Global minimizers for a p-Ginzburg-Landau-type energy in $\mathbb{R}^2$

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Abstract

Given a $p > 2$, we prove existence of global minimizers for a p-Ginzburg-Landau-type energy over maps on $\mathbb{R}^2$ with degree $d = 1$ at infinity. For the analogous problem on the half-plane we prove existence of a global minimizer when p is close to 2.

The key ingredient of our proof is the degree reduction argument that allows us to construct a map of degree $d = 1$ from an arbitrary map of degree $d > 1$ without increasing the p-Ginzburg-Landau energy.

1 Introduction

For a given $p > 2$ consider the Ginzburg-Landau-type energy

$$E_p(u) = \int_{\mathbb{R}^2} |\nabla u|^p + \frac{1}{2}\left(1 - |u|^2\right)^2 \tag{1.1}$$

over the class of maps $u \in W^{1,p}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ that satisfy $E_p(u) < \infty$ and have a degree d "at infinity". The last statement can be made precise by observing that any map $u \in W^{1,p}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ with $E_p(u) < \infty$ satisfies

- $u \in C^{\alpha}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ where $\alpha = 1 - 2/p$ (Morrey's lemma, [11]).
- $\lim_{|x|\to\infty} |u(x)| = 1$ (Section 3 below).

Therefore, there exists an $R > 0$ such that the degree $\deg\left(\frac{u}{|u|}, \partial B_r(0)\right)$ is well-defined for every $r \geq R$ and is independent of r. We use this value as the definition of the degree, $\deg(u)$.

For any integer $d \in \mathbb{Z}$, introduce the class of maps

$$\mathcal{E}_d = \left\{u \in W^{1,p}_{loc}(\mathbb{R}^2, \mathbb{R}^2) : E_p(u) < \infty,\ \deg(u) = d\right\}$$

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and define

$$I_p(d) = \inf_{u \in \mathcal{E}_d} E_p(u) \,. \tag{1.2}$$

The set $\mathcal{E}_d$ is nonempty, as can be readily seen, e.g., by verifying that the map $v(re^{i\theta}) = f(r)e^{id\theta}$ with

$$f(r) = \begin{cases} r, & r < 1, \\ 1, & r \geq 1, \end{cases}$$

is in $\mathcal{E}_d$. A natural question then is whether the infimum in (1.2) is attained. Our main result provides an affirmative answer when $d = \pm 1$—we are uncertain as to whether this conclusion remains true for $|d| \geq 2$.

Theorem 1. *For $d = \pm 1$ there exists a map realizing the infimum $I_p(d)$ in* (1.2)*.*

Note that the problem (1.2) is meaningless for the standard Ginzburg-Landau energy E_2 because it is not even clear how the class $\mathcal{E}_d$ should be defined when $p = 2$ and $d \neq 0$. In fact, by a result of Cazenave (described in [7]), the constant solutions $u = e^{i\alpha}$ with $\alpha \in \mathbb{R}$ are the *only* finite energy solutions of the associated Euler-Lagrange equation, $-\Delta u = (1 - |u|^2)u$. Clearly, the degree of these solutions is zero. The natural questions for $p = 2$ are concerned with *local minimizers*, i.e., those maps that are minimizers of the energy functional E_2 on $B_R(0)$ with respect to $C_0^\infty(B_R(0))$-perturbations for every $R > 0$. These questions were first addressed in [7]. Subsequently, Mironescu [10], relying on a result of Sandier [12], characterized these local minimizers completely by showing that, up to a translation and a rotation, they are all of the form $f(r)e^{i\theta}$. Here $f(r)$ is the unique solution of the ODE obtained by imposing rotational invariance on the Euler-Lagrange equation.

Next we turn our attention to the analogous problem on the upper half-plane

$$\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_2 > 0\} \,.$$

Once again, for $p > 2$, we are interested in minimizers of the energy functional

$$E_p(u) = \int_{\mathbb{R}^2_+} |\nabla u|^p + \frac{1}{2}(1 - |u|^2)^2 \,,$$

but this time over all maps in $W^{1,p}_{loc}(\mathbb{R}^2_+, \mathbb{R}^2)$, satisfying the boundary condition

$$|u(x_1, 0)| = 1 \,, \ \forall x_1 \in \mathbb{R} \,, \tag{1.3}$$

along with a degree condition at infinity. Here the definition of the degree can be given by a small modification of the argument we employed in the $\mathbb{R}^2$-case: we observe that the degree of $\frac{u}{|u|}$ on $\partial(B_R(0) \cap \mathbb{R}^2_+)$ does not depend on R for sufficiently large R and define $\deg(u)$ to be this integer value.

