Radially symmetric minimizers for a p-Ginzburg Landau type energy in $\mathbb{R}^2$

Yaniv Almog[*], Leonid Berlyand[†], Dmitry Golovaty[‡] and Itai Shafrir[§]

Abstract

We consider the minimization of a p-Ginzburg-Landau energy functional over the class of radially symmetric functions of degree one. We prove the existence of a unique minimizer in this class, and show that its modulus is monotone increasing and concave. We also study the asymptotic limit of the minimizers as $p \to \infty$. Finally, we prove that the radially symmetric solution is locally stable for $2 < p \leq 4$.

1 Introduction

Given $p > 2$ consider the minimization problem of the energy functional

$$E_p(u) = \int_{\mathbb{R}^2} |\nabla u|^p + \frac{1}{2}(1 - |u|^2)^2 \tag{1.1}$$

over the class of maps $u \in W^{1,p}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ that satisfy $E_p(u) < \infty$ and have a degree d "at infinity". In our previous work [1] it was shown that the notion of degree at infinity is well-defined. Hence, minimization over the homotopy class of maps with degree d is a sensible task. Moreover, in the case of degree $d = 1$ we proved that a minimizer does exist. An important open question is whether any minimizer u is necessarily radially symmetric, i.e., $u = f(r)e^{i\theta}$ for some function $f(r)$ satisfying $f(0) = 0$ (thanks to invariance with respect to translations we may assume that $u(0) = 0$). We show in the sequel that a (unique) minimizer *within the radially symmetric class* $u_p = f_p(r)e^{i\theta}$ exists. We were, however, unable to determine whether u_p is a minimizer or not. As a preliminary step towards establishing the minimality properties of u_p, we study in the present paper its *stability* properties. One of our main results (see Theorem 2 below) establishes that u_p is indeed stable if $p \in (2, 4]$. We conjecture that this result remains valid for any $p > 2$. It should be mentioned that the analogous stability problem for $p = 2$ on the disc $B_1(0)$ with the boundary condition $u(z) = \frac{z}{|z|}$ on $\partial B_1(0)$ was solved by Mironescu [9] and in a weaker

[*]Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA
[†]Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA
[‡]Department of Theoretical and Applied Mathematics, The University of Akron, Akron, Ohio 44325, USA
[§]Department of Mathematics, Technion - Israel Institute of Technology, 32000 Haifa, Israel

form, by Lieb and Loss [8]. Going back to the problem on $\mathbb{R}^2$, but again for $p = 2$, we recall that the L^2- stability of the radially symmetric solution was proved by Ovchinnikov and Sigal [11] and in a more natural energy space by del Pino, Felmer and Kowalczyk [5]. However, Mironescu [10] showed a stronger result, namely, that the radially symmetric solution is the unique (up to rotations and translations) *local minimizer* on $\mathbb{R}^2$, that is, on every disc $B_R(0)$ it is minimizing for its boundary values on $\partial B_R(0)$. Note that for $p = 2$ (in contrast with $p > 2$) only the notion of local minimizer makes sense since the admissible maps have infinite energy.

The manuscript is organized as follows. In Section 2 we establish existence and uniqueness of the minimizer $u_p = f_p(r)e^{i\theta}$ in the radially symmetric class, as well as its regularity. We also show that f_p is increasing and concave and obtain some precise estimates for $f_p(r)$ for large values of r. In Section 3 we study the limit of f_p as p tends to infinity. We show that $\lim_{p\to\infty} f_p = f_\infty$ is the piecewise linear function given by $\frac{r}{\sqrt{2}}$ for $r < \sqrt{2}$ and is identically equal to 1 for $r \geq \sqrt{2}$. Finally, Section 4 is devoted to the study of the stability of the radially symmetric solution.

2 Radially symmetric solutions

In this section we consider some of the properties of the minimizer of

$$I_p(f) = \int_0^\infty \left\{ \left[(f')^2 + \frac{f^2}{r^2} \right]^{p/2} + \frac{1}{2}(1 - f^2)^2 \right\} r dr \tag{2.1}$$

for any $p > 2$. Note that $I_p(f) = \frac{1}{2\pi}E(u)$ where $u = f(r)e^{i\theta}$.

2.1 Existence

For each $p > 2$ we define the space

$$X_p = \left\{ f \in W^{1,p}_{\mathrm{loc}}(0,\infty) \,:\, \int_0^\infty \left(|f'|^2 + \frac{f^2}{r^2} \right)^{p/2} r dr < \infty \right\}. \tag{2.2}$$

Existence of a solution will be established by minimization of $I_p(f)$ over X_p. Note that $X_p \subset C^\alpha_{\mathrm{loc}}[0,\infty)$, with $\alpha = 1 - 2/p$, since whenever $f \in X_p$, the function $F(x_1, x_2) = f(\sqrt{x_1^2 + x_2^2})$ belongs to $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)$, and then we can apply Morrey's theorem. Furthermore, for every $f \in X_p$ we must have have $f(0) = 0$. This follows from the continuity of f and the fact that

$$\int_0^1 \frac{|f|^p}{r^{p-1}} < \infty \,.$$

Proposition 2.1. *The minimum of $I_p(f)$ over X_p is attained by a function $f_p \in X_p$ satisfying $0 \leq f_p(r) \leq 1$, $\forall r \in [0,\infty)$.*

Proof. Put

$$m_p = \inf_{f \in X_p} I_p(f) \,. \tag{2.3}$$

We first note that $m_p < \infty$ since the function $g^* \in X_p$ defined by

$$g^*(r) = \begin{cases} r & r \le 1\,, \\ 1 & r > 1\,, \end{cases}$$

verifies $I_p(g^*) < \infty$. Consider a minimizing sequence $\{g_m\}$ for (1.1), i.e.,

$$\lim_{m\to\infty} I_p(g_m) = m_p\,.$$

By passing to a diagonal sequence we may assume that for any compact interval $[a,b] \subset (0,\infty)$ we have

$$g_m \rightharpoonup g \text{ weakly in } W^{1,p}(a,b)\,. \tag{2.4}$$

Since the convexity of the Lagrangian

$$L(P,Z,r) = \left\{ \left(P^2 + \frac{Z^2}{r^2}\right)^{p/2} + \frac{1}{2}(1-Z^2)^2 \right\} r$$

in the variable P implies weak lower-semi-continuity of the functional $I_p^{(a,b)}(f) := \int_a^b L(f', f, r)dr$ (see [7, Theorem 1,Sec. 8.2]), we deduce from (2.4) that

$$I_p^{(a,b)}(g) \le m_p\,. \tag{2.5}$$

Since the interval $[a,b]$ is arbitrary, we conclude from (2.5) that $g \in X_p$, $I_p(g) \le m_p$, so that necessarily $I_p(g) = m_p$, and g is a minimizer in (1.1). Since replacing g by

$$\tilde{g}(r) = \min(1, |g(r)|)\,,$$

gives a map $\tilde{g} \in X_p$ such that $I_p(\tilde{g}) \le I_p(g)$ (with strict inequality, unless $|g| \le 1$) we conclude that we may assume $0 \le g(r) \le 1$ for all r, and the result follows for $f_p = g$. □

The next lemma shows that f is positive on $(0,\infty)$.

Lemma 2.1. *$f_p > 0$ for all $r > 0$.*

Proof. We first claim that there is no interval of the form $[0,a]$, with $a > 0$ such that

$$f \equiv 0 \text{ on } [0,a]\,. \tag{2.6}$$

Indeed, suppose that (2.6) holds for some a. Fix any function $g \in C^\infty[0,a]$ satisfying $g(0) = g(a) = 0$ and $g(r) > 0$ for $r \in (0,a)$. Then, for any small $\varepsilon > 0$ consider the function h_ε defined by

$$h_\varepsilon(r) = \begin{cases} \varepsilon g(r) & 0 \le r \le a\,, \\ f_p(r) & r > a\,. \end{cases}$$

A simple computation gives

$$I_p(h_\varepsilon) = I_p^{(a,\infty)}(f_p) + \epsilon^p \int_0^a \left(|g'|^2 + (\frac{g}{r})^2\right)^{p/2} rdr + \int_0^a (1-\epsilon^2 g^2)^2 \, rdr < I_p(f_p)\,,$$

provided ϵ is chosen small enough.

Next, we turn to the proof itself and assume by negation that $f_p(r_0) = 0$ for some $r_0 > 0$. Put

$$\delta_0 = \max_{r \in [0,r_0]} f_p(r)\,.$$

By the above claim $\delta_0 > 0$. Let $\delta \in (0, \delta_0)$ and consider the set $S_\delta = \{r > 0 : f_p(r) < \delta\}$. Denote by $J = (\alpha, \beta)$ the component of S_δ containing r_0. Since $\delta < \delta_0$ we have $\alpha > 0$. There is a $\delta_1 > 0$ such that the function

$$H_r(t) = \left(\frac{t}{r}\right)^p + \frac{1}{2}(1-t^2)^2$$

is decreasing on $[0, \delta_1]$ for every $r \geq \alpha$. We may now replace δ by $\min(\delta, \delta_1)$ and set

$$\tilde{f}(r) = \begin{cases} f_p & r \notin J \\ \delta & r \in J \end{cases}\,.$$

From the monotonicity of H_r it follows that $I_p(\tilde{f}) < I_p(f_p)$. A contradiction. □

2.2 Uniqueness

Proposition 2.2. *The non-negative minimizer for $I_p(f)$ is unique.*

Proof. We use a convexity argument due to Benguria (see [4]) for the case of the Laplacian (see [4]) and by Diaz and Saá [6] and Anane [2] for the case of the p-Laplacian. More specifically, we follow the presentation of Belloni and Kawhol [3]. Assume f and g are both minimizers in (2.1). By an argument from the proof of Proposition 2.1 it follows that necessarily $f(r) \leq 1$ and $g(r) \leq 1$ for each r. Set

$$\eta = \frac{f^p + g^p}{2} \quad \text{and} \quad w = \eta^{\frac{1}{p}}\,.$$

Denote also

$$s(r) = \frac{f^p}{f^p + g^p}\,.$$

Note that

$$w' = \frac{1}{2}\eta^{\frac{1}{p}-1}\left(f^{p-1}f' + g^{p-1}g'\right).$$

Next we compute

$$
\begin{aligned}
(w'^2+\frac{w^2}{r^2})^{\frac{p}{2}} &= \Big|\eta^{\frac{2}{p}-2}(\frac{f^{p-1}f'+g^{p-1}g'}{2})^2+\frac{\eta^{\frac{2}{p}}}{r^2}\Big|^{\frac{p}{2}} = \eta\Big|(\frac{f^{p-1}f'+g^{p-1}g'}{2\eta})^2+\frac{1}{r^2}\Big|^{\frac{p}{2}} \\
&= \eta\Big|(\frac{s(r)f'}{f}+\frac{(1-s(r))g'}{g})^2+\frac{1}{r^2}\Big|^{\frac{p}{2}} \\
&\le \eta\Big(s(r)\big(\frac{|f'|^2}{f^2}+\frac{1}{r^2}\big)^{\frac{p}{2}}+(1-s(r))\big(\frac{|g'|^2}{g^2}+\frac{1}{r^2}\big)^{\frac{p}{2}}\Big) \\
&= \frac{\eta}{f^p+g^p}\big(f'^2+\frac{f^2}{r^2}\big)^{\frac{p}{2}}+\frac{\eta}{f^p+g^p}\big(g'^2+\frac{g^2}{r^2}\big)^{\frac{p}{2}} = \frac{1}{2}\Big(\big(f'^2+\frac{f^2}{r^2}\big)^{\frac{p}{2}}+\big(g'^2+\frac{g^2}{r^2}\big)^{\frac{p}{2}}\Big)
\end{aligned}
$$

