Contemporary Mathematics

An infinite-horizon variational problem on an infinite strip

Itai Shafrir and Ilya Yudovich

Abstract

This article is devoted to the study of a variational problem on an infinite strip $\Omega = (0,\infty)\times(0,L)$. It generalizes previous works which dealt with the one dimensional case, notably the one by Leizarowitz and Mizel. More precisely, given $g \in H^{\frac{3}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$ such that

$$g = 0 \quad \text{on} \quad (0,\infty)\times\{0\}\cup(0,\infty)\times\{L\}\,, \tag{0.1}$$

we seek a "minimal solution" for the functional

$$\begin{gathered} I[u] = \int_\Omega f\left(u, Du, D^2u\right)dx, \\ \text{for } u \in A_g := \left\{v \in H^2_{loc}(\Omega) : \left(v|_{\partial\Omega}, \tfrac{\partial v}{\partial \nu}|_{\partial\Omega}\right) = g\right\}, \end{gathered} \tag{0.2}$$

where $\frac{\partial v}{\partial \nu}|_{\partial\Omega}$ is the outward normal derivative on $\partial\Omega$, for a free energy integrand f satisfying some natural assumptions.

Since the infimum of $I[\cdot]$ on A_g is typically either $+\infty$ or $-\infty$, we consider the expression

$$J_{\Omega_k}[u] = \frac{1}{|\Omega_k|}\int_{\Omega_k} f\left(u, Du, D^2u\right)dx,$$

where $\Omega_k = (0,k)\times(0,L)$, for any $k>0$, and study the limit as k tends to infinity.

As $k\to\infty$, the limit of $J_{\Omega_k}[u]$ represents the average energy of u on Ω, and whenever this limit has meaning we define

$$J[u] = \liminf_{k\to\infty} J_{\Omega_k}[u]\,. \tag{0.3}$$

Our main result establishes, for any g satisfying (0.1), the existence of a *minimal solution* u for (0.2), i.e., u is a minimizer for $J[\cdot]$ and for each $k>0$ it is a minimizer for $J_{\Omega_k}[\cdot]$ among all functions satisfying $\left(v|_{\partial\Omega_k}, \frac{\partial v}{\partial\nu}|_{\partial\Omega_k}\right) = \left(u|_{\partial\Omega_k}, \frac{\partial u}{\partial\nu}|_{\partial\Omega_k}\right)$.

1. Introduction

In this article we give a certain two dimensional generalization of the analysis performed in the one dimensional case in the paper "One dimensional infinite horizon variational problems arising in continuum mechanics" by Leizarowitz and Mizel [**11**]. In [**11**] they studied a variational problem for real valued functions defined on an infinite semi-axis of the line. Namely, given $x \in \mathbb{R}^2$, minimize the functional

$$I\left(w\left(\cdot\right)\right)=\int_0^\infty f\left(w\left(s\right),\dot{w}\left(s\right),\ddot{w}\left(s\right)\right)ds, \quad \text{for} \quad w\in A_x=\left\{v\in W^{2,1}_{loc}\left(0,\infty\right):\ \left(v\left(0\right),\dot{v}\left(0\right)\right)=x\right\}. \tag{1.1}$$

Here $f=f\left(w,p,r\right)$ is a smooth function satisfying

(1) $f_{rr}\geq 0$,
(2) $f\left(w,p,r\right)\geq a\left|w\right|^{\alpha}-b\left|p\right|^{\beta}+c\left|r\right|^{\gamma}-d,\ a,b,c,d>0$,
(3) $f\left(w,p,r\right)\leq\varphi\left(w,p\right)+c'\left|r\right|^{\gamma},\ c'>0$,

where $\alpha,\gamma>1$, $\beta\geq 1$ satisfy $\beta<\alpha$, $\beta\leq\gamma$, and φ is continuous.

The original model of the one-dimensional problem was formulated by Coleman in [**4, 5**] to describe the structure of long polymeric fibers of a viscoelastic material under tension. Similar models with additional mass constraint were studied by Coleman, Marcus and Mizel in [**6**] and Marcus in [**15**]. The unconstrained problem (1.1) was investigated in [**11**] as a model for the determination of the thermodynamical equilibrium states of unidimensional bodies.

In this study we investigate the two-dimensional version of the model given in (1.1), replacing the semi-axis by the infinite strip $\Omega=(0,\infty)\times(0,L)$, for some fixed $L>0$. More precisely, given $g\in H^{\frac{3}{2}}\left(\partial\Omega\right)\times H^{\frac{1}{2}}\left(\partial\Omega\right)$ such that

$$g=0 \quad \text{on} \quad (0,\infty)\times\{0\}\cup(0,\infty)\times\{L\}. \tag{1.2}$$

We seek a "minimal solution" for the functional

$$I\left[u\right]=\int_\Omega f\left(u,Du,D^2u\right)dx, \quad \text{for} \quad u\in A_g:=\left\{v\in H^2_{loc}\left(\Omega\right):\left(v|_{\partial\Omega},\tfrac{\partial v}{\partial\nu}|_{\partial\Omega}\right)=g\right\}, \tag{1.3}$$

where $\frac{\partial v}{\partial\nu}|_{\partial\Omega}$ is the outward normal derivative on $\partial\Omega$. In (1.3) f is a smooth function satisfying convexity (with respect to its third variable) and growth conditions analogous to (1.1) to be described in Section 2.

As it happens in this kind of "infinite horizon" problems, the infimum of $I\left[\cdot\right]$ on A_g is typically either $+\infty$ or $-\infty$. In order to overcome that difficulty we consider the expression

$$J_{\Omega_k}\left[u\right]=\frac{1}{\left|\Omega_k\right|}\int_{\Omega_k}f\left(u,Du,D^2u\right)dx,$$

where $\Omega_k=(0,k)\times(0,L)$, for any $k>0$, and then study the limit as k tends to infinity.

For $u\in\left\{v\in H^2\left(\Omega_k\right):\left(v|_{\partial\Omega_k\cap\partial\Omega},\frac{\partial v}{\partial\nu}|_{\partial\Omega_k\cap\partial\Omega}\right)=g|_{\partial\Omega_k\cap\partial\Omega}\right\}$, the value $J_{\Omega_k}\left[u\right]$ may be thought of as mean energy associated with a "state" u of a body whose extent is the domain Ω_k. The state u may describe the concentration, or the mass density of the body. An equilibrium state for Ω_k is a state for which $J_{\Omega_k}[\cdot]$ obtains its infimum. As $k \to \infty$, the limit of $J_{\Omega_k}[u]$ represents the average energy of u on Ω, and whenever this limit has a meaning we define

$$J[u] = \liminf_{k\to\infty} J_{\Omega_k}[u], \tag{1.4}$$

which is the mean energy on Ω associated with u. Our problem, involving an infinite strip, is a relatively simple two dimensional generalization of the one dimensional problem. The problem involving functionals defined on the whole plane $\mathbb{R}^2$ is much more difficult. Some partial results for the latter problem were obtained by Leizarowitz and Marcus [**12**] and Leizarowitz [**10**] , in particular, concerning radially symmetric minimizers.

In this study we employ the notion of *minimal energy configuration* as an optimality criterion. It was introduced by Aubry and Le Daeron in the analysis of the discrete Frenkel-Kantorova model related to dislocations in one-dimensional crystals ([**2**]). This optimality criterion was used by Leizarowitz and Mizel [**11**] from which our notion of minimal solution is taken.

A function $u \in A_g$ is called a *minimal solution* of (1.3), if $J[\cdot]$ obtains its minimal value for u and if for every $k > 0$ and every $v \in A_g$ satisfying $\left(v|_{\partial\Omega_k}, \frac{\partial v}{\partial \nu}|_{\partial\Omega_k}\right) = \left(u|_{\partial\Omega_k}, \frac{\partial u}{\partial \nu}|_{\partial\Omega_k}\right)$ we have $J_{\Omega_k}[v] \geq J_{\Omega_k}[u]$.

We shall analyze existence and properties of such minimal solutions. Our main result is the following.

THEOREM. *For every* $g \in H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ *satisfying* (1.2)*, there exists a minimal solution for* (1.3).

The form of free energy integrand that appears in the applications is given by

$$f(w,p,r) = \Psi(w) - b|p|^2 + c|r|^2 \ \ (b,c>0) \ \ (w,p,r) \in \mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^4,$$

where Ψ is any function possessing some of the basic features of the van der Waals potential, for instance

$$\Psi(w) = a(w-w_1)^2(w-w_2)^2,$$

with $a > 0$, $w_2 > w_1$. However, we shall consider more general free energy integrands (the exact assumptions will be given in Section 2).

The article is organized as follows. In Section 2 we specify our notation and analyze the variational problem taken over a bounded rectangular domain. Section 3 contains several preliminary results. In Section 4 we define the minimal growth rate and prove its properties. In Section 5 we shall define our criterion for a solution of (1.3) to be minimal and use some of the techniques of [**9, 11, 16**] for the one dimensional case in order to prove its existence. The main new difficulty that arises in analyzing the two dimensional problem is the fact that the space of traces is infinite dimensional. Therefore, only weak compactness, and not strong one, of bounded sets is available. Another technical difficulty is that our basic domain, namely, the rectangular, is nonsmooth. In this respect, the book of Grisvard [**7**] was very helpful. It is plausible that the same techniques can be used to handle the case of a semi-infinite cylinder in higher dimension, i.e., for $\Omega = (0,\infty)\times\omega$ with ω a bounded domain in $\mathbb{R}^k$, but we didn't investigate this question.

Acknowledgment. This article is based on a M.Sc. thesis written by the second author (I.Y.) under the supervision of the first author (I.S.). The topic of the research was suggested by the late Prof. Arie Leizarowitz, who also guided the second author in the research for a substantial part of his graduate studies. He will always remember and cherish the expertise, patience and kindness of Prof. Leizarowitz. The first author (I.S.) acknowledges the support by the Israel Science Foundation (grant no. 1279/08).

2. Bounded domain problem

In pursuing our goal of analyzing the unbounded domain problem we begin by defining and proving some properties of the variational problem on a bounded domain .

Let Ω be a bounded open subset of $\mathbb{R}^2$, whose boundary is a polygon and $g=(g_0,g_1)\in H^{\frac{3}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$ where $H^{\frac{3}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$ is equipped with the norm

$$\|g\|_{H^{\frac{3}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)}=\Big(\sum_{\substack{0\le k,m\le 1\\ k+m=1}}\int_{\partial\Omega}\left|D^k g_m\right|^2 d\sigma(x)+\sum_{\substack{0\le k,m\le 1\\ k+m=1}}\iint_{\partial\Omega\times\partial\Omega}\frac{\left|D^k g_m(x)-D^k g_m(y)\right|^2}{|x-y|^2}d\sigma(x)\,d\sigma(y)\Big)^{\frac{1}{2}}.$$

Consider the following functional

$$\begin{aligned} I_\Omega[u]&=\int_\Omega f\left(u,Du,D^2u\right)dx,\\ u\in A_g&=\left\{v\in H^2(\Omega):\, v|_{\partial\Omega}=g_1,\ \tfrac{\partial v}{\partial\nu}\big|_{\partial\Omega}=g_2\right\}\end{aligned}\tag{2.1}$$

where $\frac{\partial v}{\partial\nu}\big|$ is the outward normal derivative on $\partial\Omega$.
The function $f:\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^4\to\mathbb{R}$ is assumed to be smooth and satisfies

$$\begin{gathered}\text{the mapping } r\mapsto f(w,p,r) \text{ is convex for each } w\in\mathbb{R}, p\in\mathbb{R}^2,\\ f(w,p,r)\ge a|w|^\alpha-b|p|^\beta+c|r|^2-d,\quad a,b,c,d>0\\ f(w,p,r)\le\phi(w)+b'|p|^\beta+c'|r|^2,\quad b',c'>0\end{gathered}\tag{2.2}$$

where $\alpha>1$, $\beta\ge 1$ satisfy $\beta<\alpha$,$\beta\le 2$ and ϕ is continuous.

Following Leizarowitz & Mizel [**11**], let us introduce the following result.

Theorem 2.1. *Let Ω be a bounded rectangular domain in $\mathbb{R}^2$. Then the functional I_Ω defined by* (2.1) *is bounded from below.*

The proof of Theorem 2.1 uses the following result from Adams [[**1**], Lemma 4.10].

Lemma 2.2. *Let $-\infty\le a<b\le\infty$, let $1\le p<\infty$ and let $0<\varepsilon_0<\infty$. There exists a finite constant $\hat{K}=\hat{K}(\varepsilon_0,p,b-a)$, depending continuously on $b-a$ for $0<b-a\le\infty$, such that for ε satisfying $0<\varepsilon\le\varepsilon_0$, and for every function f twice continuously differentiable on the open interval (a,b)*

$$\int_a^b |f'(t)|^p\,dt\le\hat{K}\varepsilon\int_a^b |f''(t)|^p\,dt+\hat{K}\varepsilon^{-1}\int_a^b |f(t)|^p\,dt.$$

By Adams [[**1**], Lemma 4.10], we may choose the coefficient $\hat{K}$ as

$$\hat{K}(\varepsilon_0, p, L) = \max\left\{\varepsilon_0, \varepsilon_0^{-1}\right\} \cdot 2^{p-1} 9^p \cdot \begin{cases} \max\limits_{1 \le s \le 2} \left\{\max\left\{s^p, \left(\frac{2}{s}\right)^p\right\}\right\} & if\ L \ge 1 \\ \max\limits_{L \le s \le 2} \left\{\max\left\{s^p, \left(\frac{2}{s}\right)^p\right\}\right\} & if\ 0 < L < 1 \end{cases}. \tag{2.3}$$

Proof of Theorem 2.1. Let $\Omega = (0, L_1) \times (0, L_2)$. Noting that $C^2(\Omega)$ is dense in $W^{2,\beta}(\Omega)$ and employing Lemma 2.2 for $\varepsilon_0 = 1$ yields $K = \hat{K}(1, \beta, L_1) + \hat{K}(1, \beta, L_2)$ such that for each $0 < \varepsilon \le 1$, and any $v \in W^{2,\beta}(\Omega)$

$$\int_\Omega |Dv|^\beta \, dx \le K\varepsilon \int_\Omega \left|D^2 v\right|^\beta dx + K\varepsilon^{-1} \int_\Omega |v|^\beta \, dx, \quad \forall v \in W^{2,\beta}(\Omega). \tag{2.4}$$

Putting $\eta = K\varepsilon \frac{b}{c}$, we have

$$\frac{1}{\eta} \int_\Omega b\,|Dv|^\beta \, dx \le \int_\Omega c\left|D^2 v\right|^\beta dx + c\varepsilon^{-2} \int_\Omega |v|^\beta \, dx, \quad \forall v \in W^{2,\beta}(\Omega). \tag{2.5}$$

Now for each $\alpha > \beta$ and $a > 0$ it is easy to check that $Q = \left(c\varepsilon^{-2}\right)^{\frac{\alpha}{\alpha-\beta}} \left(\frac{\beta}{a\alpha}\right)^{\frac{\beta}{\alpha-\beta}}$ satisfies

$$a\,|w|^\alpha + Q \ge c\varepsilon^{-2}\,|w|^\beta, \quad \forall w \in \mathbb{R}.$$

Moreover, for each $2 \ge \beta$ and $c > 0$ one has

$$c\,|r|^2 + c \ge c\,|r|^\beta, \quad \forall r \in \mathbb{R}^4.$$

Hence (2.5) ensures that

$$\frac{1}{\eta} \int_\Omega b\,|Dv|^\beta \, dx \le \int_\Omega \left(c\left|D^2 v\right|^2 + a\,|v|^\alpha\right) dx + (Q + c)\,|\Omega|, \quad \forall v \in W^{2,\beta}(\Omega) \tag{2.6}$$

By taking $\varepsilon < \min\left\{\frac{c}{2bK}, 1\right\}$, we get $\eta < \frac{1}{2}$ and from (2.2) and (2.6) we obtain

$$\int_\Omega f\left(v, Dv, D^2 v\right) dx \ge \int_\Omega b\,|Dv|^\beta \, dx - P\,|\Omega|, \quad \forall v \in W^{2,\beta}(\Omega) \tag{2.7}$$

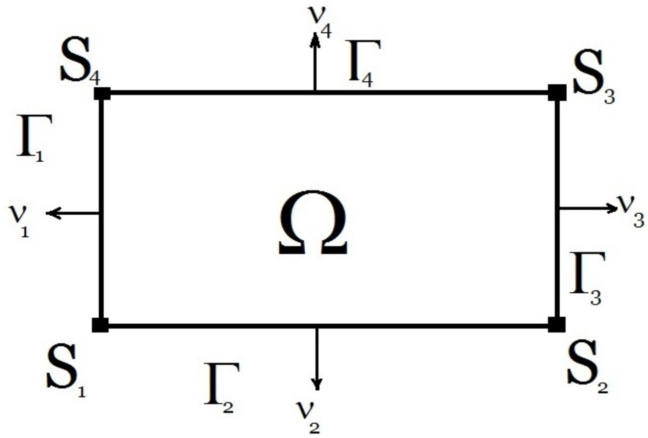
where $P = Q + c + d$. Thus I_Ω is bounded from below. □

Note that, from the growth condition (2.2) and (2.6) one also obtains the estimate

$$\int_\Omega f\left(v, Dv, D^2 v\right) dx \ge (1 - \eta) \int_\Omega \left(c\left|D^2 v\right|^2 + a\,|v|^\alpha\right) dx - (\eta\,(Q + c) + d)\,|\Omega|, \quad \forall v \in H^2(\Omega). \tag{2.8}$$

Now let us fix some notation. From now on, we consider a rectangular domain Ω, whose boundary is denoted by $\partial\Omega$. We denote each of the open edges, which constitute the boundary, by Γ_j for j ranging from 1 to 4 (see Figure 2.1). The edge Γ_{j+1} follows Γ_j according to the positive orientation, on each connected component of $\partial\Omega$. We denote by S_j the vertex which is the intersection $\bar{\Gamma}_j \cap \bar{\Gamma}_{j+1}$ (we agree that $\Gamma_5 = \Gamma_1$). We denote by ν_j the unit outward normal on Γ_j. Furthermore, we denote by γ the operator defined by $(\gamma u) = u|_{\partial\Omega}$ when u is a smooth function in $\bar{\Omega}$. (Similarly, for $j = 1, 2, 3, 4$ we denote by $\gamma_j u$ the restriction of u to Γ_j.) Furthermore, let σ be the signed distance from S_j along $\partial\Omega$, and let $x_j(\sigma)$ be the point on $\partial\Omega$ whose distance to S_j is σ. Consequently for $|\sigma| \le \delta_j$ small enough, we have $x_j(\sigma) \in \Gamma_j$ when $\sigma < 0$ and $x_j(\sigma) \in \Gamma_{j+1}$ when $\sigma > 0$.