For any $d \in \mathbb{Z}$ we set

$$\mathcal{E}^+_d = \{u \in W^{1,p}_{loc}(\mathbb{R}^2_+, \mathbb{R}^2) : |u(x_1, 0)| = 1, E_p(u) < \infty, \deg(u) = d\} \tag{1.4}$$

and define

$$I_p^+(d) = \inf_{u \in \mathcal{E}_d^+} E_p(u)\,. \tag{1.5}$$

Again, we study the question of existence of a minimizer for (1.5), but we are only able to prove a result analogous to Theorem 1 when p is sufficiently close to 2.

Theorem 2. *For $d = \pm 1$ there exists $p_0 > 2$ such that for all $p \in (2, p_0)$ the infimum $I_p^+(d)$ is attained.*

Recall that minimization problems with degree boundary conditions for the classical Ginzburg-Landau energy ($p = 2$) on perforated bounded domains were studied in [1]-[4]. Our study of the problem on a half-plane was motivated by the results in [2, 3] regarding the behavior of minimizing sequences when the H^1- capacity of the domain is sufficiently small and the minimizing sequences develop vortices approaching the boundary of the domain.

The main tool we use in the proofs of Theorems 1 and 2 is a "degree reduction" proposition proved in Section 2. In this proposition we show how we can transform any given map u of degree $D \geq 2$ (on either $\mathbb{R}^2$ or $\mathbb{R}^2_+$) to a new map $\tilde{u}$ of degree $D = 1$ so that $E_p(\tilde{u}) = E_p(u)$. Loosely speaking, the proposition establishes the intuitively clear result that "less degree implies less energy" for the infima.

In Section 3 we use the degree reduction argument to prove Theorem 1. In Section 4 we study the limit $p \to 2^+$ in the half-plane case and obtain some results needed to prove Theorem 2. The proof of this theorem is given in Section 5.

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2 A key proposition

Here we prove a key proposition that is the main ingredient of the proof of Theorem 1. A variant of it will also be used in the proof of Theorem 2. Before stating the proposition, we provide some basic properties of maps with finite energy.

Lemma 2.1. *Let $u \in W^{1,p}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ be any map with $E_p(u) < \infty$. Then $u \in C^\alpha(\mathbb{R}^2, \mathbb{R}^2)$ with $\alpha = 1 - 2/p$ and*

$$\lim_{|x| \to \infty} |u(x)| = 1\,. \tag{2.1}$$

The analogous result holds for $u \in W^{1,p}_{loc}(\mathbb{R}^2_+, \mathbb{R}^2)$ satisfying $E_p(u) < \infty$.

Proof. The first assertion is a direct consequence of Morrey's inequality [11] which asserts that, upon modifying u on a set of measure zero,

$$|u(x) - u(y)| \leq C \|\nabla u\|_{L^p(\mathbb{R}^2)} |x - y|^\alpha,\ \forall x, y \in \mathbb{R}^2\,, \tag{2.2}$$

for some constant $C > 0$ depending only on p. To prove (2.1) we employ the same argument used in the proof of the analogous result [7] in the case $p = 2$. Suppose that there exists a sequence $|x^{(n)}| \to \infty$ with $\left|u\left(x^{(n)}\right)\right| \le 1 - \delta$ for some $\delta > 0$. Then, by (2.2),

$$\int_{B_1(x^{(n)})} \left(1 - |u|^2\right)^2 \ge \eta > 0\,,$$

for all n and some constant η. But this contradicts our assumption that

$$\int_{\mathbb{R}^2} \left(1 - |u|^2\right)^2 \le E_p(u) < \infty\,.$$

In the case of $u \in W^{1,p}_{loc}(\mathbb{R}^2_+, \mathbb{R}^2)$ it suffices to extend u to a map $U \in W^{1,p}_{loc}\left(\mathbb{R}^2, \mathbb{R}^2\right)$ via reflection w.r.t. the x_2-axis, and to apply the previous argument. $\square$

Next we state and prove the main result of this section.

