_p\left(d_1,d_2\right)\,.$$

We may assume without loss of generality that $d_2\neq 0$. As in the proof of Lemma 4 we can find a point $\tilde{x}_1$ such that $u_2\left(\tilde{x}_1\right)=-u_1\left(\tilde{x}_1\right)$, and by changing the axes we may assume that $u_2\left(\tilde{x}_1\right)=N$. Denote $u_2-u_1=(v_1,v_2,v_3)$. Since $d_2\neq 0$, there exists $\tilde{x}_2\in S^2$ such that $u_2(\tilde{x}_2)=S$, implying that $v_3\left(\tilde{x}_2\right)\leq 0$. Since δ_p is attained, equalities hold in all the inequalities in (2.14). Thus, $\left\|\nabla v_1\right\|_{L^p(S^2)}=\left\|\nabla v_2\right\|_{L^p(S^2)}=0$, $\left\|\nabla v_3\right\|_{L^p(S^2)}=\frac{2}{C_p}$ and $\min_{S^2}v_3=0$, $\max_{S^2}v_3=2$. Since $v_3\left(\tilde{x}_1\right)=2$ we deduce that $v_1\left(\tilde{x}_1\right)=v_2\left(\tilde{x}_1\right)=0$. Therefore, $v_1\equiv 0$, $v_2\equiv 0$ and

$$\left|u_1^z\left(x\right)\right|=\left|u_2^z\left(x\right)\right|\quad\forall x\in S^2, \tag{2.30}$$

where u_1^z and u_2^z denote the z-components of u_1 and u_2, respectively. Since $d_2\neq 0$ and $v_3\geq 0$, we deduce from (2.30) that the set $\{v_3=0\}$ has positive measure (we must have $v_3(x)=0$ at points x where $u_2^{x_3}\left(x\right)\leq 0$). Hence, the distribution function of v_3 satisfies $\mu\left(0\right)<4\pi$, and the decreasing rearrangement of v_3 satisfies $v_3^*(s) = 0$ on the interval $(\mu(0), 4\pi)$. This contradicts Remark 2 for the function v_3. $\square$

3 Generalization to dimension n

In this short section we shall show how to generalize the results of Section 2 to maps from S^n to S^n, for every $n \geq 3$. Set

$$S^n = \left\{x \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1\right\}.$$

For $p \geq n$ each $u \in W^{1,p}(S^n, S^n)$ has a well defined degree and we may write again

$$W^{1,p}(S^n, S^n) = \bigcup_{d\in\mathbb{Z}} \mathcal{E}_d = \bigcup_{d\in\mathbb{Z}} \{u : \deg u = d\}.$$

Indeed, for $p > n$ the maps in $W^{1,p}(S^n, S^n)$ are continuous, while in the limiting case $p = n$ we refer to the VMO degree (see [4]). The distance between $\mathcal{E}_{d_1}$ and $\mathcal{E}_{d_2}$ is defined naturally by

$$\delta_p^p(d_1, d_2) = \inf\left\{\int_{S^n} |\nabla(u_1 - u_2)|^p : u_1 \in \mathcal{E}_{d_1}, u_2 \in \mathcal{E}_{d_2}\right\}. \tag{3.1}$$

Denote by $\omega_n = \frac{2\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}$ the n-dimensional area of the unit n-sphere. The generalization of Theorem 1 for higher dimensional spheres was given by Cianchi in [5].

Theorem 5. *Let $p > n$. If $v \in W^{1,p}(S^n, \mathbb{R})$, then*

$$\max_{S^n} v - \min_{S^n} v \leq C_p^{(n)} \|\nabla v\|_{L^p(S^n)}, \tag{3.2}$$

where

$$C_p^{(n)} = (\omega_{n-1})^{-1/p}\left[\sqrt{\pi}\frac{\Gamma\left(\frac{p-n}{2p-2}\right)}{\Gamma\left(\frac{2p-n-1}{2p-2}\right)}\right]^{1-1/p}.$$

Inequality (3.2) *is sharp.*

Note that in the proof of Lemma 2 we used the fact that the dimension of S^2 is even. Thus, in the generalizations of Lemma 2 and Lemma 3 to arbitrary $n \geq 2$ we should take into account the parity of n. The proof of the next Lemma requires an obvious modification of the one of Lemma 2.