FIGURE 2.1



We denote by $Y(\partial\Omega)$ the subspace of $\prod\limits_{j=1}^{4} H^{\frac{3}{2}}(\Gamma_j) \times H^{\frac{1}{2}}(\Gamma_j)$ consisting of the functions

$$\{f_j^k\}_{\substack{1\le j\le 4\\ 0\le k\le 1}} = \left\{\left(f_1^0, f_1^1\right), \left(f_2^0, f_2^1\right), \left(f_3^0, f_3^1\right), \left(f_4^0, f_4^1\right)\right\} \in \prod_{j=1}^{4} H^{\frac{3}{2}}(\Gamma_j) \times H^{\frac{1}{2}}(\Gamma_j)$$

satisfying

$$f_j^0(S_j) = f_{j+1}^0(S_j) \qquad \text{for} \quad 1 \le j \le 4$$

$$\int_0^{\delta_j} \left|D^k f_{j+1}^l(x_j(\sigma)) - D^l f_j^k(x_j(-\sigma))\right|^2 \frac{d\sigma}{\sigma} < \infty \text{ for } k+l=1,\ 1\le j\le 4.$$

DEFINITION 2.3. Let Ω be a bounded open subset of $\mathbb{R}^2$. We say that the boundary Γ is a curvilinear polygon of class C^m, m integer ≥ 1 if for every $x \in \Gamma$ there exists a neighborhood V of x in $\mathbb{R}^2$ and a mapping ψ from V in $\mathbb{R}^2$ such that

(a) ψ is injective,

(b) ψ together with ψ^{-1} (defined on $\psi(V)$) belongs to the class C^m,

(c) $\Omega \cap V$ is either $\{y \in \Omega | \psi_2(y) < 0\}$,$\{y \in \Omega | \psi_1(y) < 0$ and $\psi_2(y) < 0\}$ or $\{y \in \Omega | \psi_1(y) < 0$ or $\psi_2(y) < 0\}$ where $\psi_j(y)$ denotes the jth component of ψ.

DEFINITION 2.4. We denote by $W_0^{k,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$. When $p=2$ we denote $H_0^k(\Omega) = W_0^{k,2}(\Omega)$.

We interpret $W_0^{k,p}(\Omega)$ as comprising those functions $u \in W^{k,p}(\Omega)$ such that "$D^\alpha u = 0$ on $\partial\Omega$" for all $|\alpha| \le k-1$.

The following Trace Theorem (cf. Grisvard [[**7**], Theorem 1.5.2.8]) plays an important role in the sequel.

THEOREM 2.5. *Let Ω be a bounded open subset of $\mathbb{R}^2$ whose boundary $\partial\Omega$ is a curvilinear polygon of class C^∞. Then the mapping*

$$u \mapsto \gamma u = \left\{ \gamma_j \frac{\partial^k u}{\partial \nu_j^k} | 1 \le j \le 4, 0 \le k \le 1 \right\} \tag{2.9}$$

which is defined for $u \in C^1\left(\overline{\Omega}\right)$, has a unique continuous extension as a linear continuous operator from $H^2(\Omega)$ onto the space $Y(\partial\Omega)$ defined above.

This mapping is surjective and there exists a continuous linear right inverse

$$\left\{ f_j^k \right\}_{\substack{1 \le j \le 4 \\ 0 \le k \le 1}} \mapsto \mathcal{R} f \quad \text{of} \quad Y(\partial\Omega) \to H^2(\Omega) \tag{2.10}$$

such that

$$\gamma_j \frac{\partial^k \mathcal{R} f}{\partial \nu_j^k} = f_j^k, \quad 1 \le j \le 4, 0 \le k \le 1.$$

The existence of the right inverse $\mathcal{R}$ follows from the following remark.

REMARK 2.6. The operator $\gamma : H^2(\Omega) \to Y(\partial\Omega)$ induces an isomorphism $\tilde{\gamma} : H^2(\Omega)_{\diagup H_0^2(\Omega)} \to Y(\partial\Omega)$ (note that $H_0^2(\Omega) = \ker \gamma$). Recall that the norm of the Hilbert space $H^2(\Omega)_{\diagup H_0^2(\Omega)}$ is given by $\|[u]\|_{H^2(\Omega)_{\diagup H_0^2(\Omega)}} = \inf_{v \in [u]} \|v\|_{H^2(\Omega)} = \inf_{\gamma v = \gamma u} \|v\|_{H^2(\Omega)}$. Hence, we can define for every $g \in Y(\partial\Omega)$: $\mathcal{R} g$ is the unique $v \in H^2(\Omega)$ satisfying

$$\|v\|_{H^2(\Omega)} = \min_{\gamma u = g} \|u\|_{H^2(\Omega)}. \tag{2.11}$$

We shall apply Theorem 2.5 only in the simple case in which Ω is a rectangle.

PROPOSITION 2.7. *There exist $A = A(\Omega) > 0$ and a constant $B = B(\Omega, \gamma u) > 0$ such that*

$$I_\Omega[u] \ge A \|u\|_{H^2(\Omega)}^2 - B, \ \forall u \in H^2(\Omega). \tag{2.12}$$

It is customary to call (2.12) a coercivity property of I_Ω.

PROOF. Using Theorem 2.5, $(u - \mathcal{R}(\gamma u)) \in H_0^2(\Omega)$. Thus, by Poincare inequality there exists $K = K(\operatorname{diam}(\Omega)) > 0$ such that

$$\|u - \mathcal{R}(\gamma u)\|_{L^2(\Omega)} \le K \|Du - D(\mathcal{R}(\gamma u))\|_{L^2(\Omega)} \le K^2 \left\|D^2 u - D^2(\mathcal{R}(\gamma u))\right\|_{L^2(\Omega)}.$$

(For $\Omega = (0, L_1) \times (0, L_2)$, we may set $K = \sqrt{\frac{\min\left(L_1^2, L_2^2\right)}{2}}$ (c.f. Leoni, [[**13**],Theorem 12.17])

Hence,

$$\begin{aligned} \|u\|_{L^2(\Omega)}^2 &\le 2\left(\|\mathcal{R}(\gamma u)\|_{L^2(\Omega)}^2 + \|u - \mathcal{R}(\gamma u)\|_{L^2(\Omega)}^2\right) \\ &\le 2\left(\|\mathcal{R}(\gamma u)\|_{L^2(\Omega)}^2 + K^4 \left\|D^2 u - D^2(\mathcal{R}(\gamma u))\right\|_{L^2(\Omega)}^2\right) \\ &\le 2\|\mathcal{R}(\gamma u)\|_{L^2(\Omega)}^2 + 4K^4 \left\|D^2 u\right\|_{L^2(\Omega)}^2 + 4K^4 \left\|D^2(\mathcal{R}(\gamma u))\right\|_{L^2(\Omega)}^2 \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} \|Du\|_{L^2(\Omega)}^2 &\le 2\left(\|D(\mathcal{R}(\gamma u))\|_{L^2(\Omega)}^2 + \|Du - D(\mathcal{R}(\gamma u))\|_{L^2(\Omega)}^2\right) \\ &\le 2\left(\|D(\mathcal{R}(\gamma u))\|_{L^2(\Omega)}^2 + K^2 \left\|D^2 u - D^2(\mathcal{R}(\gamma u))\right\|_{L^2(\Omega)}^2\right) \\ &\le 2\|D(\mathcal{R}(\gamma u))\|_{L^2(\Omega)}^2 + 4K^2 \left\|D^2 u\right\|_{L^2(\Omega)}^2 + 4K^2 \left\|D^2(\mathcal{R}(\gamma u))\right\|_{L^2(\Omega)}^2. \end{aligned} \tag{2.14}$$

By (2.8), we have

$$\int_{\Omega} f\left(u, Du, D^2u\right) dx \geq (1-\eta)\, c \left\|D^2 u\right\|^2_{L^2(\Omega)} - \left(\eta\,(Q+c)+d\right)|\Omega|\,, \quad \forall u \in H^2(\Omega)\,. \tag{2.15}$$

Thus, by (2.13) – (2.15)

$$\int_{\Omega} f\left(u, Du, D^2u\right) dx \geq \frac{(1-\eta)\,c}{3}\left\|D^2 u\right\|^2_{L^2(\Omega)}$$
$$+\frac{(1-\eta)\,c}{3\cdot 4K^2}\left(\|Du\|^2_{L^2(\Omega)} - 2\left\|D\left(\mathcal{R}(\gamma u)\right)\right\|^2_{L^2(\Omega)} - 4K^2\left\|D^2\left(\mathcal{R}(\gamma u)\right)\right\|^2_{L^2(\Omega)}\right)$$
$$+\frac{(1-\eta)\,c}{3\cdot 4K^4}\left(\|u\|^2_{L^2(\Omega)} - 2\left\|\mathcal{R}(\gamma u)\right\|^2_{L^2(\Omega)} - 4K^4\left\|D^2\left(\mathcal{R}(\gamma u)\right)\right\|^2_{L^2(\Omega)}\right) - \left(\eta\,(Q+c)+d\right)|\Omega|\,.$$

Hence

$$I_{\Omega}[u] \geq \frac{(1-\eta)\,c}{3}\min\left\{1, \frac{1}{4K^4}\right\}\|u\|^2_{H^2(\Omega)}$$
$$-\left\{\frac{(1-\eta)\,c}{6}\left(4+\frac{1}{K^2}+\frac{1}{K^4}\right)\left\|\mathcal{R}(\gamma u)\right\|^2_{H^2(\Omega)} + \left(\eta\,(Q+c)+d\right)|\Omega|\right\}.$$

Thus, by setting

$$A(\Omega) = \frac{(1-\eta)\,c}{3}\min\left\{1, \frac{1}{4K^4}\right\} \tag{2.16}$$

and

$$B(\Omega, \gamma u) = \frac{(1-\eta)\,c}{6}\left(4+\frac{1}{K^2}+\frac{1}{K^4}\right)\left\|\mathcal{R}(\gamma u)\right\|^2_{H^2(\Omega)} + \left(\eta\,(Q+c)+d\right)|\Omega| \tag{2.17}$$

we obtain (2.12), as required. □

Definition 2.8. The open domain Ω has the *cone property* if there exists a finite cone C such that each point $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C.

Definition 2.9. An open and bounded domain Ω has a *locally Lipschitz boundary* (see [**1**], p.67) if each point $x \in \partial\Omega$ has a neighborhood U_x such that $\partial\Omega \cap U_x$ is a graph of a Lipschitz continuous function.

We shall use in the sequel the following version of the Rellich-Kondrashov Theorem (cf. [**1**], Theorem 6.2).

Theorem 2.10. *Let Ω be a bounded domain in $\mathbb{R}^N$.*
1. If Ω has a cone property, then the following embedding is compact:

$$W^{1+m,p}(\Omega) \to W^{1,q}(\Omega) \;\; if \;\; N = mp \text{ and } 1 \leq q < \infty.$$

2. If Ω has a locally Lipschitz boundary, then the following embedding is compact:

$$W^{m,p}(\Omega) \to C^0\left(\overline{\Omega}\right) \text{ if } mp > N.$$

When Ω is a rectangle, both parts of Theorem 2.10 hold.

Proposition 2.11. *For bounded Lipschitz domain Ω, the functional I_{Ω} is weakly lower semicontinuous on $H^2(\Omega)$.*

The proof given here is similar to Evans ([**8**],p.446).

PROOF. Consider any sequence $(u_k)_{k=1}^{\infty}$ with

$$u_k \rightharpoonup u \text{ weakly in } H^2(\Omega) \tag{2.18}$$

and set $l := \liminf_{k\to\infty} I_\Omega[u_k]$. We have to show that

$$I_\Omega[u] \leq l. \tag{2.19}$$

Note first from (2.18) and the fact that any weakly convergent sequence is bounded, that

$$\sup_k \|u_k\|_{H^2(\Omega)} < \infty. \tag{2.20}$$

Upon passing to subsequence if necessary, we may suppose also that

$$l = \lim_{k\to\infty} I_\Omega[u_k]. \tag{2.21}$$

By Theorem 2.10 we have

$$u_k \to u \text{ strongly in } W^{1,p}(\Omega),\ 1 \leq p < \infty; \tag{2.22}$$

and thus, passing if necessary to yet another subsequence, we have

$$\begin{array}{c} u_k \to u \text{ uniformly on } \Omega \\ Du_k \to Du \text{ a.e. in } \Omega \end{array}. \tag{2.23}$$

Fix $n \in \mathbb{N}$. Then by (2.23) and Egorov's Theorem

$$Du_k \to Du \text{ uniformly on } E_n \ . \tag{2.24}$$

where E_n is a measurable set with

$$|\Omega \setminus E_n| < \frac{1}{n}. \tag{2.25}$$

Set

$$F_n := \left\{x \in \Omega \middle| |u(x)| + |Du(x)| + \left|D^2u(x)\right| \leq n\right\}. \tag{2.26}$$

Then

$$|\Omega \setminus F_n| \to 0 \text{ as } n \to \infty. \tag{2.27}$$

We finally set

$$G_n = E_n \cap F_n, \tag{2.28}$$

and observe from (2.25) and (2.27) that $|\Omega \setminus G_n| \to 0$ as $n \to \infty$. Recalling (2.2), we set

$$\tilde{f}(w,p,r) := f(w,p,r) + b|p|^\beta + d. \tag{2.29}$$

It is easy to see that

$$\tilde{f}(w,p,r) \geq 0, \quad \forall (w,p,r) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^4 \tag{2.30}$$

and that the function $r \mapsto \tilde{f}(w,p,r)$ is convex for each $w \in \mathbb{R},\ p \in \mathbb{R}^2$. Consequently,

$$
\begin{aligned}
(2.31)\qquad I_\Omega[u_k] + b\,\|Du_k\|^\beta_{L^\beta(\Omega)} + d\,|\Omega| &= \int_\Omega \tilde{f}\left(u_k, Du_k, D^2u_k\right) dx \geq \int_{G_n} \tilde{f}\left(u_k, Du_k, D^2u_k\right) dx \\
&\geq \int_{G_n} \tilde{f}\left(u_k, Du_k, D^2u\right) dx + \int_{G_n} \tilde{f}_r\left(u_k, Du_k, D^2u\right)\cdot\left(D^2u_k - D^2u\right) dx,
\end{aligned}
$$

the last inequality following from the convexity of $\tilde{f}$ in its third argument. Now in view of (2.24) , (2.26) and (2.28):

$$
(2.32)\qquad \lim_{k\to\infty} \int_{G_n} \tilde{f}\left(u_k, Du_k, D^2u\right) dx = \int_{G_n} \tilde{f}\left(u, Du, D^2u\right) dx.
$$

In addition, since $\tilde{f}_r\left(u_k, Du_k, D^2u\right) \to \tilde{f}_r\left(u, Du, D^2u\right)$ uniformly on G_n and $D^2u_k \rightharpoonup D^2u$ weakly in $L^2(\Omega)$, we have

$$
(2.33)\qquad \lim_{k\to\infty} \int_{G_n} \tilde{f}_r\left(u_k, Du_k, D^2u\right)\cdot\left(D^2u_k - D^2u\right) dx = 0.
$$

By (2.32) and (2.33) we deduce from (2.31) that

$$
\lim_{k\to\infty} \int_{G_n} \tilde{f}\left(u_k, Du_k, D^2u_k\right) dx \geq \int_{G_n} \tilde{f}\left(u, Du, D^2u\right) dx.
$$

This inequality holds for each $n \in \mathbb{N}$. We now let n go to infinity, and recall (2.30), (2.22) and the Monotone Convergence Theorem to conclude

$$
l + b\,\|Du\|^\beta_{L^\beta(\Omega)} + d\,|\Omega| \geq \int_\Omega \tilde{f}\left(u, Du, D^2u\right) dx = I_\Omega[u] + b\,\|Du\|^\beta_{L^\beta(\Omega)} + d\,|\Omega|,
$$

Thus $l \geq I_\Omega[u]$ as required. □

In the rest of the section we shall denote by Ω a bounded Lipschitz domain.