Above we used the convexity of the function $t\mapsto(t^2+\frac{1}{r^2})^{\frac{p}{2}}$. Note that equality holds in the above only if $\frac{f'}{f}=\frac{g'}{g}$. If such an equality holds for all r, we conclude easily that $g=cf$ for some constant c, which then must be equal to 1. Therefore, the uniqueness claim follows from the above inequality and the convexity of the second term $(1-f^2)^2$ as a function of f^p for $p\ge 2$ and $0\le f\le 1$. □

Remark 2.1. *As a matter of fact, the only minimizers of I_p are f_p and $-f_p$. In view of lemma 2.1 a non-negative minimizer must be strictly positive. Since $I_p(|f|)=I_p(f)$, it follows that a minimizer may not change sign, and our assertion follows from the uniqueness for non-negative minimizers.*

2.3 Regularity

This subsection is devoted to the study of the regularity properties of the minimizer f_p.

Proposition 2.3. *We have $f_p\in C^\infty(0,\infty)$.*

Proof. The Euler-Lagrange equation associated with (2.1) is

$$\frac{1}{r}\big(r|\nabla u_p|^{p-2}f_p'\big)'=|\nabla u_p|^{p-2}\frac{f_p}{r^2}-\frac{2}{p}f_p(1-f_p^2)\,, \tag{2.7}$$

where

$$u_p=f_p(r)e^{i\theta}\,.$$

A direct consequence of (2.7) is that $|\nabla u_p|^{p-2}f_p'\in W^{1,\frac{p}{p-2}}_{\mathrm{loc}}(0,\infty)\subset C(0,\infty)$ and we immediately obtain that $f_p\in C^1(0,\infty)$ (using that $f_p>0$ by Lemma 2.1) . Inserting this new information into (2.7) we deuce that $f_p\in C^2(0,\infty)$. Bootstrapping gives $f_p\in C^k(0,\infty)$ for all k, as claimed. □

Our next objective is to prove the differentiability of f at 0.

Proposition 2.4. $f_p'(0)=\lim_{r\to0^+}\frac{f_p(r)}{r}$ *exists and is a positive number.*

Proof. We denote for convenience f for f_p and get from (2.7),

$$0 = f''\Big(1 + \frac{p-2}{|\nabla u_p|^2}|f'|^2\Big) + \frac{f'}{r}\Big(1 - \frac{p-2}{|\nabla u_p|^2}\frac{f^2}{r^2}\Big) - \frac{f}{r^2}\Big(1 - \frac{p-2}{|\nabla u_p|^2}|f'|^2\Big) + \frac{2}{p}|\nabla u_p|^{2-p} f(1-f^2)\,, \quad (2.8)$$

or equivalently,

$$\frac{rf''}{f'} = \frac{-\Big(|f'|^2 - (p-3)\frac{f^2}{r^2}\Big) + \frac{f}{rf'}\Big(\frac{f^2}{r^2} - (p-3)|f'|^2\Big)}{\frac{f^2}{r^2} + (p-1)|f'|^2} - \frac{2}{p}|\nabla u_p|^{2-p}\frac{f}{rf'}r^2(1-f^2)\cdot\frac{1}{1+(p-2)\frac{|f'|^2}{|\nabla u_p|^2}}\,. \quad (2.9)$$

Put

$$h = \frac{rf'}{f}\,. \quad (2.10)$$

We divide the rest of the proof into several steps.

Step 1: $-\frac{1}{p-1} < h(r) < 1$ for all $r > 0$.

We can rewrite (2.9) as

$$r\frac{f''}{f'} = \frac{-h^2 + (p-3) + h^{-1} - (p-3)h}{1+(p-1)h^2} - \frac{2}{p}|\nabla u_p|^{2-p}\frac{r^2}{h}(1-f^2)\frac{1+h^2}{1+(p-1)h^2}\,. \quad (2.11)$$

Since

$$h' = \frac{f''h}{f'} + \frac{h}{r}(1-h)\,, \quad (2.12)$$

substituting (2.11) into (2.12) yields

$$\begin{aligned} h' &= \Big(\frac{1-h}{r}\Big)\cdot\Big(\frac{1+(p-2)h+h^2}{1+(p-1)h^2} + h\Big) - \frac{2}{p}|\nabla u_p|^{2-p} r(1-f^2)\frac{1+h^2}{1+(p-1)h^2} \\ &= \frac{1+h^2}{1+(p-1)h^2}\left[\frac{(1-h)[1+(p-1)h]}{r} - \frac{2}{p}|\nabla u_p|^{2-p} r(1-f^2)\right]\,. \quad (2.13)\end{aligned}$$

By (2.13) we have

$$h' \le \frac{1}{r}F_p(h)\,, \quad (2.14)$$

where

$$F_p(h) = \frac{(1+h^2)(1-h)[1+(p-1)h]}{1+(p-1)h^2}\,. \quad (2.15)$$

We now prove that $h(r) < 1$ for all $r > 0$. Suppose to the contrary that there exists $r_0 > 0$ for which $h(r_0) \ge 1$. Then, (2.14) yields $h'(r) < 0$ and $h(r) > 1$ for all $r < r_0$. Therefore, by (2.15) also $F_p(h) < 0$ for $r < r_0$. Integrating (2.14) gives

$$\int_{h(r_0)}^{h(r)} \frac{dh}{-F_p(h)} \ge \ln\frac{r_0}{r}\,, \quad \forall r < r_0\,. \quad (2.16)$$

Since $\int_{h(r_0)}^{\infty} \frac{dh}{-F_p(h)} < \infty$, (2.16) leads to a contradiction for $r > 0$ small enough.

Finally, we show that $h(r) > -\frac{1}{p-1}$ on $(0, \infty)$. Suppose to the contrary that $h(r_0) \leq -\frac{1}{p-1}$ for some r_0. Then, from (2.14) and (2.15) it follows that

$$h(r) \leq -\frac{1}{p-1} \text{ and } h'(r) < 0\,, \quad \forall r \geq r_0\,.$$

Therefore, also $f_p'(r) < 0$ for all $r \geq r_0$, violating $I_p(f_p) < \infty$. Step 1 is established.

<u>Step 2:</u> $\frac{f_p(r)}{r}$ is strictly decreasing on $(0, \infty)$.

From Step 1 we get that

$$\left(\frac{f}{r}\right)' = \frac{f}{r^2}(h-1) < 0\,,\ \forall r > 0\,, \tag{2.17}$$

and the conclusion follows.

<u>Step 3:</u> $\lim_{r\to 0^+} h(r) = 1$.

Fix any $r_0 > 0$. By Step 2 we have,

$$|\nabla u_p|(r) \geq \frac{f(r)}{r} > \frac{f(r_0)}{r_0}\,, \quad \forall r < r_0\,.$$

Consequently, we have by (2.13),

$$h' \geq \frac{F_p(h)}{r} - C_0 r\,, \quad \forall r \in (0, r_0)\,, \tag{2.18}$$

for some positive C_0, which is independent of r. For a contradiction, we assume that $\liminf_{r\to 0^+} h(r) = a < 1$. Then, using (2.18) we can find $r_1 \in (0, r_0)$ small enough so that $h'(r_1) > 0$. Bootstrapping we obtain that $h'(r) > 0$ for all $r < r_1$. In particular, the full limit $\lim_{r\to 0^+} h(r) = a$ exists. Integration of (2.18) then yields

$$\int_{h(r)}^{h(r_1)} \frac{dh}{F_p(h)} \geq \ln\frac{r_1}{r} - C\,, \quad \forall r < r_1\,. \tag{2.19}$$

Here we used the fact that $F_p(h) > 0$ by Step 1. Passing to the limit $r \to 0^+$ in (2.19) gives $\int_a^{h(r_1)} \frac{dh}{F_p(h)} = \infty$. In view of (2.15) we must have

$$a = \lim_{r\to 0^+} h(r) = -\frac{1}{p-1}\,.$$

In particular, for r sufficiently small we have $\frac{rf'}{f} \leq -\frac{1}{2(p-1)}$, implying

$$f(r) \geq C r^{-\frac{1}{2(p-1)}}\,.$$

A contradiction.

Step 4: $f'(0)$ exists and it is a positive number.

By Step 2, the (possibly generalized) limit $\lim_{r\to 0^+}\frac{f(r)}{r}$ exists, so we only need to exclude the possibility that the limit equals $+\infty$. From Step 3 and (2.18) we get that

$$h(r) \geq 1 - cr^2\,,\ \forall r < r_0\,,$$

i.e.,

$$\frac{f'}{f} \geq \frac{1}{r} - cr\,.$$

Therefore, $f(r) \leq Cr$ for some positive constant C, independently of r, and the differentiability of f at 0 follows. Finally, $f'(0) > 0$ since $\frac{f(r)}{r}$ is decreasing. □

2.4 Monotonicity

Proposition 2.5. *$f_p' > 0$ in $(0,\infty)$.*

Proof. First we show that f_p is non-decreasing on $(0,\infty)$. Recall that $f_p'(0) > 0$ and define

$$r_1 = \sup\{r\,:\, f_p'(s) \geq 0 \text{ on } [0,r]\}\,.$$

If $r_1 = \infty$ then clearly f_p is non-decreasing on $(0,\infty)$. Assume then that $r_1 < \infty$, and then obviously

$$f_p'(r_1) = 0\,.$$

By the definition of r_1 we have also

$$f_p''(r_1) \leq 0\,. \tag{2.20}$$

Next we distinguish between two cases:

(i) There exists a right-neighborhood of r_1, $[r_1, R]$, in which $f_p' \leq 0$.