Lemma 5. *Let $d_1, d_2 \in \mathbb{Z}$, $p \geq n$ and u_1, u_2 two continuous maps in $W^{1,p}(S^n, S^n)$ such that $\deg u_i = d_i$, $i = 1, 2$. Then, there is a point $\tilde{x} \in S^n$ such that $u_2(\tilde{x}) = u_1(\tilde{x})$ in the following cases:*
(i) If n is even and $d_1 \neq -d_2$.
(ii) If n is odd and $d_1 \neq d_2$.

Next we state a generalization of Lemma 3.

Lemma 6. *For $p \geq n$ we have:*
(i) If n is even and $d_1 \neq -d_2$ then $\delta_p(d+k, d_2+k) \leq \delta_p(d_1, d_2)$, $\forall k \in \mathbb{Z}$.
(ii) If n is odd then $\delta_p(d_1+k, d_2+k) = \delta_p(d_1, d_2)$, $\forall k \in \mathbb{Z}$.

Sketch of Proof. We start with some notation. On the n-dimensional sphere

$$S^n = \left\{ x \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1 \right\}$$

define the spherical coordinates $\varphi_i \in [0, \pi]$, $i = 1, 2, \ldots, n-1$ and $\theta \in [0, 2\pi]$, where ϕ_i denotes the angle between x and e_{i+2}. Thus,

$$\begin{aligned}
x_1 &= \cos\theta \sin\varphi_1 \cdots \sin\varphi_{n-1}, \\
x_2 &= \sin\theta \sin\varphi_1 \cdots \sin\varphi_{n-1}, \\
x_3 &= \cos\varphi_1 \sin\varphi_2 \cdots \sin\varphi_{n-1}, \\
&\vdots \\
x_n &= \cos\varphi_{n-2} \sin\varphi_{n-1}, \\
x_{n+1} &= \cos\varphi_{n-1}.
\end{aligned}$$

(i) Take any $u_1 \in \mathcal{E}_{d_1}$, $u_2 \in \mathcal{E}_{d_2}$ that can be both assumed smooth, without loss of generality. Since $d_1 \neq -d_2$, Lemma 5(i) implies that there is a point $\tilde{x} \in S^n$ such that $u_1(\tilde{x}) = u_2(\tilde{x})$. We may assume without loss of generality that $u_1(S) = u_2(S) = S$. For any small $\varepsilon > 0$ define the maps $\tilde{u}_i = \tilde{u}_i^{(\varepsilon)}$, $i = 1, 2$, on S^n by generalizing the definition in (2.10) as follows:

$$\tilde{u}_i(\varphi_1, \varphi_2, \ldots, \varphi_{n-2}, \varphi_{n-1}, \theta) = \begin{cases} u_i\left(\varphi_1, \varphi_2, \ldots, \varphi_{n-2}, \frac{\pi}{\pi-\varepsilon}\varphi_{n-1}, \theta\right) & \varphi_{n-1} \in [0, \pi-\varepsilon], \\ (v_1, v_2, \ldots, v_{n+1}) & \varphi_{n-1} \in (\pi-\varepsilon, \pi], \end{cases} \tag{3.3}$$

where $v_j = v_j(\varphi_1, \varphi_2, \ldots, \varphi_{n-1}, \theta)$, $j = 1, 2, \ldots, n+1$, are defined by

$$\begin{aligned}
v_1 &= \cos(-k\theta) \sin\varphi_1 \cdots \sin\varphi_{n-2} \sin\left[\frac{\pi}{\varepsilon}(\pi - \varphi_{n-1})\right], \\
v_2 &= \sin(-k\theta) \sin\varphi_1 \cdots \sin\varphi_{n-2} \sin\left[\frac{\pi}{\varepsilon}(\pi - \varphi_{n-1})\right], \\
v_3 &= \cos\varphi_1 \sin\varphi_2 \cdots \sin\varphi_{n-2} \sin\left[\frac{\pi}{\varepsilon}(\pi - \varphi_{n-1})\right], \\
&\vdots \\
v_n &= \cos\varphi_{n-2} \sin\left[\frac{\pi}{\varepsilon}(\pi - \varphi_{n-1})\right], \\
v_{n+1} &= \cos\left[\frac{\pi}{\varepsilon}(\pi - \varphi_{n-1})\right].
\end{aligned}$$

Hence $\tilde{u}_i \in \mathcal{E}_{d_i+k},\ i=1,2$, and a direct computation, as in the proof of Lemma 3, yields

$$\lim_{\varepsilon\to 0}\int_{S^n} |\nabla(\tilde{u}_2-\tilde{u}_1)|^p \le \int_{S^n} |\nabla(u_2-u_1)|^p .$$

The result follows since u_i can be chosen arbitrarily in $\mathcal{E}_{d_i}$.