THEOREM 2.12. *There exists at least one function $u \in A_g$ satisfying*

$$
I_\Omega[u] = \inf_{v\in A_g} I_\Omega[v].
$$

PROOF. Set $m = \inf_{v\in A_g} I_\Omega[v]$. Select a minimizing sequence $(u_k)_{k=1}^\infty$. Then

$$
I_\Omega[u_k] \to m.
$$

Since m is finite and g is fixed, we conclude from (2.12) that

$$
\sup_{k\geq 1} \|u_k\|_{H^2(\Omega)} < \infty.
$$

Thus by passing to subsequence, we get

$$
u_k \rightharpoonup u \text{ weakly in } H^2(\Omega).
$$

Finally we assert that $u \in A_g$. To see this, note that since $\mathcal{R}g \in A_g$, $(u_k - \mathcal{R}g) \in H_0^2(\Omega)$. Now $H_0^2(\Omega)$ is a closed, linear subspace of $H^2(\Omega)$, and so, by Mazur's Theorem, it is weakly closed. Hence, $(u - \mathcal{R}g) \in H_0^2(\Omega)$. Consequently $u \in A_g$. In view of Proposition 2.11 we get $I_\Omega[u] \leq \liminf_{k\to\infty} I_\Omega[u_k] =$

m. But since $u \in A_g$, it follows that

$$I_\Omega[u] = m = \inf_{v \in A_g} I_\Omega[v].$$

□

DEFINITION 2.13. We denote by $V_\Omega : Y(\partial\Omega) \to \mathbb{R}$ the functional defined by

$$V_\Omega[g] = \inf_{u \in A_g} I_\Omega[u] \tag{2.34}$$

where

$$A_g = \left\{ v \in H^2(\Omega) : \left\{ \gamma_j \frac{\partial^k v}{\partial \nu_j^k} \right\}_{\substack{1 \le j \le 4 \\ 0 \le k \le 1}} = g \right\}.$$

Now, by employing Theorem 2.5 we have the following alternative definition of V_Ω:

$$V_\Omega[g] = \inf_{u \in H_0^2(\Omega)} I_\Omega[\mathcal{R}g + u], \ g \in Y(\partial\Omega) \tag{2.35}$$

where $\mathcal{R}$ is a continuous linear right inverse mapping defined in Theorem 2.5.

The following result is analogous to Leizarowitz & Mizel [[**11**],Theorem 2.2].

THEOREM 2.14. *The functional* $V_\Omega : Y(\partial\Omega) \to \mathbb{R}$ *is weakly lower semicontinuous.*

PROOF. Consider any sequence $(g_k)_{k=1}^\infty$ such that

$$g_k \rightharpoonup g \text{ weakly in } Y(\partial\Omega). \tag{2.36}$$

Let us denote by $u_k \in H_0^2(\Omega)$ the minimizer in (2.35) corresponding to g_k for $k \ge 1$. That is, for every $k \ge 1$, we have

$$V_\Omega[g_k] = I_\Omega[\mathcal{R}g_k + u_k]. \tag{2.37}$$

By (2.35)

$$I_\Omega[\mathcal{R}g_k] \ge V_\Omega[g_k]. \tag{2.38}$$

From (2.2) we have

$$\int_\Omega \left(\phi(\mathcal{R}g_k) + b' |D\mathcal{R}g_k|^\beta + c' \left|D^2\mathcal{R}g_k\right|^2 \right) dx \ge I_\Omega[\mathcal{R}g_k].$$

Since $\mathcal{R}$ is a continuous linear mapping, there exists $C_\mathcal{R} = C_\mathcal{R}(\Omega)$ such that $\|\mathcal{R}g_k\|_{H^2(\Omega)} \le C_\mathcal{R} \|g_k\|_{Y(\partial\Omega)}$. Moreover, by (2.36) the sequence $(g_k)_{k=1}^\infty$ is bounded. Thus, we get

$$\sup_{k \ge 1} \int_\Omega \left|D^2\mathcal{R}g_k\right|^2 dx < \infty.$$

Using Theorem 2.10, as in proof of Proposition 2.11, we get by extracting a subsequence and re-indexing

$$\mathcal{R}g_k \to \mathcal{R}g \text{ strongly in } W^{1,p}(\Omega) \text{ for } 1 \le p < \infty$$

$$\mathcal{R}g_k \to \mathcal{R}g \text{ strongly in } C^0\left(\bar{\Omega}\right).$$

Hence

$$\sup_{k\geq 1}\int_{\Omega}\left(\phi\left(\mathcal{R}g_k\right)+b'\left|D\mathcal{R}g_k\right|^{\beta}\right)dx<\infty \text{ for } 1\leq\beta<\infty.$$

Thus, the sequence $\left(I_{\Omega}\left[\mathcal{R}g_k\right]\right)_{k=1}^{\infty}$ is bounded.

Now we combine (2.12) , (2.37) and (2.38) to obtain

$$I_{\Omega}\left[\mathcal{R}g_k\right]\geq V_{\Omega}\left[g_k\right]=I_{\Omega}\left[\mathcal{R}g_k+u_k\right]\geq A\left\|\mathcal{R}g_k+u_k\right\|_{H^2(\Omega)}^2-B. \tag{2.39}$$

By (2.17) we have

$$B\leq\frac{(1-\eta)\,c}{6}(4+\frac{1}{K^2}+\frac{1}{K^4})C_{\mathcal{R}}^2\left\|g_k\right\|_{Y(\partial\Omega)}^2+\left(\eta\left(Q+c\right)+d\right)\left|\Omega\right|.$$

It follows from (2.39) and the boundedness of $\left(I_{\Omega}\left[\mathcal{R}g_k\right]\right)_{k=1}^{\infty}$ and $\left(g_k\right)_{k=1}^{\infty}$ that the sequence $\left(\mathcal{R}g_k+u_k\right)_{k=1}^{\infty}$ is bounded in $H^2\left(\Omega\right)$. By extracting a subsequence and re-indexing, we obtain

$$\mathcal{R}g_k+u_k\rightharpoonup\mathcal{R}g+u \text{ weakly in } H^2\left(\Omega\right)$$

and since $\mathcal{R}g_k\rightharpoonup\mathcal{R}g$ weakly in $H^2\left(\Omega\right)$, $u_k\rightharpoonup u\in H_0^2\left(\Omega\right)$ weakly in $H^2\left(\Omega\right)$.

Finally, by Proposition 2.11

$$\liminf_{k\to\infty}V_{\Omega}\left[g_k\right]=\liminf_{k\to\infty}I_{\Omega}\left[\mathcal{R}g_k+u_k\right]\geq I_{\Omega}\left[\mathcal{R}g+u\right]\geq V_{\Omega}\left[g\right].$$

□

We follow Leizarowitz & Mizel [[**11**],Theorem 2.2] once again to obtain the following theorem.

THEOREM 2.15. *The functional* $V_{\Omega}:Y\left(\partial\Omega\right)\to\mathbb{R}$ *is continuous.*

PROOF. Since we already proved in Theorem 2.14 that V_{Ω} is weakly lower semicontinuous, which implies that V_{Ω} is lower semicontinuous, it remains to prove that V_{Ω} is also upper semicontinuous. Suppose $g_k\to g$ in $Y\left(\partial\Omega\right)$ as $k\to\infty$. Let $u\in A_g$ denote a minimizer of I_{Ω} whose existence is guaranteed in Theorem 2.12. Using Theorem 2.5 we define

$$\delta_k=\mathcal{R}\left(g_k-g\right),$$

and we know that there exists $C_{\mathcal{R}}=C_{\mathcal{R}}\left(\Omega\right)$ such that

$$\left\|\delta_k\right\|_{H^2(\Omega)}=\left\|\mathcal{R}\left(g_k-g\right)\right\|_{H^2(\Omega)}\leq C_{\mathcal{R}}\left\|g_k-g\right\|_{Y(\partial\Omega)}.$$

Hence

$$\delta_k\to 0 \text{ strongly in } H^2\left(\Omega\right). \tag{2.40}$$

Now we define

$$u_k=u+\delta_k\in A_{g_k}. \tag{2.41}$$

Thus by (2.40) and (2.41)

$$u_k\to u \text{ strongly in } H^2\left(\Omega\right). \tag{2.42}$$

Using Theorem 2.10, by extracting a subsequence and re-indexing it we have

$$u_k\to u \text{ in } C^0\left(\overline{\Omega}\right) \tag{2.43}$$

$$Du_k \to Du \text{ strongly in } L^p(\Omega) \;\; 1 \le p < \infty. \tag{2.44}$$

By (2.43), there exists a number $M > 0$ such that

$$\|u_k\|_{L^\infty(\Omega)} < M.$$

Moreover, recalling ϕ from (2.2) there exists $\tilde{M} > 0$ such that

$$\|\phi(u_k)\|_{L^\infty(\Omega)} \le \tilde{M}. \tag{2.45}$$

Furthermore, using (2.44), by Brezis ([**3**],Theorem 4.9) there exist a subsequence $(u_{n_k})_{k=1}^\infty$ and a function $\hat{p} \in L^p(\Omega)$ such that

$$|Du_{n_k}| \le \hat{p}, \quad \forall k, \text{ a.e. on } \Omega. \tag{2.46}$$

Similarly, using (2.42), by Brezis ([**3**],Theorem 4.9) there exist a subsequence $\left(u_{n_{m_k}}\right)_{k=1}^\infty$ and a function $\hat{r} \in L^2(\Omega)$ such that

$$\left|D^2 u_{n_{m_k}}\right| \le \hat{r}, \quad \forall k, \text{ a.e. on } \Omega. \tag{2.47}$$

Next, we denote by $(u_k)_{k=1}^\infty$ the subsequence $\left(u_{n_{m_k}}\right)_{k=1}^\infty$. Hence (2.2), (2.45), (2.46) and (2.47) imply

$$\left|f\left(u_k, Du_k, D^2u_k\right)\right| \le |\phi(u_k)| + b' |Du_k|^\beta + c' \left|D^2 u_k\right|^2 \le \tilde{M} + b' |\hat{p}|^\beta + c' |\hat{r}|^2.$$

Consequently, it follows from the Dominated Convergence Theorem that

$$\lim_{k\to\infty} I_\Omega[u_k] = I_\Omega[u]. \tag{2.48}$$

By the definition of V_Ω

$$I_\Omega[u_k] \ge V_\Omega[g_k]. \tag{2.49}$$

Hence $$V_\Omega[g] = I_\Omega[u] = \limsup_{k\to\infty} I_\Omega[u_k] \ge \limsup_{k\to\infty} V_\Omega[g_k].$$ □

3. Some preliminary results

Given a rectangular domain $\Omega = (l_0, l_1) \times (0, L)$, we denote by

$$\begin{aligned} \Gamma_1 &= \{l_0\} \times (0, L) \\ \Gamma_2 &= (l_0, l_1) \times \{0\} \\ \Gamma_3 &= \{l_1\} \times (0, L) \\ \Gamma_4 &= (l_0, l_1) \times \{L\} \end{aligned}$$

the edges of Ω, where $0 < L,\ l_0 < l_1$.

Definition 3.1. Let $m \ge 0$ be an integer. We denote by $H^{m+\frac{1}{2}}(0, L)$, the space of all distributions f defined in $(0, L)$, such that $f \in H^m(0, L)$ and

$$\iint_{(0,L)\times(0,L)} \frac{\left|f^{(m)}(t) - f^{(m)}(s)\right|^2}{|t-s|^2} dtds < \infty,$$

with a norm

$$\|f\|_{H^{m+\frac{1}{2}}(0,L)} = \left(\|f\|^2_{H^m(0,L)} + \int\int_{(0,L)\times(0,L)} \frac{\left|f^{(m)}(t) - f^{(m)}(s)\right|^2}{|t-s|^2} dtds \right)^{\frac{1}{2}}.$$

We denote by X the following subspace of $H^{\frac{3}{2}}(0,L) \times H^{\frac{1}{2}}(0,L)$,

$$X = \{\xi = \left(\xi^0, \xi^1\right) \in H^{\frac{3}{2}}(0,L) \times H^{\frac{1}{2}}(0,L) \mid \xi^0(0) = \xi^0(L) = 0, \tag{3.1}$$
$$\int_0^L \left|\tfrac{d^l \xi^k}{dt^l}\right|^2 \tfrac{dt}{\min\{t, L-t\}} < \infty,\ k+l=1\},$$

equipped with the norm

$$\|\xi\|^2_X = \left\|\xi^0\right\|^2_{H^{\frac{3}{2}}(0,L)} + \left\|\xi^1\right\|^2_{H^{\frac{1}{2}}(0,L)} + 2\sum_{\substack{0\le l,k\le 1\\ k+l=1}} \int_{(0,L)} \left|D^l\xi^k\right|^2 \frac{dt}{t} + 2\sum_{\substack{0\le l,k\le 1\\ k+l=1}} \int_{(0,L)} \left|D^l\xi^k\right|^2 \frac{dt}{L-t}. \tag{3.2}$$

The motivation for introducing X is given by the following Proposition. It shows that the space $X \times X$ is isomorphic to the space $Z(\partial\Omega_l)$ defined by

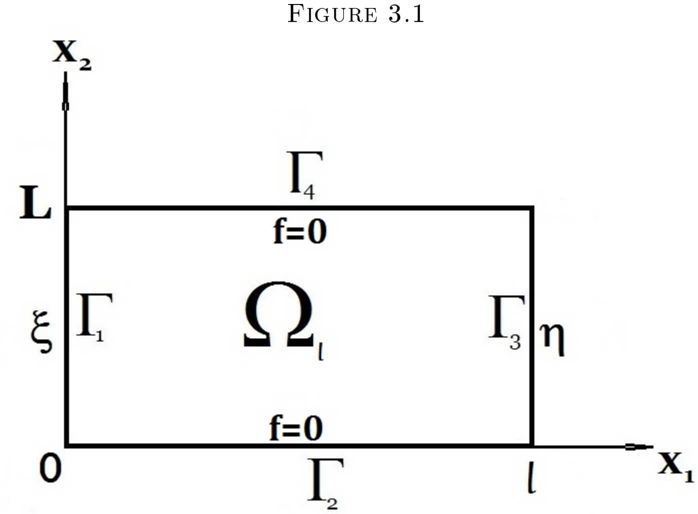
$$Z(\partial\Omega_l) = \left\{f \in Y(\partial\Omega_l) :\ \left(f_2^0, f_2^1\right) = \left(f_4^0, f_4^1\right) = (0,0)\right\}.$$

PROPOSITION 3.2. *Let $0 < c_0 < c_1$, $L > 0$ and let $\Omega_l = (0,l) \times (0,L)$ for $c_0 \le l \le c_1$. For $f \in Z(\partial\Omega_l)$, denote $\xi = \left(f_1^0, f_1^1\right)$ and $\eta = \left(f_3^0, f_3^1\right)$ (see Figure 3.1). Then following relation holds:*

$$K_1\left(\|\xi\|^2_X + \|\eta\|^2_X\right) \le \|f\|^2_{Z(\partial\Omega_l)} \le K_2\left(\|\xi\|^2_X + \|\eta\|^2_X\right) \tag{3.3}$$

where $K_1 = \min\left\{1, \arctan\left(\frac{c_0}{L}\right)\right\}$ *and* $K_2 = \left(1 + \frac{4L}{c_0^2} + \frac{\pi}{2}\right)$.