(ii) There exists no neighborhood as in (i).

Consider first case (i). Since $f_p \xrightarrow[r\to\infty]{} 1$, there must exist a maximal right-neighborhood, where $f_p' \leq 0$ which we denote by $[r_1, r_2]$. Clearly, we must have $f_p'(r_2) = 0$. From (2.8) we get that

$$\frac{r^2 f_p''}{f_p} = 1 - \frac{2}{p}\Big(\frac{f_p}{r}\Big)^{-(p-2)} r^2(1-f_p^2)\,, \quad \text{for } r = r_i\,,\ i = 1,2\,. \tag{2.21}$$

By Step 2 of the proof of Proposition 2.4 we have

$$\Big(\frac{f_p(r_2)}{r_2}\Big)^{-(p-2)} > \Big(\frac{f_p(r_1)}{r_1}\Big)^{-(p-2)}\,. \tag{2.22}$$

Furthermore, since $f_p' \leq 0$ in $[r_1, r_2]$ we have

$$(1-f_p^2)(r_2) \geq (1-f_p^2)(r_1)\,. \tag{2.23}$$

Substituting (2.22), (2.23) into (2.21) and using (2.20) yields

$$\frac{r^2 f_p''}{f_p}\Big|_{r=r_2} < \frac{r^2 f_p''}{f_p}\Big|_{r=r_1} \le 0\,,$$

i.e., $f_p''(r_2) < 0$, which clearly contradicts the definition of r_2.

Next we turn to case (ii). In this case we have $f_p''(r_1) = 0$. Differentiating the equation (2.7) at $r = r_1$ yields

$$f_p^{(3)}(r_1) = -p\frac{f_p}{r_1^3} < 0\,. \tag{2.24}$$

This implies that r_1 is a maximum point for f_p' which is obviously impossible.

Finally, we prove that $f_p' > 0$ on $[0,\infty)$ (we know already that $f_p'(0) > 0$). Suppose, for a contradiction, that there exists $r_0 > 0$ such that

$$f_p'(r_0) = f_p''(r_0) = 0\,.$$

We then obtain the same identity as in (2.24), but this time at $r = r_0$. Again we get that f_p' has a maximum at r_0, a contradiction. □

To prove monotonicity of f_p' we need the following result

Lemma 2.2. *We have*

$$h' \le 0\,, \quad \forall r > 0\,. \tag{2.25}$$

Furthermore,

$$\lim_{r\to\infty} h(r) = 0\,.$$

Proof. Suppose, for a contradiction that (2.25) does not hold. Since $\lim_{r\downarrow 0} h(r) = 1$ and $h < 1$ on $(0,\infty)$ (see Steps 1 and 4 in the proof of Proposition 2.4) h must have a minimum point at some $r = r_0$. By (2.13) we have

$$h''(r_0) = -\frac{1}{r_0^2}F_p(h) - \frac{2}{p}\frac{1+h^2}{1+(p-1)h^2}|\nabla u_p|^{2-p}\Big[-\frac{(|\nabla u_p|^2)'}{|\nabla u_p|^2}\frac{p-2}{2}r_0(1-f_p^2) + (1-f_p^2) - 2r_0 f_p f_p'\Big]\,. \tag{2.26}$$

Furthermore, as $h'(r_0) = 0$ we also have

$$\frac{1}{r_0^2}F_p(h) = \frac{2}{p}\frac{1+h^2}{1+(p-1)h^2}|\nabla u_p|^{2-p}(1-f_p^2)$$

$$(|\nabla u_p|^2)'\Big|_{r=r_0} = \left(\frac{f_p^2}{r^2}(1+h^2)\right)'\Big|_{r=r_0} = -2(1+h^2)\frac{f_p}{r_0^2}\left(\frac{f_p}{r_0} - f_p'\right)\,.$$

Substituting the above into (2.26) we obtain

$$\operatorname{sign} h''(r_0) = \operatorname{sign} g(r_0)\,, \tag{2.27}$$

where

$$g(r) := 2hf_p^2 - \{(p-2)(1-h)+2\}(1-f_p^2)\,. \tag{2.28}$$

Since r_0 is a minimum point of h, we must have $g(r_0) \geq 0$. Put

$$r_1 = \sup\{r \in (r_0,\infty)\,:\, h' \geq 0 \text{ on } (r_0,r]\}\,.$$

If $r_1 = \infty$ then, since $h < 1$, $h \xrightarrow[r\to\infty]{} h_\infty$ where $0 < h_\infty \leq 1$. But this leads to a contradiction since then also

$$rf_p' \xrightarrow[r\to\infty]{} h_\infty\,,$$

which is inconsistent with $\lim_{r\to\infty} f(r) = 1$. If $r_1 < \infty$ then necessarily $h'(r_1) = 0$ and $h''(r_1) \leq 0$, implying that $g(r_1) \leq 0$ too. But since h is non-decreasing on (r_0, r_1) while f is strictly increasing on (r_0, r_1) (by Proposition 2.5), it follows from (2.28) that g is strictly increasing on (r_0, r_1). Therefore, $g(r_1) > g(r_0) \geq 0$, implying as in (2.27) that $h''(r_1) > 0$. This contradiction completes the proof of (2.25).

Finally, as h is both positive and decreasing it must converge to a limit $h_\infty \geq 0$. From the above argument we obtain that $h_\infty = 0$. □

Corollary 2.1. *f_p' is monotone decreasing in $\mathbb{R}_+$.*

The corollary follows immediately from the fact that f_p' is a product of the positive functions h and f_p/r, the first of which is non-increasing, and the second is strictly decreasing.

2.5 Asymptotic behavior

In the following we derive the behavior of $1-f_p^2$ as $r\to\infty$. The first lemma is a well-established result in asymptotic analysis. We include the proof for the convenience of the reader.

Lemma 2.3. *Let $g(x)$ be monotone decreasing on $(0,\infty)$. Let further*

$$\int_r^\infty g(t)dt = \frac{1}{r^\alpha}[1+o(1)] \quad \textit{as } r\to\infty\,.$$

for some positive α. Then,

$$g(r) = \frac{\alpha}{r^{\alpha+1}}[1+o(1)] \quad \textit{as } r\to\infty\,.$$

Proof. Put $G(r) = \int_r^\infty g(t)\,dt$. Then, for any $h>0$,

$$hg(r) \geq \int_r^{r+h} g(t)\,dt = G(r) - G(r+h) = \frac{1+\eta(r)}{r^\alpha} - \frac{1+\eta(r+h)}{(r+h)^\alpha}\,, \tag{2.29}$$

where $\lim_{r\to\infty}\eta(r) = 0$. By (2.29),

$$\begin{aligned} hg(r) &\geq (1+\eta(r))\Big(\frac{1}{r^\alpha} - \frac{1}{(r+h)^\alpha}\Big) + \frac{\eta(r)-\eta(r+h)}{r^\alpha} \\ &\geq \frac{1}{r^\alpha}\Big((1-\eta_m)\{1-(1+\frac{h}{r})^{-\alpha}\} - 2\eta_m\Big)\,, \end{aligned}$$

where $\eta_m(r,h)=\max(|\eta(r)|,|\eta(r+h)|)$. Let $\epsilon=\frac{h}{r}$. Since for some $C>0$ we have

$$1-(1+\epsilon)^{-\alpha}\geq 1+\alpha\epsilon-C\epsilon^2\,,\ \epsilon\in[0,\frac{1}{2}]\,,$$

it follows that

$$hg(r)\geq\frac{1}{r^\alpha}\Big((1-\eta_m)(\alpha\epsilon-C\epsilon^2)-2\eta_m\Big)\,.$$

Therefore,

$$g(r)\geq\frac{1}{r^{\alpha+1}}\Big((1-\eta_m)(\alpha-C\epsilon)-2\frac{\eta_m}{\epsilon}\Big)\,. \tag{2.30}$$

Choosing $\epsilon=\eta_m^{1/2}$ we get from (2.30) (since $\lim_{r\to\infty}\sup_{h>0}\eta_m(r,h)=0$),

$$g(r)\geq\frac{\alpha}{r^{\alpha+1}}(1+o(1))\,,\quad\text{as } r\to\infty\,.$$

The second direction is proved in a similar manner. □

We use the above lemma to prove the following result

Lemma 2.4.

$$1-f_p^2\sim\frac{p}{2}\frac{1}{r^p}\quad \textit{as } r\to\infty\,, \tag{2.31a}$$

$$f_p'\sim\frac{p^2}{4}\frac{1}{r^{p+1}}\,. \tag{2.31b}$$

Proof. Integrating by parts (2.7) between r and infinity yields

$$\int_r^\infty f_p(1-f_p^2)dt=\frac{p}{2}\int_r^\infty|\nabla u_p|^{p-2}\Big[\frac{f_p}{t}-f_p'\Big]\frac{dt}{t}+\frac{p}{2}|\nabla u_p|^{p-2}f_p'\,,$$

or equivalently that

$$\int_r^\infty f_p(1-f_p^2)dt=\frac{p}{2}\int_r^\infty|1+h^2|^{(p-2)/2}(1-h)\Big(\frac{f_p}{t}\Big)^{p-1}\frac{dt}{t}+\frac{p}{2}|1+h^2|^{(p-2)/2}h\Big(\frac{f_p}{r}\Big)^{p-1}\,. \tag{2.32}$$

Applying the integral mean value theorem yields the existence of $r^*\in[r,\infty)$ such that

$$\int_r^\infty|1+h^2|^{(p-2)/2}(1-h)\Big(\frac{f_p}{t}\Big)^{p-1}\frac{dt}{t}=|1+h^2(r^*)|^{(p-2)/2}(1-h(r^*))f_p^{p-1}(r^*)\frac{r^{-(p-1)}}{p-1}\,.$$

Hence, in view of Lemma 2.2 and the fact that $f_p\xrightarrow[r\to\infty]{}1$ we obtain

$$\int_r^\infty|1+h^2|^{(p-2)/2}(1-h)\Big(\frac{f_p}{t}\Big)^{p-1}\frac{dt}{t}=\frac{r^{-(p-1)}}{p-1}[1+o(1)]\quad\text{as } r\to\infty\,. \tag{2.33}$$

Further, in view of Lemma 2.2 we have

$$|1+h^2|^{(p-2)/2}h\Big(\frac{f_p}{r}\Big)^{p-1}=o(r^{-(p-1)})\,. \tag{2.34}$$

Substituting (2.33)–(2.34) into (2.32) yields

$$\int_r^\infty f_p(1-f_p^2)dt = \frac{p}{2}\frac{r^{-(p-1)}}{p-1}[1+o(1)] \quad \text{as } r\to\infty\,.$$

As $f_p \xrightarrow[r\to\infty]{} 1$ we have

$$\int_r^\infty f_p(1-f_p^2)dt = [1+o(1)]\int_r^\infty (1-f_p^2)dt \quad \text{as } r\to\infty\,,$$

and hence

$$\int_r^\infty (1-f_p^2)dt = \frac{p}{2}\frac{r^{-(p-1)}}{p-1}[1+o(1)] \quad \text{as } r\to\infty\,.$$

The proof of (2.31a) follows immediately from Lemma 2.3 and the monotonicity of f_p.