Proposition 1. *Let $D \ge 2$ be an integer. Then, for each $u \in \mathcal{E}_D$, there exists $\tilde{u} \in \mathcal{E}_1$ such that $E_p(\tilde{u}) = E_p(u)$ and*

$$\tilde{u}_1(x) = u_1(x) \text{ and } |\tilde{u}_2(x)| = |u_2(x)|,\ \forall x \in \mathbb{R}^2\,. \tag{2.3}$$

Proof. By Lemma 2.1 there exists $R_0 > 0$ such that $|u(x)| \ge \frac{1}{2}$ for $|x| \ge R_0$. By Fubini theorem we can find $r \in (R_0, R_0 + 1)$ such that $\int_{\partial B_r(0)} |\nabla u|^p \le E_p(u)$. Therefore, by Hölder inequality,

$$\int_{\partial B_r(0)} \left|\frac{\partial u}{\partial \tau}\right| d\tau < \infty\,. \tag{2.4}$$

Here $\frac{\partial u}{\partial \tau}$ denotes the tangential derivative of u along $\partial B_r(0)$.

We start by constructing a map $\tilde{u} \in W^{1,p}(B_r(0), \mathbb{R}^2)$ (of course, also $\tilde{u} \in C(\overline{B_r(0)}, \mathbb{R}^2)$) satisfying (2.3) on $B_r(0)$ and such that $\deg(\tilde{u}, \partial B_r(0)) = 1$. Thus, until stated otherwise, we consider u on $B_r(0)$ only. Since $u = (u_1, u_2)$ is continuous, we can represent the set $B_r(0) \cap \{u_2 \neq 0\}$ as a union of its (countably many) disjoint components,

$$\{u_2 \neq 0\} \cap B_r(0) = \bigcup_{j \in \mathcal{I}_+} \omega_j^+ \cup \bigcup_{j \in \mathcal{I}_-} \omega_j^-\,,$$

where $\{\omega_j^+\}_{j \in \mathcal{I}_+}$ are the components of the set $B_r(0) \cap \{u_2 > 0\}$, while $\{\omega_j^-\}_{j \in \mathcal{I}_-}$ are the components of $B_r(0) \cap \{u_2 < 0\}$. Each index set $\mathcal{I}_\pm$ is either a finite set of integers $\{1, \dots, N\}$, or the set $\mathbb{N}$ of all positive integers. Note that both $\mathcal{I}_+$ and $\mathcal{I}_-$ are nonempty because, contrary to our assumption, the degree of u is zero if u takes values in a half-plane. Denote

$$u_{2,j}^+ = \chi_{\omega_j^+} |u_2|,\ \forall j \in \mathcal{I}_+ \quad \text{and} \quad u_{2,j}^- = \chi_{\omega_j^-} |u_2|,\ \forall j \in \mathcal{I}_-\,.$$

Then, on $B_r(0)$

$$u_2 = \sum_{j \in \mathcal{I}_+} u_{2,j}^+ - \sum_{j \in \mathcal{I}_-} u_{2,j}^-\,. \tag{2.5}$$

Next we claim that for each $\omega_j^\pm$,

$$u_{2,j}^\pm \in W^{1,p}(B_r(0))\,. \tag{2.6}$$

We pay special attention to cases where $\partial\omega_j^\pm \cap \partial B_r(0)$ is nonempty. We begin by applying a standard argument (cf. [8]) to construct an extension w of u_2 such that $w \in C_c(\mathbb{R}^2) \cap W^{1,p}(\mathbb{R}^2)$. Let Q denote the connected component of the set $\{w \neq 0\}$ that contains $\omega_j^\pm$. Then, $w \in W_0^{1,p}(Q) \cap C(Q)$, and by defining an extension map $\tilde{w}$ which is identically zero on $\mathbb{R}^2 \setminus Q$ we obtain that $\tilde{w} \in W^{1,p}(\mathbb{R}^2) \cap C_c(\mathbb{R}^2)$ (note that no regularity assumption on Q is required for this to hold, see Remarque 20 after Théorème IX.17 in [6]). Since $\tilde{w} = \chi_{\omega_j^\pm} u_2$ on $B_r(0)$, we deduce immediately that (2.6) holds.