FIGURE 3.1



PROOF. Given $f = (f_0, f_1) = \left(\left\{f_j^0\right\}_{1\le j\le 4}, \left\{f_j^1\right\}_{1\le j\le 4}\right) \in Z(\partial\Omega_l)$ we compute

$$(3.4)\qquad \|f\|^2_{Z(\partial\Omega_l)} = \|f\|^2_{H^1(\partial\Omega_l)\times L^2(\partial\Omega_l)} + \sum_{\substack{0\le k,m\le 1\\ k+m=1}} \iint\limits_{\partial\Omega_l\times\partial\Omega_l} \frac{\left|D^k f_m(x) - D^k f_m(y)\right|^2}{|x-y|^2} d\sigma(x)\, d\sigma(y)$$

$$= \sum_{\substack{0\le k,m\le 1\\ k+m=1\\ 0\le j\le 4}} \int\limits_{\Gamma_j} \left|D^k f_m\right|^2 d\sigma(x) + \sum_{\substack{0\le k,m\le 1\\ k+m=1\\ 0\le i,j\le 4}} \iint\limits_{\Gamma_j\times\Gamma_i} \frac{\left|D^k f_m(x) - D^k f_m(y)\right|^2}{|x-y|^2} d\sigma(x)\, d\sigma(y)$$

$$= \|\xi_0\|^2_{H^{\frac{3}{2}}(0,L)} + \|\xi_1\|^2_{H^{\frac{1}{2}}(0,L)} + \|\eta_0\|^2_{H^{\frac{3}{2}}(0,L)} + \|\eta_1\|^2_{H^{\frac{1}{2}}(0,L)}$$

$$+2\iint\limits_{\Gamma_1\times\Gamma_2} \frac{|\xi_0'(x)|^2}{|x-y|^2} d\sigma(x)\, d\sigma(y) + 2\iint\limits_{\Gamma_1\times\Gamma_2} \frac{|\xi_1(x)|^2}{|x-y|^2} d\sigma(x)\, d\sigma(y)$$

$$+2\iint\limits_{\Gamma_3\times\Gamma_2} \frac{|\eta_0'(x)|^2}{|x-y|^2} d\sigma(x)\, d\sigma(y) + 2\iint\limits_{\Gamma_3\times\Gamma_2} \frac{|\eta_1(x)|^2}{|x-y|^2} d\sigma(x)\, d\sigma(y)$$

$$+2\iint\limits_{\Gamma_1\times\Gamma_3} \frac{|\xi_0'(x) - \eta_0'(y)|^2}{|x-y|^2} d\sigma(x)\, d\sigma(y) + 2\iint\limits_{\Gamma_1\times\Gamma_3} \frac{|\xi_1(x) - \eta_1(y)|^2}{|x-y|^2} d\sigma(x)\, d\sigma(y)$$

$$+2\iint\limits_{\Gamma_1\times\Gamma_4} \frac{|\xi_0'(x)|^2}{|x-y|^2} d\sigma(x)\, d\sigma(y) + 2\iint\limits_{\Gamma_1\times\Gamma_4} \frac{|\xi_1(x)|^2}{|x-y|^2} d\sigma(x)\, d\sigma(y)$$

$$+2\iint\limits_{\Gamma_3\times\Gamma_4} \frac{|\eta_0'(x)|^2}{|x-y|^2} d\sigma(x)\, d\sigma(y) + 2\iint\limits_{\Gamma_3\times\Gamma_4} \frac{|\eta_1(x)|^2}{|x-y|^2} d\sigma(x)\, d\sigma(y).$$

For every $\varphi \in L^2(\Gamma_1)$ we have

$$\iint\limits_{\Gamma_1\times\Gamma_2} \frac{|\varphi(x)|^2}{|x-y|^2} d\sigma(x)\, d\sigma(y) = \int_0^L \int_0^l \frac{|\varphi(x_2)|^2}{x_1^2+x_2^2} dx_1 dx_2 = \int_0^L \arctan\left(\frac{l}{x_2}\right) \frac{|\varphi(x_2)|^2}{x_2} dx_2$$

and

$$\iint\limits_{\Gamma_1\times\Gamma_4} \frac{|\varphi(x)|^2}{|x-y|^2} d\sigma(x)\, d\sigma(y) = \int_0^L \int_0^l \frac{|\varphi(x_2)|^2}{x_1^2+x_2^2} dx_1 dx_2 = \int_0^L \arctan\left(\frac{l}{L-x_2}\right) \frac{|\varphi(x_2)|^2}{L-x_2} dx_2.$$

Since

$$\arctan\left(\frac{c_0}{L}\right) \le \arctan\left(\frac{l}{x_2}\right), \arctan\left(\frac{l}{L-x_2}\right) \le \frac{\pi}{2},\ 0 < x_2 < L$$

we obtain

$$(3.5)\qquad \arctan\left(\frac{c_0}{L}\right) \int_0^L \frac{|\varphi(x_2)|^2}{x_2} dx_2 \le \iint\limits_{\Gamma_1\times\Gamma_2} \frac{|\varphi(x)|^2}{|x-y|^2} d\sigma(x)\, d\sigma(y) \le \frac{\pi}{2} \int_0^L \frac{|\varphi(x_2)|^2}{x_2} dx_2,$$

$$(3.6)\qquad \arctan\left(\frac{c_0}{L}\right)\int_0^L \frac{|\varphi(x_2)|^2}{L-x_2}dx_2 \le \iint_{\Gamma_1\times\Gamma_4}\frac{|\varphi(x)|^2}{|x-y|^2}d\sigma(x)\,d\sigma(y) \le \frac{\pi}{2}\int_0^L \frac{|\varphi(x_2)|^2}{L-x_2}dx_2,$$

and similarly, for every $\varphi \in L^2(\Gamma_3)$ we have:

$$(3.7)\qquad \arctan\left(\frac{c_0}{L}\right)\int_0^L \frac{|\varphi(x_2)|^2}{x_2}dx_2 \le \iint_{\Gamma_3\times\Gamma_2}\frac{|\varphi(x)|^2}{|x-y|^2}d\sigma(x)\,d\sigma(y) \le \frac{\pi}{2}\int_0^L \frac{|\varphi(x_2)|^2}{x_2}dx_2,$$

$$(3.8)\qquad \arctan\left(\frac{c_0}{L}\right)\int_0^L \frac{|\varphi(x_2)|^2}{L-x_2}dx_2 \le \iint_{\Gamma_3\times\Gamma_4}\frac{|\varphi(x)|^2}{|x-y|^2}d\sigma(x)\,d\sigma(y) \le \frac{\pi}{2}\int_0^L \frac{|\varphi(x_2)|^2}{L-x_2}dx_2.$$

Additionally for every $\varphi \in H^{\frac{1}{2}}(\Gamma_1)$, $\psi \in H^{\frac{1}{2}}(\Gamma_3)$ we have

$$\iint_{\Gamma_1\times\Gamma_3}\frac{|\varphi(x)-\psi(y)|^2}{|x-y|^2}d\sigma(x)\,d\sigma(y) \le 2\int_0^L\int_0^L \frac{|\varphi(x_2)|^2+|\psi(y_2)|^2}{(x_2-y_2)^2+l^2}dx_2dy_2$$

and since

$$\frac{1}{(x_2-y_2)^2+l^2} \le \frac{1}{c_0^2},\ 0<x_2,y_2<L$$

we obtain

$$(3.9)\qquad 0 \le \iint_{\Gamma_1\times\Gamma_3}\frac{|\varphi(x)-\psi(y)|^2}{|x-y|^2}d\sigma(x)\,d\sigma(y) \le \frac{2L}{c_0^2}\left(\|\varphi\|^2_{L^2(\Gamma_1)}+\|\psi\|^2_{L^2(\Gamma_3)}\right).$$

Now, applying right hand side inequality in (3.5) – (3.9) on (3.4) yields

$$\|f\|^2_{Z(\partial\Omega_l)} \le \|\xi_0\|^2_{H^{\frac{3}{2}}(0,L)}+\|\xi_1\|^2_{H^{\frac{1}{2}}(0,L)}+\|\eta_0\|^2_{H^{\frac{3}{2}}(0,L)}+\|\eta_1\|^2_{H^{\frac{1}{2}}(0,L)}$$

$$+2\cdot\frac{\pi}{2}\int_0^L\frac{|\xi_0'(x_2)|^2}{x_2}dx_2+2\cdot\frac{\pi}{2}\int_0^L\frac{|\xi_1(x_2)|^2}{x_2}dx_2+2\cdot\frac{\pi}{2}\int_0^L\frac{|\eta_0'(x_2)|^2}{x_2}dx_2+2\cdot\frac{\pi}{2}\int_0^L\frac{|\eta_1(x_2)|^2}{x_2}dx_2$$

$$+2\cdot\frac{2L}{c_0^2}\left(\|\xi_0'\|^2_{L^2(\Gamma_1)}+\|\eta_0'\|^2_{L^2(\Gamma_3)}\right)+2\cdot\frac{2L}{c_0^2}\left(\|\xi_1\|^2_{L^2(\Gamma_1)}+\|\eta_1\|^2_{L^2(\Gamma_3)}\right)$$

$$+2\cdot\frac{\pi}{2}\int_0^L\frac{|\xi_0'(x_2)|^2}{L-x_2}dx_2+2\cdot\frac{\pi}{2}\int_0^L\frac{|\xi_1(x_2)|^2}{L-x_2}dx_2+2\cdot\frac{\pi}{2}\int_0^L\frac{|\eta_0'(x_2)|^2}{L-x_2}dx_2+2\cdot\frac{\pi}{2}\int_0^L\frac{|\eta_1(x_2)|^2}{L-x_2}dx_2$$

$$\le\left(1+\frac{4L}{c_0^2}+\frac{\pi}{2}\right)\left(\|\xi\|^2_X+\|\eta\|^2_X\right)$$

where we have used (3.2) to obtain the last inequality.

On the other hand, applying left hand side inequality in (3.5) – (3.9) on (3.4) yields

$$\|f\|^2_{Z(\partial\Omega_l)} \ge \|\xi_0\|^2_{H^{\frac{3}{2}}(0,L)}+\|\xi_1\|^2_{H^{\frac{1}{2}}(0,L)}+\|\eta_0\|^2_{H^{\frac{3}{2}}(0,L)}+\|\eta_1\|^2_{H^{\frac{1}{2}}(0,L)}$$

$$+2\arctan\left(\frac{c_0}{L}\right)\int_0^L\frac{\left|\xi_0'(x_2)\right|^2}{x_2}dx_2+2\arctan\left(\frac{c_0}{L}\right)\int_0^L\frac{\left|\xi_1(x_2)\right|^2}{x_2}dx_2$$

$$+2\arctan\left(\frac{c_0}{L}\right)\int_0^L\frac{\left|\eta_0'(x_2)\right|^2}{x_2}dx_2+2\arctan\left(\frac{c_0}{L}\right)\int_0^L\frac{\left|\eta_1(x_2)\right|^2}{x_2}dx_2$$

$$+2\arctan\left(\frac{c_0}{L}\right)\int_0^L\frac{\left|\xi_0'(x_2)\right|^2}{L-x_2}dx_2+2\arctan\left(\frac{c_0}{L}\right)\int_0^L\frac{\left|\xi_1(x_2)\right|^2}{L-x_2}dx_2$$

$$+2\arctan\left(\frac{c_0}{L}\right)\int_0^L\frac{\left|\eta_0'(x_2)\right|^2}{L-x_2}dx_2+2\arctan\left(\frac{c_0}{L}\right)\int_0^L\frac{\left|\eta_1(x_2)\right|^2}{L-x_2}dx_2$$

$$\geq\min\left\{1,\arctan\left(\frac{c_0}{L}\right)\right\}\cdot\left(\|\xi\|_X^2+\|\eta\|_X^2\right).$$

Thus, by setting $K_1=\min\left\{1,\arctan\left(\frac{c_0}{L}\right)\right\}$ and $K_2=\left(1+\frac{4L}{c_0^2}+\frac{\pi}{2}\right)$ we obtain (3.3). □

REMARK 3.3. For $c_0\leq l$ it is an immediate conclusion from Proposition 3.2 that the spaces $\left\{f\in Y\left(\partial\Omega_l\right):\ \left(f_1^0,f_1^1\right)=\left(f_2^0,f_2^1\right)=\left(f_4^0,f_4^1\right)=(0,0)\right\}$ and X are isomorphic.

COROLLARY 3.4. *Let $0<c_0$, $L>0$ and let $\Omega_l=(0,l)\times(0,L)$, $\Omega_r=(0,r)\times(0,L)$ for r,l satisfying $c_0\leq r,l$. The spaces $Z\left(\partial\Omega_l\right)$ and $Z\left(\partial\Omega_r\right)$ are isomorphic.*

Note that with a certain abuse of notation, we can view each $f\in Z\left(\partial\Omega_l\right)$ as belonging also to $Z\left(\partial\Omega_r\right)$ and vice versa.

PROOF. By (3.3), we have for every $f\in Z\left(\partial\Omega_l\right)$

$$\mathcal{K}_1\|f\|_{Z(\partial\Omega_r)}\leq\|f\|_{Z(\partial\Omega_l)}\leq\mathcal{K}_2\|f\|_{Z(\partial\Omega_r)}\tag{3.10}$$

where $\mathcal{K}_1=\sqrt{\frac{K_1}{K_2}}$ and $\mathcal{K}_2=\sqrt{\frac{K_2}{K_1}}$. □

In the sequel it will be convenient to associate with each pair $(\varphi,\psi)\in X\times X$ a function $f=\left\{f_j^k\right\}_{\substack{1\leq j\leq 4\\0\leq k\leq 1}}\in Z\left(\partial\Omega_l\right)$ given by

$$\begin{array}{c}\left(f_1^0,f_1^1\right)=\left(\varphi^0,-\varphi^1\right)\\\left(f_2^0,f_2^1\right)=(0,0)\\\left(f_3^0,f_3^1\right)=\left(\psi^0,\psi^1\right)\\\left(f_4^0,f_4^1\right)=(0,0)\end{array}.\tag{3.11}$$

Note that this association corresponds to an isomorphism between $X\times X$ and $Z\left(\partial\Omega_l\right)$. Indeed, by Proposition 3.2 we have

$$\sqrt{\frac{K_1}{2}}\left(\|\varphi\|_X+\|\psi\|_X\right)\leq\|f\|_{Z(\partial\Omega_l)}\leq\sqrt{K_2}\left(\|\varphi\|_X+\|\psi\|_X\right)\ .\tag{3.12}$$

In the sequel we shall always denote by γ^Ω the trace operator on $\partial\Omega$.

We denote the following subspace of $H^2(\Omega_l)$ defined by

$$\mathcal{A}(\Omega_l) = \left\{ v \in H^2(\Omega_l) : \left(\gamma^{\Omega_l} v, \gamma^{\Omega_l} \frac{\partial v}{\partial \nu} \right) \in Z(\partial \Omega_l) \right\}. \tag{3.13}$$

For the space $\mathcal{A}(\Omega_l)$ we have an improved coercivity result than the one of Proposition 2.7.

PROPOSITION 3.5. *There exist $A = A(\Omega_l) > 0$ and $B = B(\Omega_l) > 0$ such that*

$$I_{\Omega_l}[u] \geq A \|u\|^2_{H^2(\Omega_l)} - B, \quad \forall u \in \mathcal{A}(\Omega_l). \tag{3.14}$$

PROOF. Let $u \in \mathcal{A}(\Omega_l)$. It follows that for a.e. $t \in (0, l)$ we have $u \in H_0^2(\Gamma^t)$ where $\Gamma^t = \{t\} \times (0, L)$. Therefore by the one-dimensional Poincare inequality

$$\int_{\Gamma^t} u^2 dx_2 \leq L^2 \int_{\Gamma^t} u_{x_2}^2 dx_2 \tag{3.15}$$

and

$$\int_{\Gamma^t} u_{x_j}^2 dx_2 \leq L^2 \int_{\Gamma^t} u_{x_2 x_j}^2 dx_2, \quad j = 1, 2. \tag{3.16}$$

Integration of (3.15) and (3.16) over $t \in (0, l)$ yields the following Poincare inequalities:

$$\int_{\Omega_l} u^2 dx \leq L^2 \int_{\Omega_l} |Du|^2 dx, \tag{3.17}$$

$$\int_{\Omega_l} |Du|^2 dx \leq L^2 \int_{\Omega_l} \left|D^2 u\right|^2 dx. \tag{3.18}$$

Applying (3.17) – (3.18) to the right hand side of (2.8) yields (3.14) with

$$A(\Omega_l) = \frac{(1-\eta)c}{3} \min\left\{1, \frac{1}{L^4}\right\} \tag{3.19}$$

and

$$B(\Omega_l) = (\eta(Q + c) + d)|\Omega_l|. \tag{3.20}$$

□

DEFINITION 3.6. For $\varphi, \psi \in X$ we define

$$U_{\Omega_l}(\varphi, \psi) = V_{\Omega_l}[f]$$

where $f \in Z(\partial\Omega_l)$ as defined in (3.11).