To prove (2.31b) we first note that

$$\lim_{r\to\infty}\frac{1-f_p^2}{1-f_p} = 2\,.$$

Hence,

$$\int_r^\infty f_p' dt = \frac{p}{4r^p}[1+o(1)] \quad \text{as } r\to\infty\,.$$

Lemma 2.3 provides, once again, the closing argument for the proof. □

3 Large p

In this section we discuss the behavior of the radially symmetric solution in the large p limit. We prove the following result

Theorem 1. *Let*

$$f_\infty = \begin{cases} \frac{r}{\sqrt{2}} & r<\sqrt{2} \\ 1 & r\geq\sqrt{2} \end{cases}\,. \tag{3.1}$$

There exists $C>0$ such that for every $p>2$ we have

$$\|f_p - f_\infty\|_{L^\infty(\mathbb{R}_+)} = \|f_p - f_\infty\|_\infty \leq C\Big(\frac{\ln p}{p}\Big)^{1/2}\,. \tag{3.2}$$

To prove the theorem we shall need to prove first a few auxiliary results. We first derive a simple upper bound

Lemma 3.1. *We have*

$$I_p(f_p) \leq \left(\frac{1}{6} + C\frac{\ln p}{p}\right)\,, \quad \forall p>2\,. \tag{3.3}$$

Proof. We use the test function

$$\tilde{f} = \begin{cases} \frac{1}{\sqrt{2}}\Big(1 - \frac{\ln p}{p}\Big) r\,, & r < \frac{\sqrt{2}}{1-\frac{\ln p}{p}} \\ 1\,, & r \geq \frac{\sqrt{2}}{1-\frac{\ln p}{p}} \end{cases} .$$

It is easy to show that there exists $C > 0$, independent of p such that

$$I_p(\tilde{f}) \leq \left(\frac{1}{6} + C\frac{\ln p}{p}\right)\,, \quad \forall p > 2\,,$$

from which the lemma immediately follows. □

We first deal with the interval $[0, \sqrt{2}]$.

Proposition 3.1. *We have*

$$\exists C > 0:\ \|\nabla u_p\|_\infty \leq 1 + \frac{C}{p}\,, \quad \forall p > 2\,. \tag{3.4}$$

Proof. We first note that by Lemma 2.2 and Step 2 of the proof of Proposition 2.4 both f_p' and f_p/r are decreasing. Therefore, the same holds for $|\nabla u_p|$ and it follows that

$$\|\nabla u_p\|_\infty = |\nabla u_p(0)|\,. \tag{3.5}$$

Obviously, if we have $|\nabla u_p|-1 \gg 1/p$ over a sufficiently large right semi-neighborhood of $r = 0$, then $I_p(f)$ would become larger than the upper bound (3.3). This, however, does not eliminate the possibility of a small neighborhood of $r = 0$ where $p(|\nabla u_p| - 1)$ is large. Thus, the proof splits into two parts: at first, using regularity arguments, we bound from below the size of the above neighborhood as a function of $|\nabla u_p(0)|$. Then, we use (3.3) to bound $|\nabla u_p(0)|$ from above.

Suppose that $|\nabla u_p(0)| = a > 1$. Let

$$s = \sup\left\{r > 0\ :\ |\nabla u_p(r)| > \frac{1+a}{2}\right\}. \tag{3.6}$$

By (2.13) we have for all $r < s$ that

$$h' \geq -\frac{2}{p}\left(\frac{1+a}{2}\right)^{-(p-2)} r\,, \tag{3.7}$$

hence

$$1 - \frac{1}{p}\Big(\frac{1+a}{2}\Big)^{-(p-2)} r^2 \leq h \leq 1\,, \quad \forall r \leq s\,. \tag{3.8}$$

Assume first that

$$s^2 \leq \frac{p}{2}\Big(\frac{1+a}{2}\Big)^{p-2}\,, \tag{3.9}$$

implying by (3.8) that

$$h \geq \frac{1}{2}\,, \quad \forall r \leq s\,. \tag{3.10}$$

By (2.12) we have

$$\frac{f_p''}{f_p'} = \frac{h'}{h} - \frac{1-h}{r}\,. \tag{3.11}$$

Therefore, using (3.10) and (3.7)–(3.8) we deduce that

$$\left|\frac{f_p''}{f_p'}\right| = -\frac{f_p''}{f_p'} \leq \frac{C}{p}\left(\frac{1+a}{2}\right)^{2-p} r\,, \quad \forall r \leq s\,,$$

implying

$$\exp\Big\{ - \frac{C}{2p}\left(\frac{1+a}{2}\right)^{2-p} r^2 \Big\} \leq \frac{f_p'(r)}{f_p'(0)}\,, \quad \forall r \leq s\,. \tag{3.12}$$

Since $h < 1$,

$$\frac{|\nabla u_p(r)|^2}{|\nabla u_p(0)|^2} = \frac{(1+h^{-2})|f_p'(r)|^2}{2|f_p'(0)|^2} \geq \frac{|f_p'(r)|^2}{|f_p'(0)|^2}\,, \quad \forall r \leq s\,.$$

Consequently, by (3.12)

$$\exp\Big\{ - \frac{C}{p}\left(\frac{1+a}{2}\right)^{2-p} r^2 \Big\} \leq \frac{|\nabla u_p(r)|^2}{|\nabla u_p(0)|^2}\,, \quad \forall r \leq s\,. \tag{3.13}$$

Setting $r = s$ in (3.13) we obtain

$$s^2 \geq Cp\left(\frac{1+a}{2}\right)^{p-2} \ln\left(\frac{2a}{1+a}\right) \geq Cp\left(\frac{1+a}{2}\right)^{p-2} \frac{a-1}{a}\,.$$

If (3.9) doesn't hold, then clearly

$$s^2 > \frac{p}{2}\left(\frac{1+a}{2}\right)^{p-2}.$$

Therefore, in all cases we have

$$s^2 \geq Cp\frac{a-1}{a}\left(\frac{1+a}{2}\right)^{p-2}. \tag{3.14}$$

To conclude, we shall use the upper-bound for the energy from Lemma 3.1 in order to bound s from above. Combining (3.14) with (3.3) and (3.6) yields

$$C \geq \int_0^s |\nabla u_p|^p r dr \geq \frac{s^2}{2}\left(\frac{1+a}{2}\right)^p \geq Cp\frac{a-1}{a}\left(\frac{1+a}{2}\right)^{2(p-1)} \geq Cp(a-1)\,, \tag{3.15}$$

From (3.15) we get

$$a \leq 1 + \frac{C}{p}\,, \quad \forall p > 2\,,$$

and (3.4) follows from (3.5). □

We can now obtain L^∞ convergence of f_p to f_∞ in every compact set in $[0,\sqrt{2})$.

Proposition 3.2. *For every $b \in (0,\sqrt{2})$ there exists $C = C(b) > 0$ such that,*

$$\Big\| f_p - \frac{r}{\sqrt{2}} \Big\|_{L^\infty(0,b)} \le C\frac{\ln p}{p}\,, \quad p > 2, \tag{3.16a}$$

$$\Big\| f_p' - \frac{1}{\sqrt{2}} \Big\|_{L^\infty(0,b)} \le C\frac{\ln p}{p}\,, \quad p > 2\,. \tag{3.16b}$$

Proof. First we note that by (3.4)

$$|\nabla u_p(0)|^2 = 2f_p'(0)^2 \le 1 + \frac{C}{p} \implies f_p'(0) \le \frac{1}{\sqrt{2}} + \frac{C}{p}\,.$$

Since f_p' is decreasing, we conclude that

$$f_p'(r) \le \frac{1}{\sqrt{2}} + \frac{C}{p}\,. \tag{3.17}$$

Integrating (3.17), using $f_p(0) = 0$, yields the existence of $C > 0$ such that for every $p > 2$ we have, for all $r > 0$,

$$f_p(r) \le \frac{1}{\sqrt{2}}\Big(1 + \frac{C}{p}\Big)r\,. \tag{3.18}$$

Put

$$w(r) = \frac{r}{\sqrt{2}} - f_p(r).$$

By (3.17)–(3.18) we have

$$w'(r) \ge -\frac{C}{p} \ge -C\frac{\ln p}{p} \quad \text{and} \quad w(r) \ge -\frac{C}{p} \ge -C\frac{\ln p}{p}\,, \quad \forall r > 0\,.$$

In order to conclude, we need to prove that for each $b \in (0,\sqrt{2})$ there exists C_b such that

$$w'(r) \le C_b\frac{\ln p}{p} \quad \text{and} \quad w(r) \le C_b\frac{\ln p}{p}\,, \quad \forall r \in [0,b]\,. \tag{3.19}$$

For such b we set $\tilde{b} = \frac{b+\sqrt{2}}{2}$ and claim that

$$\int_b^{\tilde{b}} w(r)\,dr \le C_b\frac{\ln p}{p}\,. \tag{3.20}$$

To prove (3.20) we first note that

$$\frac{1}{2}\int_0^\infty \big(1 - f_p^2\big)^2 r\,dr \ge \frac{1}{2}\int_0^{\sqrt{2}} \big(1 - f_p^2\big)^2 r\,dr = \frac{1}{2}\int_0^{\sqrt{2}} \Big[1 - \frac{1}{2}r^2\Big]^2 rdr +$$
$$\int_0^{\sqrt{2}} \Big[1 - \frac{1}{2}r^2\Big]\Big[\frac{1}{2}r^2 - f_p^2\Big] r\,dr + \frac{1}{2}\int_0^{\sqrt{2}} \Big[\frac{1}{2}r^2 - f_p^2\Big]^2 r\,dr\,.$$

Since

$$\frac{1}{2}\int_0^{\sqrt{2}}\left[1-\frac{1}{2}r^2\right]^2 rdr=\frac{1}{6}\,,$$

we deduce, using (3.3), that

$$\int_0^{\sqrt{2}}\left[1-\frac{1}{2}r^2\right]\left[\frac{1}{2}r^2-f_p^2\right]r\,dr\le C\frac{\ln p}{p}\,.$$

Therefore

$$C_b\frac{\ln p}{p}\ge(1-\frac{\tilde{b}^2}{2})\int_b^{\tilde{b}}w(r)(\frac{r}{\sqrt{2}}+f_p)\,rdr\,,$$

and (3.20) follows. Finally, using the convexity of w in conjunction with (3.20) gives

$$w(b)(\tilde{b}-b)+\frac{w'(b)}{2}(\tilde{b}-b)^2=\int_b^{\tilde{b}}(w(b)+(r-b)w'(b))\,dr\le\int_b^{\tilde{b}}w(r)\,dr\le C_b\frac{\ln p}{p}\,,$$

implying, in particular, that

$$w'(b)\le C_b\frac{\ln p}{p}\,. \tag{3.21}$$

Since w' is increasing we deduce the first inequality in (3.19). The second one follows by integration of the first one. □

We now improve the estimates (3.16). We start by deriving a Pohozaev-type identity.