It follows from (2.6) that for every pair of maps, $\gamma_+ : \mathcal{I}_+ \to \{-1,+1\}, \gamma_- : \mathcal{I}_- \to \{-1,+1\}$ the function

$$u_2^{(\gamma_+,\gamma_-)} = \sum_{j\in\mathcal{I}_+} \gamma_+(j) u_{2,j}^+ + \sum_{j\in\mathcal{I}_-} \gamma_-(j) u_{2,j}^- \tag{2.7}$$

belongs to $W^{1,p}(B_r(0))$ and the map $\tilde{u} = (u_1, u_2^{(\gamma_+,\gamma_-)})$ satisfies (2.3) on $B_r(0)$. We now show that it is possible to choose γ_+ and γ_- in such a way that the resulting $\tilde{u}$ will have degree equal to 1.

First, we claim that one can assume that $\mathcal{I}_+$ is finite. Indeed, if $\mathcal{I}_+ = \mathbb{N}$, then we define a sequence of maps $v^{(N)} = (v_1^{(N)}, v_2^{(N)})$ by

$$v_2^{(N)} = \sum_{j=1}^{N} u_{2,j}^+ - \sum_{j=N+1}^{\infty} u_{2,j}^+ - \sum_{j\in\mathcal{I}_-} u_{2,j}^- \quad \text{and} \quad v_1^{(N)} = u_1\,,$$

where $N \geq 1$. By dominated convergence, it can be easily seen that $v^{(N)} \to u$ in $W^{1,p}(B_r(0))$, hence also in $C(\overline{B_r(0)})$. By the continuity of the degree, we obtain

$$\lim_{N\to\infty} \deg\left(v^{(N)}, \partial B_r(0)\right) = \deg(u, \partial B_r(0)) = D\,.$$

Therefore, for sufficiently large N, we have $\deg\left(v^{(N)}\right) = D$. Since we can replace u by $v^{(N)}$, the claim follows. We will assume in the sequel that u is such that $\mathcal{I}_+ = \{1, \dots, N\}$ for some $N \in \mathbb{N}$.

Next, we claim that one can effectively assume that $N = 1$. From now on, we assume the positive orientation (i.e., counter clockwise) of $\partial B_r(0)$. The map $U = \frac{u}{|u|}$ is well-defined on $\partial B_r(0)$ and, thanks to (2.4), it belongs to $W^{1,1}(\partial B_r(0), S^1)$. For $j = 1, ..., N$ set

$$A_j = \overline{\omega_j^+} \cap \partial B_r(0) \text{ and } a_j = \int_{A_j} U \wedge U_\tau \, d\tau\,,$$

so that a_j equals the change of phase of U on A_j. Further, denote

$$b = \int_{\partial B_r(0) \setminus \bigcup_{j=1}^{N} A_j} U \wedge U_\tau \, d\tau\,.$$

Clearly

$$2\pi D = b + \sum_{j=1}^{N} a_j\,. \tag{2.8}$$

But since replacing $u = u_1 + iu_2$ by its complex conjugate $\bar{u} = u_1 - iu_2$ on $\bigcup_{j=1}^{N} \omega_j^+$ (without changing u elsewhere) would result with a map of degree zero (since it takes its values only in the lower half-plane), we must also have

$$0 = -\sum_{j=1}^{N} a_j + b\,. \tag{2.9}$$

From (2.8)–(2.9) we get

$$b = \sum_{j=1}^{N} a_j = \pi D\,. \tag{2.10}$$

It follows from (2.10) that there exists j_0 for which

$$b + a_{j_0} - \sum_{j \neq j_0} a_j = 2a_{j_0} > 0\,. \tag{2.11}$$

From (2.11) we deduce that the map $v = (v_1, v_2) \in W^{1,p}(B_r(0))$ with $v_1 = u_1$ and v_2 given by:

$$v_2(x) = \begin{cases} -|u_2(x)| & x \notin \omega_{j_0}^+, \\ u_2(x) & x \in \omega_{j_0}^+, \end{cases}$$

has degree $d > 0$. If $d = 1$ then the proposition is proved, thus we assume in the sequel that $d \geq 2$.