REMARK 3.7. Recall that $V_\Omega[g] = \min\limits_{u \in A_g} I_\Omega[u]$ where

$$A_g(\Omega) = \left\{ v \in H^2(\Omega) : \left\{ \gamma_j^\Omega \frac{\partial^k v}{\partial \nu_j^k} \right\}_{\substack{1 \leq j \leq 4 \\ 0 \leq k \leq 1}} = g \right\}$$

($\gamma_j^\Omega \frac{\partial v}{\partial \nu_j}$ denotes the normal derivative on Γ_j oriented outwards of Ω for $1 \leq j \leq 4$). Thus by choosing $g \in Z(\partial\Omega)$ we confine ourselves to functions $u \in H^2(\Omega)$ for which

$$\gamma^\Omega u = \begin{cases} \gamma_1^\Omega u = \varphi^0 \\ \gamma_2^\Omega u = 0 \\ \gamma_3^\Omega u = \psi^0 \\ \gamma_4^\Omega u = 0 \end{cases} \quad \text{and} \quad \gamma^\Omega \frac{\partial u}{\partial \nu} = \begin{cases} \gamma_1^\Omega \frac{\partial u}{\partial \nu_1} = -\varphi^1 \\ \gamma_2^\Omega \frac{\partial u}{\partial \nu_2} = 0 \\ \gamma_3^\Omega \frac{\partial u}{\partial \nu_3} = \psi^1 \\ \gamma_4^\Omega \frac{\partial u}{\partial \nu_4} = 0 \end{cases} .$$

Definition 3.8. For Ω a rectangular domain we define the mapping $\mathcal{T}^\Omega : \mathcal{A}(\Omega) \to X \times X$ by

$$\mathcal{T}^\Omega[u] = \left(\mathcal{T}_1^\Omega[u], \mathcal{T}_2^\Omega[u]\right) := \left(\left(\gamma_1^\Omega u, -\gamma_1^\Omega \frac{\partial u}{\partial \nu_1}\right), \left(\gamma_3^\Omega u, \gamma_3^\Omega \frac{\partial u}{\partial \nu_3}\right)\right).$$

Remark 3.9. For $l_0 < l_1 < l_2$, we set $\Omega_1 = (l_0, l_1) \times (0, L)$, $\Omega_2 = (l_1, l_2) \times (0, L)$. Using integration by parts (cf. **[7]** Lemma 1.5.3.2) it is easy to show that given $u_1 \in \mathcal{A}(\Omega_1)$ and $u_2 \in \mathcal{A}(\Omega_2)$ such that $\mathcal{T}_2^{\Omega_1}[u_1] = \mathcal{T}_1^{\Omega_2}[u_2]$ we have $(u_1\chi_{\Omega_1} + u_2\chi_{\Omega_2}) \in \mathcal{A}((l_0, l_2) \times (0, L))$.

These two remarks allow us to prove the following Lemma.

Lemma 3.10. *Let $l_0 < l_1$, we denote $\Omega = (l_0, l_1) \times (0, L)$. Let $u \in \mathcal{A}(\Omega)$ for which $\mathcal{T}^\Omega[u] = (\varphi, \psi)$,* such that $I_\Omega[u] = U_\Omega(\varphi, \psi)$.

Then, for every $l_0 < c < l_1$ we have

$$U_\Omega(\varphi, \psi) = U_{\Omega_1}(\varphi, \xi) + U_{\Omega_2}(\xi, \psi)$$

where $\Omega_1 = (l_0, c) \times (0, L)$, $\Omega_2 = (c, l_1) \times (0, L)$ and $\xi = \mathcal{T}_2^{\Omega_1}[u] = \mathcal{T}_1^{\Omega_2}[u]$.

Proof. It is easy to see that

$$U_\Omega(\varphi, \psi) \geq U_{\Omega_1}(\varphi, \xi) + U_{\Omega_2}(\xi, \psi).$$

Now assume that

$$U_\Omega(\varphi, \psi) > U_{\Omega_1}(\varphi, \xi) + U_{\Omega_2}(\xi, \psi).$$

Thus without loss of generality we may as well assume that

$$I_{\Omega_1}[u|_{\Omega_1}] > U_{\Omega_1}(\varphi, \xi).$$

From Theorem 2.12 we know that there exists $v \in \mathcal{A}(\Omega_1)$ such that $\mathcal{T}^{\Omega_1}[v] = (\varphi, \xi)$ for which $I_{\Omega_1}[v] = U_{\Omega_1}(\varphi, \xi)$. Hence

$$I_\Omega[v\chi_{\Omega_1} + u\chi_{\Omega_2}] < U_\Omega(\varphi, \psi)$$

which is a contradiction. □

Lemma 3.11. *Let $0 < c_0 < c_1$ and $L > 0$. We denote $\Omega_l = (0, l) \times (0, L)$. For every $c_0 \leq l \leq c_1$ the following relation holds:*

1. There exists $C_{\mathcal{R}} = C_{\mathcal{R}}(c_0, c_1, L) > 0$ such that

$$\|\mathcal{R}f\|_{H^2(\Omega_l)} \leq C_{\mathcal{R}} \|f\|_{Z(\partial\Omega_l)}, \ f \in Z(\partial\Omega_l).$$

2. There exists $C = C(c_0, c_1, L) > 0$ such that

$$\|u\|_{C^0(\overline{\Omega}_l)} \leq C \|u\|_{H^2(\Omega_l)}, \ u \in H^2(\Omega_l).$$

PROOF. For every $c_0 \leq l \leq c_1$ set

$$C_{\mathcal{R}}(\Omega_l) = \sup_{f \in Z(\partial\Omega_l)} \frac{\left\|\mathcal{R}^l f\right\|_{H^2(\Omega_l)}}{\|f\|_{Z(\partial\Omega_l)}},$$

which is the operator norm of $\mathcal{R}^l$, the inverse mapping of the trace operator for Ω_l, introduced in Theorem 2.5, restricted to $Z(\partial\Omega_l)$. Fix any $g \neq 0$ in $Z(\partial\Omega_l)$. We can view g also as belonging to $Z(\partial\Omega_{c_1})$ and take $v = \mathcal{R}^{c_1} g$.

Define $u_l \in A_g(\Omega_l)$ by $u_l(x_1, x_2) = v\left(\frac{c_1}{l} x_1, x_2\right)$. Next we compute

$$\|u_l\|^2_{H^2(\Omega_l)} = \int_0^L \int_0^l \left(u_l^2 + (u_l)^2_{x_1} + (u_l)^2_{x_2} + (u_l)^2_{x_1x_1} + 2(u_l)^2_{x_1x_2} + (u_l)^2_{x_2x_2}\right) dx_1 dx_2$$

$$= \int_0^L \int_0^l \left(v^2 + \left(\frac{c_1}{l}\right)^2 v^2_{x_1} + v^2_{x_2} + \left(\frac{c_1}{l}\right)^4 v^2_{x_1x_1} + 2\left(\frac{c_1}{l}\right)^2 v^2_{x_1x_2} + v^2_{x_2x_2}\right)\left(\frac{c_1}{l}x_1, x_2\right) dx_1 dx_2$$

$$= \int_0^L \int_0^{c_1} \left(v^2 + \left(\frac{c_1}{l}\right)^2 v^2_{x_1} + v^2_{x_2} + \left(\frac{c_1}{l}\right)^4 v^2_{x_1x_1} + 2\left(\frac{c_1}{l}\right)^2 v^2_{x_1x_2} + v^2_{x_2x_2}\right) \frac{l}{c_1} dx_1 dx_2$$

$$\leq \left(\frac{c_1}{l}\right)^3 \|v\|^2_{H^2(\Omega_{c_1})} \leq \left(\frac{c_1}{c_0}\right)^3 \|v\|^2_{H^2(\Omega_{c_1})}.$$

Therefore, recalling the definition of $\mathcal{R}$ and using Corollary 3.4 we get,

$$\frac{\left\|\mathcal{R}^l g\right\|_{H^2(\Omega_l)}}{\|g\|_{Z(\partial\Omega_l)}} \leq \frac{\|u_l\|_{H^2(\Omega_l)}}{\|g\|_{Z(\partial\Omega_l)}} \leq \frac{\left(\frac{c_1}{c_0}\right)^{\frac{3}{2}} \|v\|_{H^2(\Omega_{c_1})}}{\mathcal{K}_1 \|g\|_{Z(\partial\Omega_{c_1})}} = \frac{\left(\frac{c_1}{c_0}\right)^{\frac{3}{2}}}{\mathcal{K}_1} C_{\mathcal{R}}(\Omega_{c_1}). \tag{3.21}$$

Taking the supremum on g in (3.21) yields

$$C_{\mathcal{R}}(\Omega_l) \leq C_{\mathcal{R}}(c_0, c_1, L) := \frac{\left(\frac{c_1}{c_0}\right)^{\frac{3}{2}}}{\mathcal{K}_1} C_{\mathcal{R}}(\Omega_{c_1}).$$

This completes the proof of Assertion 1.

Next we will prove Assertion 2. By Theorem 2.10 there is $u^l \in H^2(\Omega_l)$ such that

$$C(\Omega_l) := \max_{v \in H^2(\Omega_l)} \frac{\|v\|_{C^0(\overline{\Omega}_l)}}{\|v\|_{H^2(\Omega_l)}} = \frac{\left\|u^l\right\|_{C^0(\overline{\Omega}_l)}}{\|u^l\|_{H^2(\Omega_l)}}.$$

We define $w \in H^2(\Omega_{c_1})$ by $u^l(x_1, x_2) = w\left(\frac{c_1}{l} x_1, x_2\right)$ and compute:

$$\left\|u^l\right\|^2_{H^2(\Omega_l)} = \int_0^L \int_0^{c_1} \left(w^2 + \left(\frac{c_1}{l}\right)^2 w^2_{x_1} + w^2_{x_2} + \left(\frac{c_1}{l}\right)^4 w^2_{x_1x_1}\right.$$

$$+2\left(\frac{c_1}{l}\right)^2 w_{x_1x_2}^2 + w_{x_2x_2}^2\right) \frac{l}{c_1} dx_1 dx_2 \geq \frac{l}{c_1} \|w\|^2_{H^2(\Omega_{c_1})} \geq \frac{c_0}{c_1} \|w\|^2_{H^2(\Omega_{c_1})}.$$

Since $\|w\|_{C^0(\overline{\Omega}_{c_1})} = \left\|u^l\right\|_{C^0(\overline{\Omega}_l)}$, by the above calculation and the definition of $C(\Omega_{c_1})$ we finally obtain,

$$C(\Omega_l) = \frac{\left\|u^l\right\|_{C^0(\overline{\Omega}_l)}}{\left\|u^l\right\|_{H^2(\Omega_l)}} \leq \frac{\|w\|_{C^0(\overline{\Omega}_{c_1})}}{\left(\frac{c_0}{c_1}\right)^{\frac{1}{2}} \|w\|_{H^2(\Omega_{c_1})}} \leq \left(\frac{c_1}{c_0}\right)^{\frac{1}{2}} C(\Omega_{c_1}) := C(c_0, c_1, L).$$

□

The following four propositions and theorem have been introduced by Zaslavski in the one-dimensional setting (see [**16**]).

PROPOSITION 3.12. *Let $M_1 > 0$, $0 < c_0 < c_1$ and $L > 0$ be given. Then there exists a number $M_2 > 0$ such that for each pair of numbers l_1, l_2 satisfying*

$$0 \leq l_1 < l_2,\ l_2 - l_1 \in [c_0, c_1] \tag{3.22}$$

and each function $u \in \mathcal{A}((l_1, l_2) \times (0, L))$ which satisfies

$$I_{(l_1,l_2)\times(0,L)}[u] \leq M_1, \tag{3.23}$$

the following inequality holds:

$$\left\|\mathcal{T}_2^{(l_1,c)\times(0,L)}[u]\right\|_X \leq M_2,\ l_1 \leq c \leq l_2. \tag{3.24}$$

PROOF. First, let us denote $\Omega = (l_1, l_2) \times (0, L)$ and

$$\tilde{\Omega}_c = \begin{cases} \left(c - \frac{c_0}{2}, c\right) \times (0, L), & c \geq \frac{l_1 + l_2}{2} \\ \left(c, c + \frac{c_0}{2}\right) \times (0, L), & c < \frac{l_1 + l_2}{2} \end{cases}.$$

By the coercivity of I_Ω (see (3.14)), taking into account the explicit expression for A and B in (3.19) – (3.20), and using assertion 1 of Lemma 3.11 we deduce that there exist $\tilde{A} = \tilde{A}(c_0, c_1, L)$ and $\tilde{B} = \tilde{B}(c_0, c_1, L)$ such that

$$\tilde{A}\|u\|^2_{H^2(\Omega)} - \tilde{B} \leq M_1,\ u \in \mathcal{A}(\Omega).$$

Hence

$$\|u|_{\Omega_c}\|^2_{H^2(\Omega_c)} \leq \|u\|^2_{H^2(\Omega)} \leq \left(\frac{M_1 + \tilde{B}}{\tilde{A}}\right).$$

Now we may assume without loss of generality that $\frac{l_1+l_2}{2} \leq c \leq l_2$. By (3.12), and Theorem 2.5 we obtain for some $C_{\mathcal{T}} = C_{\mathcal{T}}(c_0, L)$

$$\left\|\mathcal{T}_2^{\Omega_c}[u]\right\|^2_X \leq \frac{1}{\left(\frac{K_1}{2}\right)} \left\|\left(\gamma^{\Omega_c} u, \gamma^{\Omega_c} \frac{\partial u}{\partial \nu}\right)\right\|^2_{Y(\partial\Omega_c)} \leq \frac{C_{\mathcal{T}}}{\left(\frac{K_1}{2}\right)} \|u|_{\Omega_c}\|^2_{H^2(\Omega_c)}.$$

Finally we define

$$M_2 = \left(\frac{C_{\mathcal{T}}\left(M_1 + \tilde{B}\right)}{\left(\frac{K_1}{2}\right)\tilde{A}}\right)^{\frac{1}{2}}$$

as required.

□

PROPOSITION 3.13. *For all $0 < c_0 < c_1$, $L > 0$ and $c_2 > 0$ the set*

$$\begin{gathered}\{U_{(l_1,l_2)\times(0,L)}(\varphi,\psi) : l_1\in[0,\infty), l_2\in[l_1+c_0,l_1+c_1],\\ \varphi,\psi\in X,\ \|\varphi\|_X, \|\psi\|_X\le c_2\}\end{gathered}$$

is bounded.

Proof. For $0<l_1<l_2$ such that $c_0\le l_2-l_1\le c_1$, we set $\Omega=(l_1,l_2)\times(0,L)$. Now, given $\varphi,\psi\in X$ satisfying $\|\varphi\|_X, \|\psi\|_X\le c_2$ we define $f\in Z(\partial\Omega)$ as in (3.11). Thus, by (3.12) we have

$$\|f\|_{Y(\partial\Omega)}\le\sqrt{K_2}\left(\|\varphi\|_X+\|\psi\|_X\right)\le 2\sqrt{K_2}c_2.$$

Now, from Lemma 3.11 we have

$$\|\mathcal{R}f\|_{H^2(\Omega)}\le C_{\mathcal{R}}\|f\|_{Y(\partial\Omega)}\le C_{\mathcal{R}}\cdot 2\sqrt{K_2}c_2.$$

By Theorem 2.10 and Lemma 3.11 there exists $C=C(c_0,c_1,L)>0$ such that

$$\|\mathcal{R}f\|_{C(\bar\Omega)}\le C\|\mathcal{R}f\|_{H^2(\Omega)}\le C\cdot C_{\mathcal{R}}\cdot 2\sqrt{K_2}c_2.$$

Thus, using the continuity of ϕ, there exists $M>0$ such that $\|\phi(\mathcal{R}f)\|_{C(\overline{\Omega})}\le M$. Finally, since

$$|Du|^\beta\le\frac{\beta}{2}|Du|^2+\left(1-\frac{\beta}{2}\right),$$

recalling (2.2) and the lower bound of I_Ω(see (2.7)) we have

$$-Pc_1L\le U_\Omega(\varphi,\psi)\le I_\Omega(\mathcal{R}f)\le Mc_1L+(b'+c')\,C_{\mathcal{R}}\cdot 2\sqrt{K_2}c_2+b'c_1L.$$

Recalling the definition of P, there exists $\tilde P=\tilde P(c_0,c_1,L)$ >0 such that $P\le\tilde P$. Thus, we proved the Lemma. □

Next, we fix $\bar\xi\in X$ and let $0<c_0<c_1<\infty$, $L>0$. By Proposition 3.13 there exists a number M_0 such that

$$\begin{gathered}M_0\ge\sup\{|U_{(l_1,l_2)\times(0,L)}(\varphi,\psi)| : l_1\in[0,\infty), l_2\in[l_1+c_0,l_1+c_1],\\ \varphi,\psi\in X,\ \|\varphi\|_X, \|\psi\|_X\le 2\|\bar\xi\|_X+1\}.\end{gathered}\tag{3.25}$$

By Proposition 3.12 there exists a positive number M_1 such that

$$\begin{gathered}\inf\{U_{(l_1,l_2)\times(0,L)}(\varphi,\psi) : l_1\in[0,\infty), l_2\in[l_1+c_0,l_1+c_1],\\ \varphi,\psi\in X,\ \|\varphi\|_X+\|\psi\|_X\ge M_1\}>2M_0+1.\end{gathered}\tag{3.26}$$

Proposition 3.14. *For a positive number M_1 satisfying (3.26) and any $M_2>0$ there exists an integer $N>2$ such that:*

1. For each $l_1\in[0,\infty)$, each $\triangle l\in[c_0,c_1]$, each pair of integers k_1,k_2 satisfying $0\le k_1<k_2$, $k_2-k_1\ge N$ and each sequence $(\xi_i)_{i=k_1}^{k_2}\subset X$ satisfying $\{i\in\{k_1,\dots,k_2\} : \|\xi_i\|_X\le M_1\}=\{k_1,k_2\}$ the following relation holds:

$$\sum_{i=k_1}^{k_2-1}\left[U_{\Omega_i}(\xi_i,\xi_{i+1})-U_{\Omega_i}(\varphi_i,\varphi_{i+1})\right]\ge M_2\tag{3.27}$$

where $\varphi_i=\xi_i,\ i=k_1,k_2,\ \varphi_i=\bar\xi,\ i=k_1+1,\dots,k_2-1$ and $\Omega_i=(l_1+i\triangle l,l_1+(i+1)\triangle l)\times(0,L)$.