Proposition 3.3. *We have*

$$\int_0^\infty|\nabla u_p|^p r\,dr=\frac{2}{p}m_p\,, \tag{3.22}$$

where m_p is defined in (2.3).

Proof. Let $f_p^{(\alpha)}(r)=f_p(\alpha r)$ and $J=\int_0^\infty|\nabla u_p|^p r\,dr$. Clearly,

$$M_\alpha=I_p(f_p^{(\alpha)})=\alpha^{p-2}J+\frac{1}{\alpha^2}(m_p-J)\,.$$

Hence,

$$\frac{dM_\alpha}{d\alpha}=(p-2)\alpha^{p-3}J-\frac{2}{\alpha^3}(m_p-J)\,.$$

Since M_α must have a global minimum at $\alpha=1$, (3.22) follows. □

Corollary 3.1. *We have*

$$\liminf_{p\to\infty}p|\nabla u_p(0)|^p\ge\frac{1}{3}\,. \tag{3.23}$$

Proof. Since $f_p' < f_p/r < 1/r$ we have

$$|\nabla u_p| \leq \frac{\sqrt{2}}{r}\,.$$

Thus, for every $l > 0$,

$$\int_l^\infty |\nabla u_p|^p r dr \leq \frac{2^{p/2}}{(p-2)l^{p-2}}\,,$$

from which we get

$$\lim_{p\to\infty} \int_l^\infty |\nabla u_p|^p\, r dr = 0\,, \quad \forall l > \sqrt{2}\,. \tag{3.24}$$

By (3.16) we have

$$\liminf_{p\to\infty} m_p \geq \sup_{b\in(0,\sqrt{2})} \liminf_{p\to\infty} \frac{1}{2}\int_0^b (1-f_p^2)^2 r dr = \frac{1}{6}\,.$$

Thus, by (3.24) and (3.22) we have for all $l > \sqrt{2}$,

$$\liminf_{p\to\infty} p \int_0^l |\nabla u_p|^p r dr \geq \frac{1}{3}\,.$$

As $|\nabla u_p(r)| \leq |\nabla u_p(0)|$ we deduce that

$$\liminf_{p\to\infty} p|\nabla u_p(0)|^p \frac{l^2}{2} \geq \frac{1}{3} \quad \forall l > \sqrt{2}\,,$$

from which (3.23) readily follows. □

Lemma 3.2. *Let* $g = |\nabla u_p|^p$, *and*

$$g_0 = \frac{1}{p}\Big(1 - \frac{1}{2}r^2\Big)^2\,.$$

Then,

$$\lim_{p\to\infty} p\|g - g_0\|_{L^\infty(0,a)} = 0\,, \quad \forall a < \sqrt{2}\,. \tag{3.25}$$

Proof. Multiplying (2.7) by rf_p and integrating over $[0, r]$, we obtain

$$\frac{p}{4}r^2 g(1-\alpha_p) - \frac{p}{2}\int_0^r g(t)tdt + h(r) = 0\,, \tag{3.26a}$$

in which

$$\alpha_p = 1 - \frac{2\frac{f_p}{r}f_p'}{|\nabla u_p|^2} > 0\,, \tag{3.26b}$$

and

$$h(r) = \int_0^r f_p^2(1-f_p^2)tdt\,. \tag{3.26c}$$

We may write

$$h(r) = h_0(r)(1+\beta_p) \quad \text{with} \quad h_0(r) = \int_0^r \frac{t^2}{2}\Big(1-\frac{t^2}{2}\Big)t\,dt = \frac{1}{8}\Big(r^4 - \frac{1}{3}r^6\Big)\,.$$

Set

$$\epsilon_p(a) = \max\big(\|\alpha_p\|_{L^\infty(0,a)}, \|\beta_p\|_{L^\infty(0,a)}\big)\,. \tag{3.27}$$

By (3.16) there exists $C>0$ such that

$$\epsilon_p(a) \le C\frac{\ln p}{p}\,, \tag{3.28}$$

for all fixed $a<\sqrt{2}$.

Set

$$G(r) = \int_0^r g(t)t\,dt\,,$$

to obtain from (3.26) that

$$G' - \gamma_p G + \frac{2}{p}\gamma_p h = 0\,, \tag{3.29}$$

where

$$\gamma_p = \frac{2}{r(1-\alpha_p)}\,.$$

Solving (3.29) and then evaluating G' once again from (3.29) yields the general solution of (3.26):

$$g(r) = -\frac{2}{p}\frac{\gamma_p}{r}\left[h + \int_0^r \exp\Big\{\int_t^r \gamma_p(s)ds\Big\}\gamma_p(t)h(t)dt + C_0\exp\Big\{-\int_r^a \gamma_p(t)dt\Big\}\right]\,, \tag{3.30}$$

where C_0 is arbitrary.

First we compute

$$\frac{\gamma_p}{r}\exp\Big\{-\int_r^a \gamma_p(t)dt\Big\} \le \frac{\gamma_p}{r}\exp\Big\{-2\int_r^a \frac{dt}{t}\Big\} = \frac{\gamma_p r}{a^2}\,.$$

On the other hand, a similar computation gives

$$\frac{\gamma_p}{r}\exp\Big\{-\int_r^a \gamma_p(t)dt\Big\} \ge \frac{\gamma_p}{r}\Big(\frac{r}{a}\Big)^{\frac{2}{1-\epsilon_p}}\,.$$

Therefore,

$$\frac{2}{a^2}\Big(\frac{r}{a}\Big)^{2\left((1-\epsilon_p)^{-1}-1\right)} \le \frac{\gamma_p}{r}\exp\Big\{-\int_r^a \gamma_p(t)dt\Big\} \le \frac{2}{a^2}\big(1+C\epsilon_p\big)\,.$$

Similarly,

$$\frac{\gamma_p}{r}\int_0^r \exp\left\{\int_t^r \gamma_p(s)\,ds\right\}\gamma_p(t)h(t)\,dt \geq \frac{\gamma_p}{r}\int_0^r \left(\frac{r}{t}\right)^2 \gamma_p(t)h(t)\,dt$$
$$\geq 4\int_0^r \frac{h(t)}{t^3}\,dt \geq \left(\frac{1}{4}r^2 - \frac{1}{24}r^4\right)(1-\epsilon_p)\,,$$

and

$$\frac{\gamma_p}{r}\int_0^r \exp\left\{\int_t^r \gamma_p(s)\,ds\right\}\gamma_p(t)h(t)\,dt \leq \left(\frac{1}{4}r^2 - \frac{1}{24}r^4\right)(1+C\epsilon_p)\,.$$

Combining the above with (3.30) we obtain that

$$\frac{2}{p}\left[\tilde{C}_0 r^{2\left((1-\epsilon_p)^{-1}-1\right)} - \frac{1}{2}r^2 + \frac{1}{8}r^4 - C\epsilon_p\right] \leq g \leq \frac{2}{p}\left[\tilde{C}_0 - \frac{1}{2}r^2 + \frac{1}{8}r^4 + C\epsilon_p\right]. \quad (3.31)$$

Note that the above lower bound is unsatisfactory in some neighborhood of $r=0$ where

$$1 - r^{2\left((1-\epsilon_p)^{-1}-1\right)} \sim \mathcal{O}(\epsilon_p)\,,$$

which is valid for $r \sim \mathcal{O}(1)$ as $p \to \infty$.

We defer the proof of convergence near $r=0$ to a later stage and instead prove first the existence of $\lim_{p\to\infty}\tilde{C}_0(p)$, and then obtain its value. Clearly,

$$\liminf_{p\to\infty}\tilde{C}_0(p) \geq \frac{1}{2},$$

otherwise g would become negative, for some sufficiently large p and a fixed $r_0 < \sqrt{2}$ - a contradiction. Suppose now to the contrary, that a sequence $\{p_k\}_{k=1}^\infty$ exists such that $\tilde{C}_0(p_k) = C_k \to b$, where $b \in (\frac{1}{2}, \infty]$. By (3.4) we have

$$\|g(\cdot, p_k)\|_{L^\infty(\mathbb{R}_+)} \leq C\,,$$

where C is independent of k. Hence, by (3.31) we have

$$C_k \leq C p_k\,. \quad (3.32)$$

Set

$$g_{0,k} = 2\left[C_k - \frac{1}{2}r^2 + \frac{1}{8}r^4\right].$$

Note that by our supposition $\lim g_{0,k}(r) > 0$ in $[0, \sqrt{2}+\delta]$ for some $\delta > 0$. It follows from (3.31) and (3.32) that

$$\frac{\ln(g_{0,k}-\epsilon_k)}{p_k} - 2\left|\frac{\ln(g_{0,k}-\epsilon_k)}{p_k}\right|^2 \leq |\nabla u_p| - 1 + \frac{\ln p_k}{p_k} \leq \frac{\ln(g_{0,k}+\epsilon_k)}{p_k} + 2\left|\frac{\ln(g_{0,k}+\epsilon_k)}{p_k}\right|^2 \quad (3.33)$$

where $\epsilon_k(a) = \epsilon(p_k)(a)$.