Consider the set $\Omega^- = \{x \in B_r(0) \,:\, v_2(x) < 0\}$ and write it as a disjoint countable union of its components,

$$\Omega^- = \bigcup_{j \in \mathcal{J}} \Omega_j^-\,.$$

Set

$$V = \frac{v}{|v|} \quad \text{on } \partial B_r(0)\,.$$

By (2.4)

$$\int_{\partial B_r(0)} \left| \frac{\partial V}{\partial \tau} \right| d\tau < \infty\,. \tag{2.12}$$

Define

$$G_+ = \{x \in \partial B_r(0) \,:\, v_2(x) > 0\} \ \text{ and } \ G_- = \{x \in \partial B_r(0) \,:\, v_2(x) < 0\}\,.$$

As above

$$\int_{G_+} V \wedge V_\tau \, d\tau = \int_{G_-} V \wedge V_\tau \, d\tau = \pi d\,. \tag{2.13}$$

We can write each of G_+ and G_- (which are (relatively) open subsets of $\partial B_r(0)$) as a countable union of open segments on the circle $\partial B_r(0)$:

$$G_+ = \bigcup_{i\in\mathcal{K}_+} J_+^i \quad\text{and}\quad G_- = \bigcup_{i\in\mathcal{K}_-} J_-^i$$

Clearly, each segment J_-^i satisfies

$$J_-^i \subset \overline{\Omega_{\zeta(i)}^-} \quad\text{for a unique } \zeta(i)\in\mathcal{J}\,. \tag{2.14}$$

Of course, also $J_+^i \subset \overline{\omega_{j_0}^+}$ for each i. Since for each segment $J_\pm^i$, $v_2(\partial J_\pm^i)=0$, i.e., $V(\partial J_\pm^i)\subset\{-1,1\}$, we clearly have

$$\int_{J_\pm^i} V\wedge V_\tau\,d\tau \in \{-\pi,0,\pi\}\,. \tag{2.15}$$

Invoking (2.12) we deduce that the number of intervals $J_\pm^i$ for which $\int_{J_\pm^i} V\wedge V_\tau\,d\tau\neq 0$ is *finite*. We denote them (ordered according to the positive orientation) by

$$J_+^{j_1},\dots,J_+^{j_{\kappa_+}} \quad\text{and}\quad J_-^{l_1},\dots,J_-^{l_{\kappa_-}}\,.$$

Then our assumption that $d\geq 2$ in conjunction with (2.15) and (2.13) implies that $\kappa_+\geq 2$.

Given an $s=1,\dots,\kappa_+$, denote by $r\exp(i\theta_{1,j_s})$ and $r\exp(i\theta_{2,j_s})$ the end points of $J_+^{j_s}$ so that

$$J_+^{j_s} = \{re^{i\theta}\,:\,\theta\in(\theta_{1,j_s},\theta_{2,j_s})\}\,.$$

We claim that there exists at least one pair of two consecutive segments, w.l.o.g. $J_+^{j_1}$ and $J_+^{j_2}$, such that for the intermediate segment

$$I=\{re^{i\theta}\,:\,\theta\in(\theta_{2,j_1},\theta_{1,j_2})\}$$

we have

$$\delta := \int_I V\wedge V_\tau\,d\tau \neq 0\,.$$

Indeed, this follows immediately from the fact that the total change of phase of V over all such intermediate segments equals πd by (2.13).

Next, set

$$I_+=\{x\in I\,:\,v_2(x)>0\} \quad\text{and}\quad I_-=\{x\in I\,:\,v_2(x)<0\}\,.$$

From the definitions of $J_+^{j_s}$ and δ it follows that

$$\int_{I_-} V\wedge V_\tau=\delta \quad\text{and}\quad \int_{I_+} V\wedge V_\tau = 0\,. \tag{2.16}$$

Furthermore, it is easy to see that

(i) $V(r\exp(i\theta_{2,j_1})) = \pm 1$ and $V(r\exp(i\theta_{1,j_2})) = -V(r\exp(i\theta_{2,j_1}))$,

(ii) $\delta = \pm\pi$,

(iii) $I \cap \bigcup_{s=1}^{\kappa_-} J_-^{l_s} \neq \emptyset$, and there is an odd number of segments $J_-^{l_{\sigma_1}}, \ldots, J_-^{l_{\sigma_{2k+1}}}$ such that $J_-^{l_{\sigma_i}} \subset I$ for every $i = 1, \ldots, 2k+1$ and some $k \in \mathbb{N}$ (see Figure 1).