2. For each $l_1 \in [0,\infty)$ *, each* $\triangle l \in [c_0, c_1]$*, each pair of integers* k_1, k_2 *satisfying* $0 \leq k_1 < k_2$*,* $k_2 - k_1 \geq N$ *and each sequence* $(\xi_i)_{i=k_1}^{k_2} \subset X$ *satisfying* $\{i \in \{k_1, \dots, k_2\} : \|\xi_i\|_X \leq M_1\} = \{k_1\}$ *relation* (3.27) *holds with* $\varphi_{k_1} = \xi_{k_1}$ *,* $\varphi_i = \bar{\xi}$*,* $i = k_1 + 1, \dots, k_2$.

Proof. By Proposition 3.13 there exists a number $M_3 > 0$ such that

$$M_3 \geq \sup \left\{ \left| U_{(l_1,l_2)\times(0,L)}(\varphi,\psi) \right| : l_1 \in [0,\infty), l_2 \in [l_1 + c_0, l_1 + c_1], \right.$$
$$\left. \varphi,\psi \in X, \|\varphi\|_X, \|\psi\|_X \leq 2\left\|\bar{\xi}\right\|_X + 1 + 2M_1 \right\}.$$

Fix an integer $N \geq 2M_3 + M_2 + 1$. The validity of the proposition now follows from the definition of $U_{(l_1,l_2)\times(0,L)}, M_3, N$ and (3.25),(3.26). □

Proposition 3.15. *Assume that the positive number* M_1 *satisfies* (3.26) *and let* $M_3 > 0$*. Then there exists a number* $M_4 > M_1$ *such that:*

1. For each $l_1 \in [0,\infty)$ *, each* $\triangle l \in [c_0, c_1]$*, each pair of integers* k_1, k_2 *satisfying* $0 \leq k_1 < k_2$*, and each sequence* $(\xi_i)_{i=k_1}^{k_2} \subset X$ *satisfying*

$$\max\{\|\xi_{k_1}\|_X, \|\xi_{k_2}\|_X\} \leq M_1, \quad \max\{\|\xi_i\|_X : i = k_1+1, \dots, k_2-1\} > M_4 \tag{3.28}$$

there is a sequence $(\varphi_i)_{i=k_1}^{k_2} \subset X$ *that satisfies* $\varphi_{k_i} = \xi_{k_i}$*,* $i = 1, 2$*,*

$$\sum_{i=k_1}^{k_2-1} \left[U_{\Omega_i}(\xi_i, \xi_{i+1}) - U_{\Omega_i}(\varphi_i, \varphi_{i+1}) \right] \geq M_3 \tag{3.29}$$

where $\Omega_i = (l_1 + i\triangle l, l_1 + (i+1)\triangle l) \times (0, L)$.

2. For each $l_1 \in [0,\infty)$ *, each* $\triangle l \in [c_0, c_1]$*, each pair of integers* k_1, k_2 *satisfying* $0 \leq k_1 < k_2$*, and each sequence* $(\xi_i)_{i=k_1}^{k_2} \subset X$ *satisfying*

$$\|\xi_{k_1}\|_X \leq M_1, \quad \max\{\|\xi_i\|_X : i = k_1+1, \dots, k_2\} > M_4 \tag{3.30}$$

there is a sequence $(\varphi_i)_{i=k_1}^{k_2} \subset X$ *that satisfies* $\varphi_{k_1} = \xi_{k_1}$ and (3.29).

Proof. There exists an integer $N > 2$ such that Proposition 3.14 holds with $M_2 = 4(M_3 + 1)$. By Proposition 3.13 there exists a number r such that

$$r > \sup \left\{ \left| U_{(l_1,l_2)\times(0,L)}(\varphi,\psi) \right| : l_1 \in [0,\infty), l_2 \in [l_1 + c_0, l_1 + c_1], \varphi,\psi \in X, \right.$$
$$\left. \|\varphi\|_X, \|\psi\|_X \leq \left\|\bar{\xi}\right\|_X + 1 + M_1 \right\}. \tag{3.31}$$

By Proposition 3.12 there exists a positive number $M_4 > M_1$ such that

$$\inf \left\{ U_{(l_1,l_2)\times(0,L)}(\varphi,\psi) : l_1 \in [0,\infty), l_2 \in [l_1 + c_0, l_1 + c_1], \varphi,\psi \in X, \right.$$
$$\left. \|\varphi\|_X + \|\psi\|_X \geq M_4 \right\} > rN + M_3 + \tilde{P}(N-1)c_1 L \tag{3.32}$$

(recall $\tilde{P}$ as in proof of Theorem 3.13).

We will prove assertion 1 first. Let $l_1 \in [0,\infty)$, $\triangle l \in [c_0, c_1]$, $0 \leq k_1 < k_2$, $(\xi_i)_{i=k_1}^{k_2} \subset X$ be given.

Assume that (3.28) holds. Then there is $j \in \{k_1 + 1, \dots, k_2 - 1\}$ such that $\|\xi_j\|_X \geq M_4$. Set

$$i_1 = \max\{i \in \{k_1, \dots, j\} : \|\xi_i\|_X \leq M_1\},$$
$$i_2 = \min\{i \in \{j, \dots, k_2\} : \|\xi_i\|_X \leq M_1\}.$$

If $i_2 - i_1 \geq N$ then by the definition of N and Proposition 3.14 there exists a sequence $(\varphi_i)_{i=k_1}^{k_2} \subset X$ which satisfies (3.29) and $\varphi_{k_i} = \xi_{k_i},\ i = 1, 2$.

Assume that $i_2 - i_1 < N$ and define a sequence $(\varphi_i)_{i=k_1}^{k_2} \subset X$ by

$$\varphi_i = \xi_i,\ i \in \{k_1, \dots, i_1\} \cup \{i_2, \dots, k_2\},\ \varphi_i = \bar{\xi},\ i = i_1 + 1, \dots, i_2 - 1. \tag{3.33}$$

It follows from (3.33), (3.31), the lower bound of I_Ω (2.7) and the definition of i_1, i_2, j that

$$\begin{aligned} \sum_{i=k_1}^{k_2-1} \left[U_{\Omega_i}(\xi_i, \xi_{i+1}) - U_{\Omega_i}(\varphi_i, \varphi_{i+1})\right] &= \sum_{i=i_1}^{i_2-1} \left[U_{\Omega_i}(\xi_i, \xi_{i+1}) - U_{\Omega_i}(\varphi_i, \varphi_{i+1})\right] \\ &\geq U_{\Omega_{j-1}}(\xi_{j-1}, \xi_j) + (i_2 - i_1 - 1)\left(-\tilde{P}c_1 L\right) - (i_2 - i_1)\, r. \end{aligned} \tag{3.34}$$

By this relation and the definition of j, M_4 (see (3.32))

$$\sum_{i=k_1}^{k_2-1} \left[U_{\Omega_i}(\xi_i, \xi_{i+1}) - U_{\Omega_i}(\varphi_i, \varphi_{i+1})\right] \geq M_3. \tag{3.35}$$

This completes the proof of Assertion 1.

Next, we will prove Assertion 2. Let $l_1 \in [0, \infty)$, $\triangle l \in [c_0, c_1]$, $0 \leq k_1 < k_2$, $(\xi_i)_{i=k_1}^{k_2} \subset X$.

Assume that (3.30) holds. Then there is $j \in \{k_1, \dots, k_2\}$ such that $\|\xi_j\|_X \geq M_4$. Set $i_1 = \max\{i \in \{k_1, \dots, j\} : \|\xi_i\|_X \leq M_1\}$. There are two cases:

1) $\|\xi_i\|_X > M_1,\ i = j, \dots, k_2$;

2) $\min\{\|\xi_i\|_X :\ i = j, \dots, k_2\} \leq M_1$.

Consider the first case. We set $\varphi_i = \xi_i,\ i = k_1, \dots, i_1,\ \varphi_i = \bar{\xi},\ i = i_1 + 1, \dots, k_2$. If $k_2 - i_1 \geq N$ then (3.29) follows from the definition of N and Proposition 3.14. If $k_2 - i_1 < N$ then (3.29) follows from the definition of $(\varphi_i)_{i=k_1}^{k_2}, i_1, j, M_4$ and (3.31) (see (3.34), (3.35) with $i_2 = k_2$).

Consider the second case. Set $i_2 = \min\{i \in \{j, \dots, k_2\} : \|\xi_i\|_X \leq M_1\}$. If $i_2 - i_1 \geq N$ then by definition of N and Proposition 3.14 there exists a sequence $(\varphi_i)_{i=k_1}^{k_2} \subset X$ that satisfies (3.29) and $\varphi_{k_i} = \xi_{k_i},\ i = 1, 2$. If $i_2 - i_1 < N$ we define a sequence $(\varphi_i)_{i=k_1}^{k_2} \subset X$ by (3.33). Then (3.34) and (3.35) follows from (3.33), the definition of $i_1, i_2, j, M_4, (3.31)$. Assertion 2 is proved. □

THEOREM 3.16. *Let M_1, M_2, c, L be positive numbers. Then there exists a number $S > 0$ such that for each $l_1 \in [0, \infty)$ and each $l_2 \in [l_1 + c, \infty)$ the following holds:*

(i) For each $\varphi, \psi \in X$ satisfying $\|\varphi\|_X, \|\psi\|_X \leq M_1$ and each $u \in \mathcal{A}((l_1, l_2) \times (0, L))$ satisfying

$$\mathcal{T}^{(l_1,l_2)\times(0,L)}[u] = (\varphi, \psi),$$
$$I_{(l_1,l_2)\times(0,L)}[u] \leq U_{(l_1,l_2)\times(0,L)}(\varphi, \psi) + M_2$$

the following relation holds:

$$\left\|\mathcal{T}_2^{(l_1,l')\times(0,L)}[u]\right\|_X \leq S,\ l_1 \leq l' \leq l_2. \tag{3.36}$$

(ii) For each $\varphi \in X$ satisfying $\|\varphi\|_X \leq M_1$ and each $u \in \mathcal{A}((l_1, l_2) \times (0, L))$ satisfying

$$\mathcal{T}_1^{(l_1,l_2)\times(0,L)}[u] = \varphi,$$
$$I_{(l_1,l_2)\times(0,L)}[u] \leq U_{(l_1,l_2)\times(0,L)}\left(\varphi, \mathcal{T}_2^{(l_1,l_2)\times(0,L)}[u]\right) + M_2$$

relation (3.36) *holds.*

Proof. Fix $\bar{\xi} \in X$. By Proposition 3.13 there exists a number

$$M_0 \geq \sup \left\{ \left| U_{(l_1,l_2)\times(0,L)}\left(\varphi,\psi\right) \right| : l_1 \in [0,\infty), l_2 \in [l_1+c, l_1+2c+2], \right. \\ \left. \varphi,\psi \in X,\ \|\varphi\|_X + \|\psi\|_X \leq 2\left\|\bar{\xi}\right\|_X + 1 \right\}. \tag{3.37}$$

By Proposition 3.12 we may assume that

$$\inf \left\{ U_{(l_1,l_2)\times(0,L)}\left(\varphi,\psi\right) : l_1 \in [0,\infty), l_2 \in [l_1+c, l_1+2c+2], \right. \\ \left. \varphi,\psi \in X,\ \|\varphi\|_X + \|\psi\|_X \geq M_1 \right\} > 2M_0 + 1. \tag{3.38}$$

Indeed, we may replace if necessary M_1 by a larger number $S_1 > M_1$ such that Proposition 3.15 holds with

$$M_3 = M_2 + 2,\ M_4 = S_1 > M_1,\ c_0 = c,\ c_1 = 2c+2. \tag{3.39}$$

By Proposition 3.13 there exists a number $M_3 > 0$ such that

$$M_3 > \sup \left\{ \left| U_{(l_1,l_2)\times(0,L)}\left(\varphi,\psi\right) \right| : l_1 \in [0,\infty), l_2 \in [l_1+c, l_1+2c+2], \right. \\ \left. \varphi,\psi \in X,\ \|\varphi\|_X, \|\psi\|_X \leq S_1 \right\}. \tag{3.40}$$

By Proposition 3.12 there exists $S > S_1 + 1$ such that

$$\left\| \mathcal{T}_2^{(l_1,l')\times(0,L)}[u] \right\|_X \leq S,\ l_1 \leq l' \leq l_2$$

for each $l_1 \in [0,\infty)$, $l_2 \in [l_1+c, l_1+2c+2]$ and each function $u \in \mathcal{A}\left((l_1,l_2)\times(0,L)\right)$ satisfying $I_{(l_1,l_2)\times(0,L)}[u] \leq 2M_3 + 2M_2 + 2$.

Assume that $l_1 \in [0,\infty)$, $l_2 \geq l_1 + c$. We will show that property (i) holds.

Let $\varphi,\psi \in X$, $\|\varphi\|_X, \|\psi\|_X \leq M_1$ and let $u \in \mathcal{A}\left((l_1,l_2)\times(0,L)\right)$ be a function which satisfies

$$\begin{array}{c} \mathcal{T}^{(l_1,l_2)\times(0,L)}[u] = (\varphi,\psi) \\ I_{(l_1,l_2)\times(0,L)}[u] \leq U_{(l_1,l_2)\times(0,L)}\left(\varphi,\psi\right) + M_2 \end{array}. \tag{3.41}$$

There is a positive integer p such that $pc \leq l_2 - l_1 \leq (p+1)c$. Set $\triangle l = \frac{l_2-l_1}{p}$. Hence $\triangle l \in [c, 2c]$.

By (3.41) and Theorem 2.12

$$\sum_{i=0}^{p-1} \left[U_{\Omega_i}\left(\mathcal{T}^{\Omega_i}[u]\right) - U_{\Omega_i}\left(\varphi_i,\varphi_{i+1}\right) \right] \leq \sum_{i=0}^{p-1} \left[I_{\Omega_i}[u] - U_{\Omega_i}\left(\varphi_i,\varphi_{i+1}\right) \right] \leq M_2 \tag{3.42}$$

for each sequence $(\varphi_i)_{i=0}^{p} \subset X$ satisfying $\varphi_0 = \mathcal{T}_1^{\Omega_0}[u]$, $\varphi_p = \mathcal{T}_2^{\Omega_{p-1}}[u]$,
where $\Omega_i = (l_1 + i\triangle l, l_1 + (i+1)\triangle l) \times (0,L)$, $i = 0,..,p-1$.