We argue from here by bootstrapping. Let $a \in (0, \sqrt{2}+\delta]$ be such that

$$\limsup \epsilon_k(a) \leq \limsup \frac{g_{0,k}(a)}{2}\,. \tag{3.34}$$

For sufficiently large k we have, in view of (3.33) and (3.32) and the fact that $\epsilon_k(\sqrt{2}+\delta)$ is bounded , that

$$2\frac{f_k}{r}f_k' \leq |\nabla u_p|^2 \leq 1 + \frac{C}{p_k} \quad \forall r \in [0, \sqrt{2}+\delta]\,,$$

where $f_k = f_{p_k}$, from which we obtain that

$$\frac{f_k}{r} \leq \frac{1}{\sqrt{2}} + \frac{C}{p_k} \quad \forall r \in [0, \sqrt{2}+\delta] \ \ . \tag{3.35}$$

Consequently, by (3.33) and (3.34), we have for sufficiently large k that

$$f_k' \geq \frac{1}{\sqrt{2}} - C\frac{\ln p_k}{p_k} \quad \forall r \in [0, a]\,, \tag{3.36}$$

where C is independent of a. Since $f_k' \leq f_k/r$, we have by (3.26b), for sufficiently large k, that

$$\alpha_{p_k} \leq \frac{\ln p_k}{p_k} \quad \forall r \in [0, a]\,, \tag{3.37}$$

where C is independent of a. Furthermore, by (3.26c,d), (3.35), (3.36), and the fact that $f_k/r > f_k'$ there exists $C > 0$ which is independent of both k and a such that

$$\beta_{p_k}(r) \leq C\frac{\ln p_k}{p_k}$$

for all $r \leq a$. Combining the above and (3.37) we obtain for sufficiently large k

$$\limsup \epsilon_k(a) \leq \lim \frac{g_{0,k}(a)}{2} \Rightarrow \limsup \frac{p_k}{\ln p_k}\epsilon_k \leq C\,, \tag{3.38}$$

where C is independent of a. From (3.16) we thus have

$$\limsup \frac{p_k \epsilon_k}{\ln p_k} \leq C$$

for all $a < \sqrt{2}$.

Let then a_0 be such that

$$\limsup \epsilon_k(a_0) = \lim \frac{g_{0,k}(a_0)}{2}\,.$$

Since by (3.38) we have $\lim g_{0,k}(a_0) = 0$, it follows that $a_0 > \sqrt{2}+\delta$. Hence

$$\limsup \frac{p_k}{\ln p_k}\epsilon_k(\sqrt{2}+\delta) \leq C\,.$$

Substituting into (3.31) we obtain that

$$\lim \frac{p_k}{\ln p_k} g(\sqrt{2}+\delta) > 0\,.$$

Let $l > \sqrt{2}$. Then, $f_p'(l) < f_p(l)/l < 1/l$, and hence $g(l) \leq (\sqrt{2}/l)^p$. Consequently, $g(l)$ is exponentially small for all $l > \sqrt{2}$ as $p \to \infty$, and in particular at $l = \sqrt{2}+\delta$ - a contradiction. Hence, we obtain that $\lim_{p\to\infty} \tilde{C}_0(p) = 1/2$.

To complete the proof of (3.25) we need to extend (3.31) to every neighborhood of $r = 0$. Since obtaining an $\mathcal{O}(\epsilon_p)$ accuracy in this neighborhood is a difficult task, we allow for an error of larger magnitude. Thus, requiring that

$$\frac{2}{p}\Big[\tilde{C}_0 r^{2\left((1-\epsilon_p)^{-1}-1\right)} - \frac{1}{2}r^2 + \frac{1}{8}r^4 - C\epsilon_p^{\frac{1}{2}}\Big] \leq g \leq \frac{2}{p}\Big[\tilde{C}_0 - \frac{1}{2}r^2 + \frac{1}{8}r^4 + C\epsilon_p^{\frac{1}{2}}\Big]\,. \quad (3.39)$$

It is easy to show that the lower bound in (3.39) provides an estimate which is $O(\epsilon_p^{\frac{1}{2}})$-accurate whenever $r^2 > e^{-\epsilon_p^{-\frac{1}{2}}}$. To complete the proof of (3.25), we just need to obtain an $O(\epsilon_p^{\frac{1}{2}})$-accurate estimate for g, valid for $r^2 \leq e^{-\epsilon_p^{-\frac{1}{2}}}$.

We argue again by bootstrapping. We may regroup the terms in (2.8) to get

$$-\frac{2}{p}|\nabla u_p|^{2-p} f_p(1-f_p^2) = f_p''\Big(1+\frac{p-2}{|\nabla u_p|^2}|f_p'|^2\Big) + \Big(\frac{f_p'}{r}-\frac{f_p}{r^2}\Big)\Big(1+\frac{p-2}{|\nabla u_p|^2}\frac{f_p f_p'}{r}\Big)\,. \quad (3.40)$$

By Step 1 in the proof of Proposition 2.4 we have $\frac{f_p'}{r} - \frac{f_p}{r^2} > 0$, and by Corollary 2.1, $f_p'' < 0$. Hence,

$$f_p'' \geq -\frac{2}{p}|\nabla u_p|^{2-p} f_p(1-f_p^2)\,.$$

It follows that as long as

$$g \geq g(0)\big(1-\epsilon_p^{\frac{1}{2}}\big)\,,$$

we must have, by (3.16) that

$$f_p'' \geq -\frac{2}{p}\frac{r}{\Big[g(0)\big(1-\epsilon_p^{\frac{1}{2}}\big)\Big]^{(p-2)/p}}\,.$$

Integrating the above yields, in view of (3.23),

$$f_p'(r) \geq f_p'(0) - 4r^2\big(1+2\epsilon_p^{\frac{1}{2}}\big)\,.$$

Note that we can replace the constant 4 by any other constant greater than 3. Consequently,

$$|\nabla u_p| \geq \sqrt{2}|f_p'| \geq \sqrt{2}\big|f_p'(0) - 4r^2\big(1+2\epsilon_p^{\frac{1}{2}}\big)\big| \geq \sqrt{2}|f_p'(0)|\bigg|1 - \frac{4\sqrt{2}r^2}{|\nabla u_p(0)|}^{2}\big(1+2\epsilon_p^{\frac{1}{2}}\big)\bigg|\,.$$

Hence,

$$g(r) \ge g(0)\big(1-\epsilon_p^{\frac{1}{2}}\big) \Rightarrow g(r) \ge g(0)\left[1-\frac{4\sqrt{2}r^2}{|\nabla u_p(0)|}\big(1+2\epsilon_p^{\frac{1}{2}}\big)\right]^p$$

Applying again (3.23) we obtain that as long as

$$r^2 < \frac{1}{8p}\epsilon_p^{\frac{1}{2}},$$

we have

$$g(r) \ge g(0)\big(1-\epsilon_p^{\frac{1}{2}}\big). \tag{3.41}$$

On the other hand,

$$g(r) \le g(0). \tag{3.42}$$

Since (3.41), (3.42), and (3.39) are simultaneously satisfied at $r^2 = 2e^{-\epsilon_p^{-\frac{1}{2}}}$, we obtain

$$\begin{cases} \frac{2}{p}\tilde{C}_0(1+C\epsilon_p^{\frac{1}{2}}) \ge g(0)(1-\epsilon_p^{\frac{1}{2}}) \\ \frac{2}{p}\tilde{C}_0(1-C\epsilon_p^{\frac{1}{2}}) \le g(0) \end{cases}.$$

Consequently,

$$\left|\tilde{C}_0 - \frac{p}{2}g(0)\right| \le C\epsilon_p^{\frac{1}{2}}.$$

Furthermore, in view of (3.41) and (3.39) we can safely state that

$$\frac{2}{p}\left[\tilde{C}_0-\frac{1}{2}r^2+\frac{1}{8}r^4\tilde{C}_0-C\epsilon_p^{\frac{1}{2}}\right] \le g \le \frac{2}{p}\left[\tilde{C}_0-\frac{1}{2}r^2+\frac{1}{8}r^4+C\epsilon_p^{\frac{1}{2}}\right], \quad \text{in } [0,a], \forall a\in(0,\sqrt{2}). \tag{3.43}$$

Or

$$\|p(g-g_0)\| \le |\tilde{C}_0(p)-1| + C\epsilon_p^{1/2}.$$

□

Remark 3.1. *From* (3.25) *we can obtain the next two terms in the asymptotic expansion of* f_p *in the large* p *limit*

$$f_p = \frac{r}{\sqrt{2}}\left[1-\frac{\ln p}{p}+\frac{\ln g_0(r)}{p}+o\left(\frac{1}{p}\right)\right]. \tag{3.44}$$

The above expansion is valid in $[0,a]$ *for every* $a<\sqrt{2}$.

Remark 3.2. *Note that* (3.43) *is valid for all* $r>0$. *It is only because of* (3.28) *that we have to confine the validity of* (3.25) *to closed intervals in* $[0,\sqrt{2})$ *whose edges do not depend on* p. *Note further that by* (3.27) *we have that* $\epsilon_p \le 2$ *for all* $r\le\sqrt{2}$.

We can now extend the validity of the above estimate to $[0,\sqrt{2}-\mathcal{O}(\sqrt{\ln p/p})]$.

Proposition 3.4. *There exists $C > 0$, which is independent of p, such that the estimate* (3.25) *holds for every $r \in [0, \sqrt{2} - C(\ln p/p)^{1/2}]$.*

Proof. Let

$$g_{0,C} = \tilde{C}_0 - \frac{1}{2}r^2 + \frac{1}{8}r^4 .$$

Suppose that $\tilde{C}_0$ is such that $g_{0,C}(\sqrt{2} - \Delta_p) = 0$. From the previous lemma we have that $\Delta_p \to 0$ as $p \to \infty$. It is easy to show that,

$$g_{0,C}\Big(\sqrt{2} - \frac{2}{3}\Delta_p\Big) \leq -C\Delta_p^2 ,$$

for all $C < 1/6$ and for sufficiently large p.