(ii) Clearly, we may assume that $d_1 \neq d_2$. By Lemma 5 (ii) there is a point $\tilde{x} \in S^n$ such that $u_1(\tilde{x}) = u_2(\tilde{x})$. As in the proof of (i) we get $\delta_p(d_1+k, d_2+k) \le \delta_p(d_1,d_2)$, $\forall k \in \mathbb{Z}$. Since $d_1+k \neq d_2+k$, we can apply again the proof of (i) to obtain $\delta_p(d_1+k-k, d_2+k-k) \le \delta_p(d_1+k, d_2+k)$. □

Our main result for general $n \ge 2$, generalizing Theorem 2 , Theorem 3 and Theorem 4 is:

Theorem 6. *The distance between homotopy classes in the space $W^{1,p}(S^n,S^n)$ $(p \ge n)$, satisfies:*
(i) $\delta_n(d_1,d_2) = 0$ for every $d_1,d_2 \in \mathbb{Z}$.
(ii) If $p > n$ then $\delta_p(d_1,d_2) = \frac{2}{C_p^{(n)}}$ for every $d_1 \neq d_2$ and $\delta_p(d_1,d_2)$ is not attained.

Sketch of Proof. (i) It is enough to show that $\delta_n(0,m) = 0$ for any m, and then apply Lemma 6 to get the result for any pair d_1,d_2. As in the proof of Lemma 6 above, we slightly modify the construction in the proof of Theorem 2 by letting only the θ and φ_{n-1} coordinates to be "active". For any small $\varepsilon > 0$ define the functions $\Phi_i = \Phi_i^{(\varepsilon)} : [0,\pi] \to [0,\pi]$, $i=1,2$, by (2.11). It is easy to verify that

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi} \left(\Phi_i'(\varphi)\right)^n \sin^{n-1}\varphi\, d\varphi = 0 . \tag{3.4}$$

Using these functions define the maps $u_i = u_i^{(\varepsilon)},\ i=1,2$, from S^n to S^n by $u_i = (v_1^{(i)}, v_2^{(i)}, \ldots, v_{n+1}^{(i)})$ where $v_j^{(i)} = v_j^{(i)}(\varphi_1,\varphi_2,\ldots,\varphi_{n-1},\theta)$ are defined by

$$\begin{aligned}
v_1^{(i)} &= \cos(m\theta)\sin\varphi_1\cdots\sin\varphi_{n-2}\sin\Phi_i(\varphi_{n-1}),\\
v_2^{(i)} &= \sin(m\theta)\sin\varphi_1\cdots\sin\varphi_{n-2}\sin\Phi_i(\varphi_{n-1}),\\
v_3^{(i)} &= \cos\varphi_1\sin\varphi_2\cdots\sin\varphi_{n-2}\sin\Phi_i(\varphi_{n-1}),\\
&\vdots\\
v_n^{(i)} &= \cos\varphi_{n-2}\sin\Phi_i(\varphi_{n-1}),\\
v_{n+1}^{(i)} &= \cos\Phi_i(\varphi_{n-1}).
\end{aligned}$$

Using (3.4) we can easy verify that $\lim_{\varepsilon\to 0}\int_{S^n}|\nabla(u_1^{(\varepsilon)} - u_2^{(\varepsilon)})|^n = 0$, and the result of (i) follows.

(ii) For the proof of the lower bound take any $u_1 \in \mathcal{E}_{d_1}$, $u_2 \in \mathcal{E}_{d_2}$. Since $d_1 \neq d_2$ Lemma 5 applied to u_1 and $-u_2$ implies that there is a point $\tilde{x}_1 \in S^n$ such that $u_2(\tilde{x}_1) = -u_1(\tilde{x}_1)$ (alternatively, we can see directly that such a point exists because otherwise the map $I : [0,1]\times S^n \to S^n$ given by

$$I(t,x) = \frac{tu_1(x) + (1-t)u_2(x)}{\|tu_1(x) + (1-t)u_2(x)\|}$$

would be a homotopy between u_1 and u_2). The rest of the proof is the same as in the proof of Lemma 4. The proof that $\delta_p(d_1, d_2)$ is not attained uses the same argument as in the proof of Theorem 4 □

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