It follows from this, (3.42), (3.39), (3.41) and Proposition 3.15 that

$$\left\| \mathcal{T}_1^{\Omega_0}[u] \right\|_X, \left\| \mathcal{T}_2^{\Omega_i}[u] \right\|_X \leq S_1,\ i = 0,..,p-1.$$

By this relation, (3.41) and (3.40) for $i = 0,..,p-1$

$$I_{\Omega_i}[u] \leq U_{\Omega_i}\left(\mathcal{T}^{\Omega_i}[u]\right) + M_2 < M_3 + M_2.$$

It follows from this relation and the definition of S that

$$\left\| \mathcal{T}_2^{(l_1,l')\times(0,L)}[u] \right\|_X \le S,\ l_1 \le l' \le l_2.$$

Therefore property (i) holds. Analogously to this we can show that property (ii) holds. □

4. Minimal growth rate of energy

We will treat the problem defined in (2.1) as a minimization, in the limit as $L_1 \to \infty$, of the following functional

$$\begin{aligned} & I_{(0,L_1)\times(0,L)}[w] = \int\limits_{(0,L_1)\times(0,L)} f\left(w, Dw, D^2 w\right) dx, \\ & w \in \mathcal{A}_\xi := \Big\{ v \in H^2_{loc}\left((0,\infty)\times(0,L)\right) : \mathcal{T}_1^{(0,L_1)\times(0,L)}[v] = \xi, \\ & \quad v \in \mathcal{A}\left((0,L_1)\times(0,L)\right),\ \forall L_1 > 0 \Big\} \end{aligned} \tag{4.1}$$

where $\xi \in X$ and f is the same as in (2.2).

Our objective is to minimize the "average energy over large domains", that is to minimize the functional J defined by

$$J[w] = \liminf_{L_1 \to \infty} \frac{1}{L_1 L} I_{(0,L_1)\times(0,L)}[w],\ w \in \mathcal{A}_\xi. \tag{4.2}$$

The number

$$\mu = \inf_{w \in \mathcal{A}_\xi} J[w], \tag{4.3}$$

is called the *minimal growth rate* of the energy.

Remark 4.1. It is easy to see that μ is independent of the initial $\xi \in X$. Let $\eta \in X$, and fix $L_1 > 0$. By Theorem 2.12 there exists $u \in \mathcal{A}\left((0,L_1)\times(0,L)\right)$ for which

$$\begin{array}{c} I_{(0,L_1)\times(0,L)}[u] = U_{(0,L_1)\times(0,L)}(\eta,\xi) \\ \mathcal{T}^{(0,L_1)\times(0,L)}[u] = (\eta,\xi) \end{array}.$$

Now, let $(w_k)_{k=1}^\infty \in \mathcal{A}_\xi$ be such that $\lim\limits_{k\to\infty} J[w_k] = \mu$. Define

$$u_k(x_1,x_2) = \begin{cases} u(x_1,x_2),\ 0 < x_1 < L_1 \\ w_k(x_1 - L_1, x_2),\ L_1 \le x_1 \end{cases},\ k \ge 1.$$

Clearly, $(u_k)_{k=1}^\infty \in \mathcal{A}_\eta$. Finally we calculate,

$$\begin{aligned} \lim_{k\to\infty} J[u_k] &= \lim_{k\to\infty} \left(\liminf_{L_2\to\infty} \frac{1}{L_2 L} I_{(0,L_2)\times(0,L)}[u_k] \right) \\ &= \lim_{k\to\infty} \left(\liminf_{L_2\to\infty} \frac{1}{L_2 L} \left(U_{(0,L_1)\times(0,L)}(\eta,\xi) + I_{(0,L_2-L_1)\times(0,L)}[w_k] \right) \right) \\ &= \lim_{k\to\infty} \left(\liminf_{L_2\to\infty} \frac{1}{L_2 L} I_{(0,L_2-L_1)\times(0,L)}[w_k] \right) = \mu. \end{aligned}$$

Next we state a result analogous to a result of Leizarowitz [[**9**],Theorem 3.1] in the discrete case.

THEOREM 4.2. *Let $L, c, R > 0$. There exist a constant μ and a constant $M > 0$ such that for every $\varphi, \psi \in \mathcal{B}_R = \{\xi \in X : \|\xi\|_X \leq R\}$ and every domain $\Omega_A = (0, L_1) \times (0, L)$ where $L_1 \in [2c, \infty)$, the inequality*

$$|U_{\Omega_A}(\varphi, \psi) - \mu L_1 L| \leq M$$

holds.

To prove Theorem 4.2 we need the following three lemmas in which we follow closely Lemma 3.4, Lemma 3.5 and Lemma 3.6 of Leizarowitz (cf. [**9**]). Denote by $\lambda(A)$ the minimal averaged rate over all x_1-periodic boundary conditions of period $\frac{A}{L}>0$, namely

$$\lambda(A) = \min_{\varphi \in \mathcal{B}_R} \left\{ \frac{U_{\Omega_A}(\varphi, \varphi)}{A} : \Omega_A = \left(0, \frac{A}{L}\right) \times (0, L) \right\}. \tag{4.4}$$

The existence of the minimal value in (4.4) is ensured by Theorem 2.14 and the compactness of $\mathcal{B}_R$, with respect to the weak topology.

Recall $S = S(R, L, c) > 0$ as given by Theorem 3.16 (for $M_2 = 1$, say).

LEMMA 4.3. *The following relation holds:*

$$\mu = \inf_A \lambda(A).$$

PROOF. On the one hand, it is easy to see that $\mu \leq \lambda(A)$. On the other hand, given $\varepsilon > 0$ there are $\xi \in \mathcal{B}_R$ and by Theorem 3.16, $\eta \in \mathcal{B}_S$ with A arbitrary large such that $\frac{U_{\Omega_A}(\xi,\eta)}{A} < \mu + \varepsilon$. Therefore since for $\tilde{\Omega}_{cL} = \left(\frac{A}{L}, \frac{A}{L} + c\right) \times (0, L)$, $U_{\tilde{\Omega}_{cL}}(\eta, \xi) \leq M$ by Proposition 3.13,

$$\frac{U_{\Omega_A}(\xi, \eta) + U_{\tilde{\Omega}_{cL}}(\eta, \xi)}{A + cL} < \mu + 2\varepsilon$$

for large A. Hence $\lambda(A + cL) < \mu + 2\varepsilon$ for large A, and the result follows. ☐

In the sequel we shall use $\mu = \inf_A \lambda(A)$ as the definition of μ.

LEMMA 4.4. $\lim_{A\to\infty} \lambda(A) = \mu$.

PROOF. From (2.7) we have $\frac{I_\Omega[u]}{|\Omega|} \geq -P$, and although P depends on $|\Omega|$, for A large enough, P is constant (see (2.3)). Thus, $\lambda(A)$ is bounded from below. Set

$$\alpha = \liminf_{A\to\infty} \lambda(A).$$

We want to prove that $\limsup_{A\to\infty} \lambda(A) \leq \alpha$ which would imply that $\lim_{A\to\infty} \lambda(A)$ exists. Given $\varepsilon > 0$, there is a domain $\Omega_A = \left(0, \frac{A}{L}\right) \times (0, L)$, such that $\lambda(A) < \alpha + \varepsilon$. Since every function which is periodic of period $\frac{A}{L}$ is also periodic of period $k\frac{A}{L}$ for every integer k, and as the averaged rate over $\left(0, k\frac{A}{L}\right) \times (0, L)$ is the same as that over $\left(0, \frac{A}{L}\right) \times (0, L)$ it follows from (4.4) that

$$\lambda(kA) \leq \lambda(A) \text{ for every } k \geq 1, A \geq cL. \tag{4.5}$$

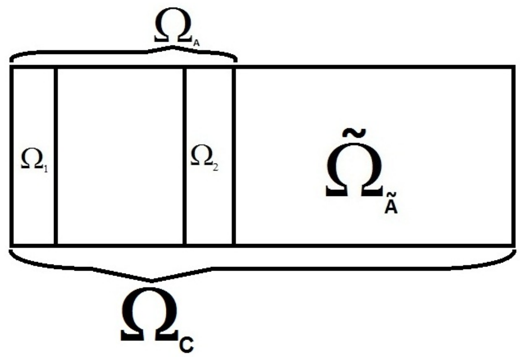
Using Proposition 3.13, we denote

$$\begin{array}{c} a = \sup\left\{U_{(0,c)\times(0,L)}(\varphi, \psi) : \ \varphi, \psi \in \mathcal{B}_{R+S}\right\} \\ b = \inf\left\{U_{(0,c)\times(0,L)}(\varphi, \psi) : \ \varphi, \psi \in \mathcal{B}_{R+S}\right\} \end{array}. \tag{4.6}$$

Set $\Omega_A = \left(0, \frac{A}{L}\right) \times (0, L)$, $\tilde{\Omega}_{\tilde{A}} = \left(\frac{A}{L}, \frac{A}{L} + \frac{\tilde{A}}{L}\right) \times (0, L)$, $\Omega_C = \left(0, \frac{A}{L} + \frac{\tilde{A}}{L}\right) \times (0, L)$ as well as $\Omega_1 = (0, c) \times (0, L), \Omega_2 = \left(\frac{A}{L} - c, \frac{A}{L}\right) \times (0, L)$ (see Figure 4.1). Thus, $C = A + \tilde{A}$. Let $\xi \in \mathcal{B}_R$ be the trace of the function for which $\lambda(A)$ is attained and $\eta \in \mathcal{B}_R$ be the analogous one for $\lambda\left(\tilde{A}\right)$. There exist $u_A \in \mathcal{A}(\Omega_A), u_{\tilde{A}} \in \mathcal{A}\left(\tilde{\Omega}_{\tilde{A}}\right), u_1 \in \mathcal{A}(\Omega_1), u_2 \in \mathcal{A}(\Omega_2)$ such that

$$\begin{array}{c} \mathcal{T}^{\Omega_A}[u_A] = (\xi, \xi),\ I_{\Omega_A}[u_A] = U_{\Omega_A}(\xi, \xi) \\ \mathcal{T}^{\tilde{\Omega}_{\tilde{A}}}[u_{\tilde{A}}] = (\eta, \eta),\ I_{\tilde{\Omega}_{\tilde{A}}}[u_{\tilde{A}}] = U_{\tilde{\Omega}_{\tilde{A}}}(\eta, \eta) \\ \mathcal{T}^{\Omega_1}[u_1] = \left(\eta, \mathcal{T}_2^{\Omega_1}[u_A]\right),\ I_{\Omega_1}[u_1] = U_{\Omega_1}\left(\eta, \mathcal{T}_2^{\Omega_1}[u_A]\right) \\ \mathcal{T}^{\Omega_2}[u_2] = \left(\mathcal{T}_1^{\Omega_2}[u_A], \eta\right),\ I_{\Omega_2}[u_2] = U_{\Omega_2}\left(\mathcal{T}_1^{\Omega_2}[u_A], \eta\right) \end{array}.$$

FIGURE 4.1



We define $u \in \mathcal{A}(\Omega_C)$ by

$$u = u_1\chi_{\Omega_1} + u_A\chi_{\Omega_A \setminus (\Omega_1 \cup \Omega_2)} + u_2\chi_{\Omega_2} + u_{\tilde{A}}\chi_{\tilde{\Omega}_{\tilde{A}}}.$$

Since $\mathcal{T}^{\Omega_C}[u] = (\eta, \eta)$, using (4.6) we get,

$$C\lambda(C) \leq A\lambda(A) + \tilde{A}\lambda\left(\tilde{A}\right) + 2(a - b).$$

In particular, for $C = kA + S$ with $S \in [0, A)$, we have

$$C\lambda(C) \leq kA\lambda(kA) + S\lambda(S) + 2(a - b),$$

or equivalently,

$$\lambda(C) \leq \frac{kA\lambda(kA)}{kA + S} + \frac{S\lambda(S)}{kA + S} + \frac{2(a - b)}{kA + S}.$$

For k large enough the second term in the last inequality is less than ε, and so by (4.5) and the way we chose A we get

$$\lambda(C) \leq (\alpha + \varepsilon)\frac{kA}{kA + S} + \varepsilon < \alpha + 3\varepsilon$$

and this holds for all large k and every $S < A$. We conclude that $\limsup_{A \to \infty} \lambda(A) \leq \alpha$ since the preceding inequality is true for every $\varepsilon > 0$.

Finally we claim that $\alpha = \mu$. Clearly, $\mu \leq \alpha$. On the other hand, if we had $\lambda(C) < \alpha$ for some C, it would imply using (4.5) that

$$\alpha = \liminf_{A\to\infty} \lambda(A) \leq \liminf_{k\to\infty} \lambda(kC) \leq \lambda(C) < \alpha$$

a contradiction. □

LEMMA 4.5. *The following inequality holds:*

$$\limsup_{A\to\infty} A(\lambda(A) - \mu) < \infty.$$

PROOF. Denote by $\rho(A)$ the minimal averaged rate over $\Omega_A = \left(0, \frac{A}{L}\right) \times (0, L)$, namely:

$$\rho(A) = \min_{\xi,\eta\in\mathcal{B}_R} \frac{U_{\Omega_A}(\xi,\eta)}{A}. \tag{4.7}$$

The existence of the minimal value in (4.7) is ensured by Theorem 2.14 and the compactness of $\mathcal{B}_R$ with respect to the weak topology. Recall the definition of a and b in (4.6). We claim that $\rho(A) \leq \mu + \frac{2(a-b)}{A}$ for every $A \geq cL$. Suppose to the contrary that $\rho(A) > \mu + \frac{2(a-b)}{A}$ for some A. For every integer k, there exists $u_{kA} \in \mathcal{A}(\Omega_{kA})$ such that

$$\rho(kA) = \frac{U_{\Omega_{kA}}\left(\mathcal{T}^{\Omega_{kA}}[u_{kA}]\right)}{kA}. \tag{4.8}$$

For $1 \leq i \leq k$, we set $\Omega_i = \left((i-1)\frac{A}{L}, i\frac{A}{L}\right) \times (0, L)$. Recalling $S > 0$, as given by Theorem 3.16, we have

$$\left\|\mathcal{T}_j^{\Omega_i}[u_{kA}]\right\|_X \leq S,\ j = 1, 2 \text{ and } 2 \leq i \leq k-1.$$

Additionally, there exist $\xi_A, \eta_A \in \mathcal{B}_R$ such that

$$\rho(A) = \frac{U_{\Omega_i}(\xi_A, \eta_A)}{A} \tag{4.9}$$

for $1 \leq i \leq k$. Thus, by (4.8), (4.9) and (4.6) we have

$$\rho(kA) = \frac{1}{kA}\sum_{i=1}^{k} U_{\Omega_i}\left(\mathcal{T}^{\Omega_i}[u_{kA}]\right) \geq \frac{1}{kA}\sum_{i=1}^{k}\left[U_{\Omega_i}(\xi_A, \eta_A) - 2(a-b)\right] = \rho(A) - \frac{2(a-b)}{A}.$$

Also from the definition of ρ, and λ in (4.7) and (4.4), $\rho(kA) \leq \lambda(kA)$, so we get $\mu + \frac{2(a-b)}{A} < \rho(A) \leq \rho(kA) + \frac{2(a-b)}{A} \leq \lambda(kA) + \frac{2(a-b)}{A}$ for all $k \geq 1$. Letting $k \to \infty$ we get a contradiction since $\lambda(kA) \underset{k\to\infty}{\to} \mu$ by Lemma 4.6.

Let $A \geq cL$ and let $\xi, \eta \in \mathcal{B}_R$ for which $\rho(A)$ in (4.7) is attained. Let $u_A \in \mathcal{A}(\Omega_A)$ be such that

$$I_{\Omega_A}[u_A] = A\rho(A),\ \mathcal{T}^{\Omega_A}[u_A] = (\xi, \eta)\ .$$

In $\Omega = (0, c) \times (0, L)$ let $u \in \mathcal{A}(\Omega)$ be such that

$$I_\Omega[u] = U_\Omega\left(\eta, \mathcal{T}_2^{\Omega_A}[u_A]\right),\ \mathcal{T}^\Omega[u] = \left(\eta, \mathcal{T}_2^\Omega[u_A]\right).$$

Modifying u_A to a map $\tilde{u} \in \mathcal{A}(\Omega_A)$ by letting $\tilde{u} = u$ on Ω, $\tilde{u} = u_A$ on $\Omega_A \setminus \Omega$ yields a $\frac{A}{L}$−periodic function with $I_{\Omega_A}[\tilde{u}] \leq I_{\Omega_A}[u] + a - b$. Hence for every $A \geq cL$,

$$A\lambda(A) - A\rho(A) \leq a - b.$$

Combining this with $\rho(A) \leq \mu + \frac{2(a-b)}{A}$ leads to

$$\limsup_{A\to\infty} A(\lambda(A) - \mu) \leq \limsup_{A\to\infty} A\left(\lambda(A) - \rho(A) + \frac{2(a-b)}{A}\right) \leq 3(a-b).$$

□

Proof of Theorem 4.2. Note first that from Lemma 4.3 it follows that $\frac{U_{\Omega_A}(\xi,\xi)}{A} \geq \mu$ for all $\xi \in X$. The argument used in the proof of Lemma 4.5 shows that for every $\xi, \eta \in \mathcal{B}_R$ and every $A \geq cL$ the inequality $U_{\Omega_A}(\xi,\eta) - \mu A \geq -(a-b)$ holds (recall (4.6)).