Let

$$|\nabla u_p|\Big(\sqrt{2} - \frac{2}{3}\Delta_p\Big) = 1 - \delta_p .$$

Since $f_p/r \leq 1/\sqrt{2}$, we have $f_p'\big(\sqrt{2} - 2\Delta_p/3\big) \geq 1 - C\delta_p$. From here it is easy to show that $\epsilon_p(\sqrt{2} - 2\Delta_p/3) \leq C\delta_p$. By (3.40) we have

$$f_p'' \leq -\frac{C}{p^2}|1-\delta_p|^{2-p}(1 - \frac{r}{\sqrt{2}}) + Cp^{1/2} \quad \forall r \in [\sqrt{2} - 2\Delta_p/3, \sqrt{2} - \Delta_p/3] , \tag{3.45}$$

where we have taken into account the fact that $|\nabla u_p|$ is decreasing and that

$$\frac{1 + \frac{p-2}{|\nabla u_p|^2}\frac{f_p' f_p}{r}}{1 + \frac{p-2}{|\nabla u_p|^2}|f_p'|^2} \leq Cp^{1/2} .$$

Integrating (3.45) over $[\sqrt{2} - 2\Delta_p/3, \sqrt{2} - \Delta_p/3]$ yields

$$-\frac{1}{\sqrt{2}} \leq -C\frac{\Delta_p^2}{p^2}|1-\delta_p|^{-p} + Cp^{1/2}\Delta_p ,$$

from which we obtain

$$(1-\delta_p)^p \geq C\frac{\Delta_p^2}{p^{5/2}} .$$

Consequently,

$$\delta_p \leq \frac{5}{2}\frac{\ln p}{p} - 2\frac{\ln \Delta_p}{p} + \frac{C}{p} .$$

We conclude from here that

$$\epsilon_p(\sqrt{2} - 2\Delta_p/3) \leq C\delta_p \leq C\frac{\ln p}{p} . \tag{3.46}$$

Since g is positive we obtain by (3.31) that

$$\Delta_p \leq C\Big[\frac{\ln p}{p}\Big]^{1/2} .$$

Since $\epsilon_p(a)$ is an increasing function of a (3.25) must be valid in $[0, \sqrt{2} - 2\Delta_p/3]$. $\square$

Proof of Theorem 1. In view of proposition 3.4 there exists $C > 0$ such that (3.16a) and hence (3.44) hold for sufficiently large p whenever $r < \sqrt{2} - C(\ln p/p)^{1/2}$. From the monotonicity of f_p it follows that

$$f_p(\sqrt{2} - C/(\ln p/p)^{1/2}) \leq f_p(r) \leq 1\,.$$

□

4 Stability of the radial solution

In this section we prove our main stability result for $u_p = f_p(r)e^{i\theta}$, the degree one radially symmetric solution of

$$\frac{p}{2}\nabla \cdot (|\nabla u|^{p-2}\nabla u) + u(1 - |u|^2) = 0\,. \tag{4.1}$$

A simple computation gives the second variation of E_p at u_p:

$$J_2(\phi) = \int_{\mathbb{R}^2} \left\{ \frac{p}{2}|\nabla u_p|^{p-2}\left[|\nabla \phi|^2 + (p-2)\frac{|\Re(\nabla u_p \cdot \nabla\bar{\phi})|^2}{|\nabla u_p|^2}\right] + 2|\Re(u_p\bar{\phi})|^2 - (1 - |u_p|^2)|\phi|^2 \right\}. \tag{4.2}$$

Because of (4.2) and analogously to [5], we consider perturbations in the "natural" Hilbert space $\mathcal{H}$ consisting of functions $\phi \in H^1_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ for which

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left\{ \frac{p}{2}|\nabla u_p|^{p-2}\left[|\nabla \phi|^2 + (p-2)\frac{|\Re(\nabla u_p \cdot \nabla\bar{\phi})|^2}{|\nabla u_p|^2}\right] + 2|\Re(u_p\bar{\phi})|^2 + (1 - |u_p|^2)|\phi|^2 \right\} < \infty\,.$$

Note that $\mathcal{H}$ contains all "admissible perturbations" ϕ, i.e., any ϕ for which $E_p(u_p + \phi) < \infty$. Note also that in contrast with the case $p = 2$, in our case $p > 2$, constant functions do belong to $\mathcal{H}$. Thanks to the invariance of the functional E_p with respect to rotations and translations (see [11]) we have

$$J_2(\phi) = 0 \text{ for } \phi = \begin{cases} \frac{\partial u_p}{\partial \theta} = i f_p e^{i\theta}\,, \\ \frac{\partial u_p}{\partial x_1} = \frac{1}{2}(f_p' - \frac{f_p}{r})e^{2i\theta} + \frac{1}{2}(f_p' + \frac{f_p}{r})\,, \\ \frac{\partial u_p}{\partial x_2} = -\frac{i}{2}(f_p' - \frac{f_p}{r})e^{2i\theta} + \frac{i}{2}(f_p' + \frac{f_p}{r})\,. \end{cases} \tag{4.3}$$

Indeed, this leads to the equality cases in the next theorem.

Theorem 2. *For every $2 < p \leq 4$ the radially symmetric solution u_p is stable in the sense that $J_2(\phi) \geq 0$ for all $\phi \in \mathcal{H}$. Moreover, we have $J_2(\phi) = 0$ if and only if*

$$\phi = c_0\frac{\partial u_p}{\partial \theta} + c_1\frac{\partial u_p}{\partial x_1} + c_2\frac{\partial u_p}{\partial x_2}, \text{ for some constants } c_0, c_1, c_2 \in \mathbb{R}. \tag{4.4}$$

Following [9] we represent each ϕ by its Fourier expansion

$$\phi = \sum_{n=-\infty}^{\infty} \phi_n(r) e^{in\theta} \,. \tag{4.5}$$

Substituting into (4.2) we obtain

$$\frac{1}{2\pi} J_2(\phi) = E_1(\phi_1) + \sum_{n=2}^{\infty} E_n(\phi_n, \phi_{2-n}) \,, \tag{4.6}$$

in which

$$E_1(\phi_1) = \int_0^\infty \left\{ \frac{p}{2} |\nabla u_p|^{p-2} \left[|\phi_1'|^2 + \frac{1}{r^2}|\phi_1|^2 + (p-2) \frac{\left|\Re\left(f_p'\phi_1' + \frac{f_p\phi_1}{r^2}\right)\right|^2}{|\nabla u_p|^2} \right] \right.$$
$$\left. + 2f_p^2 |\Re\phi_1|^2 - (1-f_p^2)|\phi_1|^2 \right\} r dr \,, \tag{4.7a}$$

and

$$E_n(\phi_n, \phi_{2-n}) = \int_0^\infty \left\{ \frac{p}{2} |\nabla u_p|^{p-2} \left[|\phi_n'|^2 + |\phi_{2-n}'|^2 + \frac{n^2}{r^2}|\phi_n|^2 + \frac{(2-n)^2}{r^2}|\phi_{2-n}|^2 + \right. \right.$$
$$\left. \frac{1}{2}(p-2) \frac{\left| f_p'(\bar{\phi}_n' + \phi_{2-n}') + \frac{f_p}{r^2}(n\bar{\phi}_n + (2-n)\phi_{2-n}) \right|^2}{|\nabla u_p|^2} \right]$$
$$\left. + f_p^2 |(\bar{\phi}_n + \phi_{2-n})|^2 - (1-f_p^2)(|\phi_n|^2 + |\phi_{2-n}|^2) \right\} r dr \,. \tag{4.7b}$$

A necessary and sufficient condition for the positive definiteness of J_2 is that the E_n's are all positive definite. An appropriate Hilbert space for the study of the functionals $\{E_n\}$ is

$$\mathcal{S} = \{\phi \in H^1_{loc}(\mathbb{R}_+, \mathbb{C}) \cap L^2_r(\mathbb{R}_+, \mathbb{C}) \,:\, \int_0^\infty \frac{p}{2}|\nabla u_p|^{p-2} \Big[|\phi'|^2 + \frac{1}{r^2}|\phi|^2 \Big] \, r dr < \infty \} \,.$$

We also denote by $\tilde{\mathcal{S}}$ the space of real-valued functions in $\mathcal{S}$.

4.1 $n \neq 2$

We consider first the case $n = 1$.

Lemma 4.1.

$$\inf_{\phi \in \mathcal{S}} E_1(\phi) = 0 \,. \tag{4.8}$$

Furthermore, the minimum in (4.8) *is attained only for* $\phi = cif_p$*, for any real constant* c*.*

Proof. Since $E_1(i|\phi|) \le E_1(\phi)$ for every ϕ for which $E_1(\phi) < \infty$, with strict inequality unless ϕ takes only purely imaginary values, we may consider instead of E_1 the following functional

$$\tilde{E}_1(\phi) = \int_0^\infty \left\{ \frac{p}{2} |\nabla u_p|^{p-2} \Big[|\phi'|^2 + \frac{1}{r^2}|\phi|^2 \Big] - (1 - f_p^2)|\phi|^2 \right\} r dr \,,$$

over $\tilde{\mathcal{S}}$. Consider first $\phi \in C_c^\infty(0,\infty)$ and set $\phi = f_p w$. Integration by parts, with the aid of (2.7) yields

$$F_1(w) = \tilde{E}_1(f_p w) = \int_0^\infty \frac{p}{2} |\nabla u_p|^{p-2} f_p^2 |w'|^2 r dr \,. \tag{4.9}$$

A standard use of cut-off functions yields that (4.9) holds also for smooth $\phi = f_p w$ with compact support in $[0,\infty)$ (i.e, the support may contain the origin). Finally, by density of smooth maps with compact support in $[0,\infty)$ in $\tilde{\mathcal{S}}$ it follows that (4.9) continues to hold for $\phi = f_p w \in \tilde{\mathcal{S}}$. Therefore, $\tilde{E}_1(\phi) \ge 0$ for all $\phi \in \tilde{\mathcal{S}}$ and $F_1(w) = 0$ if and only if $w \equiv$ const. □

We now consider the case $n \ge 3$.

Proposition 4.1. *For each $n \ge 3$ we have*

$$E_n(u_1, u_2) > 0 \text{ for all } (u_1, u_2) \in \tilde{\mathcal{S}} \times \tilde{\mathcal{S}} \setminus \{(0,0)\} \,.$$

Proof. The result follows right away from the previous lemma and the inequality

$$E_n(u_1, u_2) \ge \tilde{E}_1(|u_1|) + \tilde{E}_1(|u_2|) \,,$$

with strict inequality, unless $u_j \equiv 0$, $j = 1, 2$. □

4.2 $n = 2$

It is easy to reduce the analysis of E_2 to that of a functional acting on real-valued functions. Indeed, writing a complex-valued function ϕ as $\phi = \phi^R + i\phi^I$, we have

$$E_2(\phi_2, \phi_0) = E_2^R(\phi_2^R, \phi_0^R) + E_2^I(\phi_2^I, \phi_0^I) \,,$$

where

$$E_2^R(\phi_2^R, \phi_0^R) = E_2(\phi_2^R, \phi_0^R), \quad E_2^I(\phi_2^I, \phi_0^I) = E_2(i\phi_2^I, i\phi_0^I) \,.$$

Clearly,

$$E_2(i\phi_2^I, i\phi_0^I) = E_2^R(-\phi_2^I, \phi_0^I) \,.$$

Hence,

$$E_2(\phi_2, \phi_0) = E_2^I(-\phi_2^R, \phi_0^R) + E_2^I(\phi_2^I, \phi_0^I) \,, \tag{4.10}$$

and it suffices to study the minimization to the functional E_2^I over $\tilde{\mathcal{S}} \times \tilde{\mathcal{S}}$.

From (4.3) and (4.6) it follows that the functions

$$\Phi_0 = f_p' + \frac{f_p}{r} \text{ and } \Phi_2 = -f_p' + \frac{f_p}{r}\,, \tag{4.11}$$

satisfy

$$E_2^R(-\Phi_2, \Phi_0) = E_2^I(\Phi_2, \Phi_0) = 0\,.$$

We next claim:

Proposition 4.2. *For $p \in (2,4]$ we have $E_2^I(\phi_2, \phi_0) \geq 0$ for every $(\phi_2, \phi_0) \in \tilde{\mathcal{S}} \times \tilde{\mathcal{S}}$ with equality if and only if $(\phi_2, \phi_0) = c(\Phi_2, \Phi_0)$ for some $c \in \mathbb{R}$ (see* (4.11)*).*

For the proof of Proposition 4.2 we shall need some preliminary results. First, by (4.7b) we have

$$\begin{aligned} E_2^I(\phi_2, \phi_0) = \int_0^\infty \Bigg\{ \frac{p}{2} |\nabla u_p|^{p-2} \Bigg[(\phi_2')^2 + (\phi_0')^2 + \frac{4}{r^2}(\phi_2)^2 \\ + \frac{1}{2}(p-2) \frac{\big(f_p'(\phi_0' - \phi_2') - 2\frac{f_p}{r^2}\phi_2)\big)^2}{|\nabla u_p|^2} \Bigg] \\ + f_p^2(\phi_0 - \phi_2)^2 - (1 - f_p^2)\big((\phi_2)^2 + (\phi_0)^2\big) \Bigg\} r dr\,. \end{aligned} \tag{4.12}$$

It is more convenient to consider an alternative form by applying the transformation

$$A = \phi_0 + \phi_2, \quad B = \phi_0 - \phi_2,$$

to obtain

$$\begin{aligned} E_2^I(\phi_0, \phi_2) = F_2(A, B) := \int_0^\infty \Bigg\{ \frac{p}{4} |\nabla u_p|^{p-2} \Bigg[(A')^2 + (B')^2 + \frac{2}{r^2}(A - B)^2 \\ + (p-2) \frac{\big(f_p'B' - \frac{f_p}{r^2}(A - B)\big)^2}{|\nabla u_p|^2} \Bigg] \\ + f_p^2 B^2 - \frac{1}{2}(1 - f_p^2)(A^2 + B^2) \Bigg\} r dr\,. \end{aligned} \tag{4.13}$$

Clearly,

$$F_2(f_p/r, f_p') = 0\,. \tag{4.14}$$

The "problematic term" in (4.13) is the one involving the mixed product AB'. The difficulty in handling this term is the obstacle for determining the positivity of F_2 for every $p > 2$. We were able to overcome this difficulty only in the case $p \in (2,4]$ thanks to the following lemma.

Lemma 4.2. *We have*

$$F_2(A,B) = G_2(A,B) + \int_0^\infty \frac{p(p-2)}{4}|\nabla u_p|^{p-2}\frac{\big(hA' - \frac{1}{r}(h^2A - B)\big)^2}{1+h^2} r dr\,, \quad (4.15)$$

with

$$\begin{aligned} G_2(A,B) = \int_0^\infty \Bigg\{ &\frac{p}{4}|\nabla u_p|^{p-2}\Bigg[(A')^2 + (B')^2 + \frac{2}{r^2}(A-B)^2 \\ &+ (p-2)\frac{h^2((B')^2-(A')^2) + \frac{1-h^4}{r^2}A^2 - \frac{2}{r^2}(1-h^2)AB}{1+h^2}\Bigg] \\ &+ \frac{p(p-2)}{4r}[H'(2AB - B^2) - (h^2H)'A^2] \\ &+ f_p^2B^2 - \frac{1}{2}(1-f_p^2)(A^2+B^2)\Bigg\} r dr\,, \quad (4.16) \end{aligned}$$

where

$$H = \frac{h}{1+h^2}|\nabla u_p|^{p-2} \quad \text{and} \quad h = rf_p'/f_p \text{ (as in (2.10))}.$$

Moreover, $G_2(f_p/r, f_p') = 0$ *and the pair* $(f_p/r, f_p')$ *solves the Euler-Lagrange equations associated with* G_2.

Proof. First, a direct computation gives the identity

$$\begin{aligned} &\frac{(f_p'B' - \frac{f_p}{r^2}(A-B))^2}{|\nabla u_p|^2} \\ &= \frac{h^2(|B'|^2 - |A'|^2) - \frac{2h}{r}[(AB)' - BB' - h^2AA'] + \frac{1-h^4}{r^2}|A|^2 - \frac{2}{r^2}(1-h^2)AB}{1+h^2} \\ &\qquad + \frac{(hA' - \frac{1}{r}(h^2A-B))^2}{1+h^2}\,. \quad (4.17) \end{aligned}$$

Next, integration by parts yields

$$\begin{aligned} &\int_0^\infty |\nabla u_p|^{p-2}\Big\{\frac{-\frac{2h}{r}[(AB)' - BB' - h^2AA']}{1+h^2}\Big\} r\,dr \\ &\qquad = \int_0^\infty \Big\{H'(2AB - B^2) - (h^2H)'A^2\Big\}\,dr \quad (4.18) \end{aligned}$$

Using (4.17)–(4.18) in conjunction with (4.13) leads to (4.15)–(4.16). Finally, a direct computation shows that the integrand in the integral on the right-hand-side of (4.15) is identically zero for $A = f_p/r$ and $B = f_p'$, and the last assertion of the lemma follows. □

Proof of Proposition 4.2. In view of Lemma 4.2 it suffices to show that

$$G_2(u,v)\geq 0,\ \forall (u,v)\in \tilde{\mathcal{S}}\times\tilde{\mathcal{S}},\ \text{with equality iff:}\quad u=\phi:=f_p/r \text{ and } v=\psi:=f_p'\,. \tag{4.19}$$

We write G_2 in the form

$$G(u,v)=\int_0^\infty \big(\alpha(r)u'^2+\beta(r)v'^2+a(r)u^2+2b(r)uv+c(r)v^2\big)\,dr\,.$$

The properties of the coefficients which are important to us are

$$\alpha(r),\beta(r)>0 \text{ and } b(r)<0,\quad \text{for } r>0\,. \tag{4.20}$$

Indeed, clearly $\beta(r)>0$. Next,

$$\alpha(r)=\frac{p}{4}|\nabla u_p|^{p-2}r\Big(1-(p-2)\frac{h^2}{1+h^2}\Big)>0\,,$$

provided $p\leq 4$, since $0<h<1$ by Step 1 of Proposition 2.4 and Proposition 2.5. Finally,

$$b=r\Big\{\frac{p}{4}|\nabla u_0|^{p-2}\big[-\frac{2}{r^2}-\frac{(p-2)}{r^2}(1-h^2)\big]+\frac{p(p-2)}{4r}H'\Big\}<0\,,$$

since $0<h(r)<1$, and

$$H'=|\nabla u_p|^{p-2}\frac{(1-h^2)h'}{(1+h^2)^2}+(p-2)\frac{h}{1+h^2}\Big(\big(\frac{f_p}{r}\big)\big(\frac{f_p}{r}\big)'+f_p'f_p''\Big)<0\,,$$

since $h'\leq 0$ by Lemma 2.2 and both f_p' and f_p/r are decreasing (as we noted already before, by Lemma 2.2 and Step 2 of the proof of Proposition 2.4).

By Lemma 4.2 we know that ϕ and ψ satisfy

$$\begin{cases} -(\alpha\phi')'+a\phi+b\psi=0,\\ -(\beta\psi')'+c\psi+b\phi=0.\end{cases} \tag{4.21}$$

□

We consider first $u,v\in C_c^\infty(0,\infty)$. By Picone's identity

$$(u')^2-\big(\frac{u^2}{\phi}\big)'\phi'=(u'-(u/\phi)\phi')^2\geq 0 \tag{4.22}$$

$$(v')^2-\big(\frac{v^2}{\psi}\big)'\psi'=(v'-(v/\psi)\psi')^2\geq 0\,. \tag{4.23}$$

Multiplying (4.22)–(4.23) by α and β respectively, applying integration by parts and using (4.21) we obtain

$$\begin{aligned} 0 &\leq \int_0^\infty \alpha(u')^2 - \alpha(\frac{u^2}{\phi})'\phi' + \beta(v')^2 - \beta(\frac{v^2}{\psi})' \\ &= \int_0^\infty \alpha(u')^2 + \frac{u^2}{\phi}(a\phi + b\psi) + \beta(v')^2 + \frac{v^2}{\psi}(c\psi + b\phi) \\ &= \int_0^\infty \alpha u'^2 + \beta v'^2 + au^2 + cv^2 + b(u^2\frac{\psi}{\phi} + v^2\frac{\phi}{\psi}) \\ &= G(u,v) + \int_0^\infty b\Big(u(\frac{\psi}{\phi})^{1/2} - v(\frac{\phi}{\psi})^{1/2}\Big)^2 . \end{aligned} \tag{4.24}$$

From (4.24) and a density argument we conclude that

$$G(u,v) \geq \int_0^\infty (-b)\Big(u(\frac{\psi}{\phi})^{1/2} - v(\frac{\phi}{\psi})^{1/2}\Big)^2 , \ \forall u,v \in \tilde{\mathcal{S}} ,$$

and (4.19) follows.

Next we are ready to present the proof of our main stability theorem.

Proof of Theorem 2. Representing each $\phi \in \mathcal{H}$ by its Fourier expansion (4.5), we have by (4.6), Lemma 4.1, Proposition 4.1, (4.10) and Proposition 4.2 that $J_2(\phi) \geq 0$. Furthermore, by the equality cases in Lemma 4.1, Proposition 4.1 and Proposition 4.2 we have $J_2(\phi) = 0$ iff $\phi = \phi_0 + \phi_1 e^{i\theta} + \phi_2 e^{2i\theta}$ where

$$\phi_1 = a_1 i f_p\,, \quad (\phi_2^I, \phi_0^I) = a_2(\Phi_2, \Phi_0) \ \text{ and } (-\phi_2^R, \phi_0^R) = a_3(\Phi_2, \Phi_0)\,, \qquad \text{with } a_1, a_2, a_3 \in \mathbb{R}\,.$$

It is easy to verify that these relations are equivalent to (4.4). □

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