We prove now that $U_{\Omega_A}(\varphi,\psi) - \mu A \leq M$ for all $\varphi, \psi \in \mathcal{B}_R$ and $A \geq 2cL$. Using Lemma 4.5 we choose $\alpha > 0$ so that for all $A \geq 2cL$ we have

$$A(\lambda(A) - \mu) < \alpha. \tag{4.10}$$

Let $\xi_A \in \mathcal{B}_R$ and $u_A \in \mathcal{A}(\Omega_A)$ be chosen so that $\lambda(A) = \frac{U_{\Omega_A}(\xi_A,\xi_A)}{A} = \frac{I_{\Omega_A}[u_A]}{A}$ where $\mathcal{T}^{\Omega_A}[u_A] = (\xi_A,\xi_A)$. By (4.10) we get $U_{\Omega_A}(\xi_A,\xi_A) - \mu A < \alpha$ for all $A \geq 2cL$. We claim that for all $A \geq 2cL$

$$U_{\Omega_A}(\varphi,\psi) - \mu A \leq \alpha + 2(a-b), \ \forall \varphi,\psi \in \mathcal{B}_R. \tag{4.11}$$

We denote $\Omega_1 = (0,c) \times (0,L)$, $\Omega_2 = \left(c, \frac{A}{L} - c\right) \times (0,L)$ and $\Omega_3 = \left(\frac{A}{L} - c, \frac{A}{L}\right) \times (0,L)$. Clearly, $\Omega_A = \Omega_1 \cup \Omega_2 \cup \Omega_3$. By Theorem 3.16, $\left\|\mathcal{T}_2^{\Omega_1}[u_A]\right\|_X, \left\|\mathcal{T}_2^{\Omega_2}[u_A]\right\|_X \leq S$. Hence

$$\begin{aligned}
U_{\Omega_A}(\varphi,\psi) - \mu A &\leq U_{\Omega_1}\left(\varphi, \mathcal{T}_2^{\Omega_1}[u_A]\right) \\
&+ U_{\Omega_2}\left(\mathcal{T}_2^{\Omega_1}[u_A], \mathcal{T}_2^{\Omega_2}[u_A]\right) + U_{\Omega_3}\left(\mathcal{T}_2^{\Omega_2}[u_A], \psi\right) - \mu A \\
&\leq U_{\Omega_1}\left(\xi_A, \mathcal{T}_2^{\Omega_1}[u_A]\right) + (a-b) + U_{\Omega_2}\left(\mathcal{T}_2^{\Omega_1}[u_A], \mathcal{T}_2^{\Omega_2}[u_A]\right) \\
&+ U_{\Omega_3}\left(\mathcal{T}_2^{\Omega_2}[u_A], \xi_A\right) + (a-b) - \mu A \\
&\leq U_{\Omega_A}(\xi_A,\xi_A) - \mu A + 2(a-b) < \alpha + 2(a-b).
\end{aligned}$$

Thus we proved the theorem with $M = \alpha + 2(a-b)$. □

5. Existence of a minimal energy configuration

In this section we will prove the existence, for each $\xi \in X$, of a minimal solution in $\mathcal{A}_\xi$. Recall that the space X is defined in (3.1).

Definition 5.1. A function $w^* \in H^2_{loc}((0,\infty) \times (0,L))$ is called a *locally minimal energy configuration* if $I_{(0,L_1)\times(0,L)}[w^*] \leq I_{(0,L_1)\times(0,L)}[w]$ for each $0 < L_1$ and each $w \in H^2((0,L_1) \times (0,L))$ satisfying

$$\left(\gamma^{(0,L_1)\times(0,L)} w^*, \gamma^{(0,L_1)\times(0,L)} \frac{\partial w^*}{\partial \nu}\right) = \left(\gamma^{(0,L_1)\times(0,L)} w, \gamma^{(0,L_1)\times(0,L)} \frac{\partial w}{\partial \nu}\right).$$

If in addition to the above property w^* also provides the minimal growth rate of energy, then w^* is called a *minimal energy configuration.*

Now we will construct for each $\xi \in X$ a $w^* \in \mathcal{A}_\xi$ which is a minimal energy configuration (which constitutes the minimal solution in our case).

Given $L>0$ and $\xi \in X$, we denote $\Omega_l = (0,l)\times(0,L)$ for $l \geq 1$. Consider, for each integer $k \geq 1$, $u_k \in \mathcal{A}(\Omega_k)$ satisfying

$$
\begin{array}{c} I_{\Omega_k}[u_k] = U_{\Omega_k}(\xi,\xi) \\ \mathcal{T}^{\Omega_k}[u_k] = (\xi,\xi) \end{array}. \tag{5.1}
$$

Denote by $u_k|_{\Omega_m}$ the restriction of u_k to $\Omega_m = (0,m)\times(0,L)$ for $m \leq k$. From (5.1) and Lemma 3.10, we have

$$
I_{\Omega_m}[u_k|_{\Omega_m}] = U_{\Omega_m}\left(\xi, \mathcal{T}_2^{\Omega_m}[u_k]\right). \tag{5.2}
$$

Thus, by Theorem 3.16 there exists $S = S(m, L, \|\xi\|_X) > 0$ (with $M_2 = 1$, say) satisfying

$$
\left\|\mathcal{T}_2^{\Omega_m}[u_k]\right\|_X < S,\ \forall k \geq m. \tag{5.3}
$$

Hence, by Proposition 3.13, the sequence $(I_{\Omega_m}[u_k|_{\Omega_m}])_{k\geq m}$ is bounded. Consequently, it follows from the coercivity property (3.14), that the sequence $(u_k|_{\Omega_m})_{k\geq m}$ is bounded in $\mathcal{A}(\Omega_m)$. Thus, we can suppose, by extracting a subsequence and re-indexing, that for some $u^m \in \mathcal{A}(\Omega_m)$

$$
u_k|_{\Omega_m} \rightharpoonup u^m \quad \text{weakly in } \mathcal{A}(\Omega_m). \tag{5.4}
$$

Using (5.4), we denote by $\left(u_k^1\right)_{k\geq 1}$ the subsequence of $(u_k)_{k=1}^{\infty}$ satisfying

$$
u_k^1|_{\Omega_1} \rightharpoonup u^1 \quad \text{weakly in } \mathcal{A}(\Omega_1).
$$

Next, for every $m \in \mathbb{N}$ we define recursively $\left(u_k^{m+1}\right)_{k\geq m+1}$ as a subsequence of $(u_k^m)_{k\geq m}$ satisfying

$$
u_k^{m+1}|_{\Omega_{m+1}} \rightharpoonup u^{m+1} \quad \text{weakly in } \mathcal{A}(\Omega_{m+1}). \tag{5.5}
$$

Consequently, from (5.5), we have

$$
u^n|_{\Omega_m} = u^m,\ \forall n \geq m. \tag{5.6}
$$

In light of (5.6) we define $u^* \in \mathcal{A}_\xi$ by

$$
u^* = \sum_{m\geq 1} u^m \chi_{[m-1,m)\times(0,L)}. \tag{5.7}
$$

We proceed to demonstrate that this construction does provide a minimal energy configuration.

THEOREM 5.2. *For every $\xi \in X$ there is a minimal energy configuration for $I[\cdot]$ in $\mathcal{A}_\xi$.*

PROOF. We will show that the function u^* defined in (5.7) is a minimal energy configuration for problem (4.1). By Lemma 3.10 and (5.7), in order to prove the minimality property of u^*, in the sense of Definition 5.1, it will suffice to show that for every $m \in \mathbb{N}$

$$
I_{\Omega_m}[u^m] = U_{\Omega_m}\left(\xi, \mathcal{T}_2^{\Omega_m}[u^m]\right). \tag{5.8}
$$

Consider the diagonal sequence $\left(u_k^k\right)_{k=1}^{\infty}$. Clearly,

$$
u_k^k|_{\Omega_m} \rightharpoonup u^m = u^*|_{\Omega_m} \quad \text{weakly in } \mathcal{A}(\Omega_m)\ \forall m \in \mathbb{N}. \tag{5.9}
$$

Now, for each $m \in \mathbb{N}$, by (5.9), there exists a $M>0$ satisfying

$$
\|u_k^k|_{\Omega_{m+1}}\|^2_{H^2(\Omega_{m+1})} < M. \tag{5.10}
$$

Set $\Gamma^{x_1} = \{x_1\} \times (0, L)$. By (5.10) and Fubini's Theorem,

$$h_k(x_1) := \int\limits_{\Gamma^{x_1}} \left(\left|u_k^k\right|^2 + \left|Du_k^k\right|^2 + \left|D^2 u_k^k\right|^2 \right) dx_2 \in L^1(m, m+1). \tag{5.11}$$

Hence by Fatou's Lemma

$$\begin{aligned} &\int\limits_{(m,m+1)} \liminf_{k\to\infty} h_k(x_1)\, dx_1 \\ &= \int\limits_{(m,m+1)} \left\{ \liminf_{k\to\infty} \int\limits_{\Gamma^{x_1}} \left(\left|u_k^k\right|^2 + \left|Du_k^k\right|^2 + \left|D^2 u_k^k\right|^2 \right) dx_2 \right\} dx_1 \\ &\leq \liminf_{k\to\infty} \left\| u_k^k|_{(m,m+1)\times(0,L)} \right\|^2_{H^2((m,m+1)\times(0,L))} < M. \end{aligned}$$

Consequently, there exists $c \in (m, m+1)$ satisfying

$$\liminf_{k\to\infty} \int\limits_{\Gamma^c} \left(\left|u_k^k\right|^2 + \left|Du_k^k\right|^2 + \left|D^2 u_k^k\right|^2 \right) dx_2 < M.$$

Hence, by extracting a subsequence and re-indexing it, we might as well suppose that

$$\int\limits_{\Gamma^c} \left(\left|u_k^k\right|^2 + \left|Du_k^k\right|^2 + \left|D^2 u_k^k\right|^2 \right) dx_2 < M \ \forall k \in \mathbb{N}. \tag{5.12}$$

We denote

$$\xi_k = \left(\xi_k^0, \xi_k^1\right) := \left(u_k^k|_{\Gamma^c}, \frac{\partial}{\partial x_1} u_k^k|_{\Gamma^c} \right). \tag{5.13}$$

Since $\xi_k \in H^2(\Gamma^c) \times H^1(\Gamma^c)$, by Theorem 2.10, $\frac{\partial}{\partial x_2}\xi_k^0, \xi_k^1 \in C\left(\overline{\Gamma^c}\right)$. Moreover, since $\xi_k \in X$, by the definition of X in (3.1), we have

$$\xi^0(0) = \xi^0(L) = 0.$$

Furthermore, since $\frac{\partial}{\partial x_2}\xi_k^0, \xi_k^1$ are continuous, in order for the integrals in (3.1) to be finite, we also have

$$\begin{aligned} \tfrac{\partial}{\partial x_2}\xi_k^0(0) = \tfrac{\partial}{\partial x_2}\xi_k^0(L) = 0, \\ \xi_k^1(0) = \xi_k^1(L) = 0. \end{aligned}$$

Thus,

$$\xi_k \in H_0^2(\Gamma^c) \times H_0^1(\Gamma^c). \tag{5.14}$$

Using (5.12), there exist $v^0 \in H_0^2(\Gamma^c)$ and $v^1 \in H_0^1(\Gamma^c)$ such that for a subsequence of $\left(u_k^k\right)_{k=1}^{\infty}$ the following holds:

$$\begin{matrix} \xi_k^0 \rightharpoonup v^0 & \text{weakly in } H_0^2(\Gamma^c) \\ \xi_k^1 \rightharpoonup v^1 & \text{weakly in } H_0^1(\Gamma^c) \end{matrix}. \tag{5.15}$$

By Kondrashov Theorem (cf. [[**7**],Theorem 1.4.3.2]) the embeddings $H^2(\Gamma^c) \hookrightarrow H^{\frac{3}{2}}(\Gamma^c)$ and $H^1(\Gamma^c) \hookrightarrow H^{\frac{1}{2}}(\Gamma^c)$ are compact. Therefore, by passing to a subsequence we may assume that

$$\mathcal{T}_2^{\Omega_c}\left[u_k^k\right] \to (v_0, v_1) \quad \text{strongly in } X. \tag{5.16}$$

Moreover, since the trace operator γ is continuous and using (5.6) we deduce that

$$\mathcal{T}_2^{\Omega_c}\left[u^*\right] = (v_0, v_1).$$

Since

$$\mathcal{T}_1^{\Omega_c}\left[u_k^k\right]=\mathcal{T}_1^{\Omega_c}\left[u^*\right]=\xi,\ \forall k,$$

it follows from (3.12) that

$$\left(\gamma^{\Omega_c}u_k^k,\gamma^{\Omega_c}\frac{\partial u_k^k}{\partial\nu}\right)\to\left(\gamma^{\Omega_c}u^*,\gamma^{\Omega_c}\frac{\partial u^*}{\partial\nu}\right)\quad\text{strongly in } Y\left(\partial\Omega_c\right).$$

Hence by Theorem 2.15 we conclude that

$$\lim_{k\to\infty}U_{\Omega_c}\left(\xi,\mathcal{T}_2^{\Omega_c}\left[u_k^k\right]\right)=U_{\Omega_c}\left(\xi,\mathcal{T}_2^{\Omega_c}\left[u^*\right]\right). \tag{5.17}$$

Recalling (5.2), by Proposition 2.11 one has

$$\lim_{k\to\infty}U_{\Omega_c}\left(\xi,\mathcal{T}_2^{\Omega_c}\left[u_k^k\right]\right)=\lim_{k\to\infty}I_{\Omega_c}\left[u_k^k|_{\Omega_c}\right]\geq I_{\Omega_c}\left[u^*|_{\Omega_c}\right].$$

Consequently, (5.17) implies that $I_{\Omega_c}\left[u^*|_{\Omega_c}\right]=U_{\Omega_c}\left(\xi,\mathcal{T}_2^{\Omega_c}\left[u^*\right]\right)$. Finally, using Lemma 3.10 we conclude that for each m:

$$I_{\Omega_m}\left[u^*|_{\Omega_m}\right]=U_{\Omega_m}\left(\xi,\mathcal{T}_2^{\Omega_m}\left[u^*\right]\right). \tag{5.18}$$

We proceed to show that

$$J\left[u^*\right]=\mu.$$

By Theorem 3.16 there exists $S>0$ such that $\left\|\mathcal{T}_2^{\Omega_m}\left[u^*\right]\right\|_X\leq S$ for every $m\geq 1$. Recalling (5.18), by Theorem 4.2 we have, for some constant $M>0$,

$$\left|I_{\Omega_m}\left[u^*\right]-\mu mL\right|\leq M.$$

Thus,

$$\frac{I_{\Omega_m}\left[u^*\right]}{mL}\to\mu\ as\ m\to\infty,$$

which concludes the proof.

□

References

[1] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.

[2] S. Aubry, P.Y. Le Daeron, The discrete Frenkel-Kontorova model and its extensions, Physica 8D (1983), 381-422.

[3] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, New York, 2011.

[4] B.D. Coleman, Necking and drawing in polymeric fibers under tension, Arch. Rational Mech. Anal. 83 (1983), 115-137.

[5] B.D. Coleman, On the cold drawing of polymers, Comp. & Maths. with Appl. 11(1985), 35-65.

[6] B.D. Coleman ,M. Marcus, V.J. Mizel, On the thermodynamics of periodic phases, Arch. Rational Mech. Anal. 117 (1992), 321-347.

[7] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Pitman Advanced Publishing Program, Boston, 1985.

[8] L.C. Evans, Partial Differential Equations, American Mathematical Society, 1998.

[9] A. Leizarowitz, Infinite horizon autonomous systems with unbounded cost, Appl. Math. Optim. 13 (1985), 19-43.

[10] A. Leizarowitz, Radially symmetric minimizers of variational problems in the plane, Advances in Differential Equations, Volume 10, Number 3, (2004), 241-258.

[11] A. Leizarowitz, V.J. Mizel, One dimensional infinite horizon variational problems arising in continuum mechanics, Arch. Rational Mech. Anal. 106 (1989), 161-194.

[12] A. Leizarowitz, M. Marcus, On radially symmetric minimizers of second order two-dimensional variational problems, Calculus of Variations and Differential Equations, Chapman & Hall, CRC, Res. Notes Math., 410 (2000), 136-147.

[13] G. Leoni, A First Course in Sobolev Spaces, American Mathematical Society, 2009.

[14] J.L. Lions, E. Magenes, Non-homogeneous boundary value problems and Applications I, Grundlehren #181, Springer, Berlin, 1972.

[15] M. Marcus, Uniform Estimates for a Variational Problems with Small Parameters, Arch. Rational Mech. Anal. 124 (1993), 67-98.

[16] A.J. Zaslavski, Existence and uniform boundedness of optimal solutions of variational problems, Abstr. Appl. Anal. Volume 3, Number 3-4 (1998), 265-292.

Department of Mathematics, Technion - I.I.T., 32000 Haifa, ISRAEL