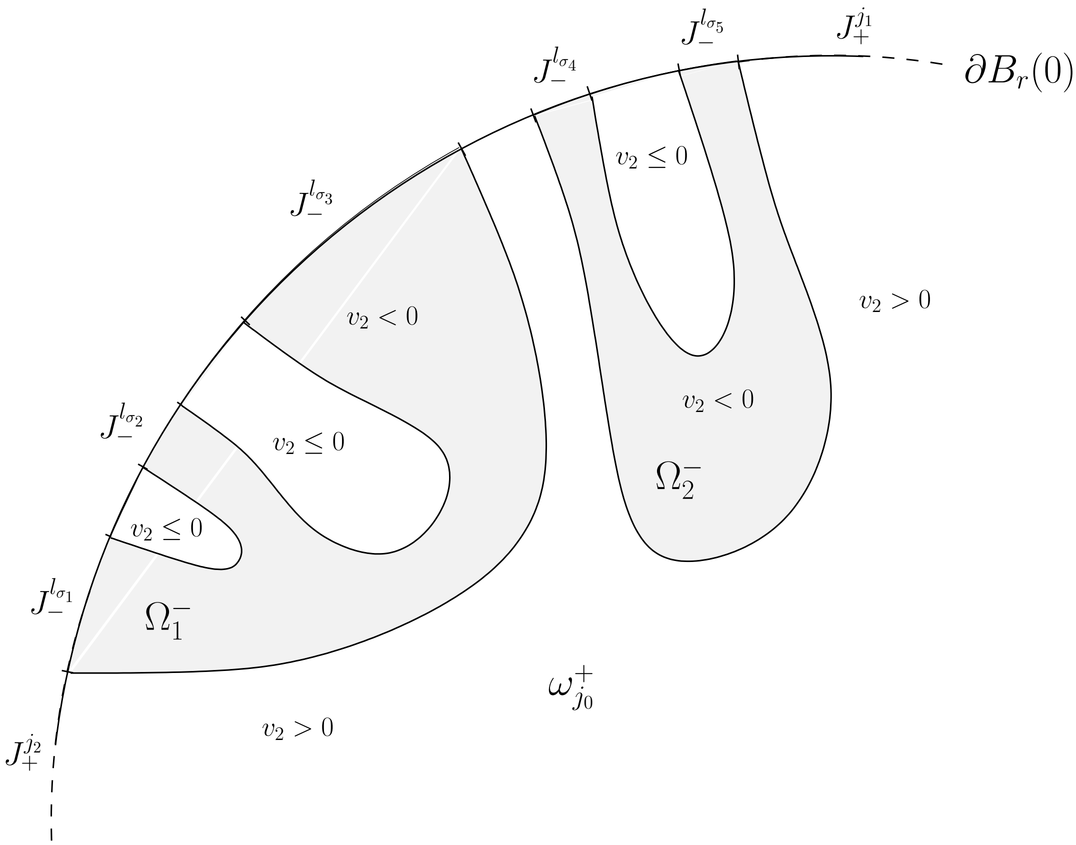

Figure 1: In this example, there are five "negative segments", $J_-^{l_{\sigma_1}}, \ldots, J_-^{l_{\sigma_5}}$, between the two "positive segments" $J_+^{j_1}$ and $J_+^{j_2}$.

Consider the components $\Omega^-_{\zeta(l_{\sigma_i})}$ corresponding to $J_-^{l_{\sigma_i}}$ for $i = 1, \ldots, 2k+1$ (see (2.14)). We now claim that, for each $i = 1, \ldots, 2k+1$, the set $\Omega^-_{\zeta(l_{\sigma_i})}$ satisfies

$$\overline{\Omega^-_{\zeta(l_{\sigma_i})}} \cap G_- \subset I\,. \tag{2.17}$$

Indeed, assume by negation that (2.17) doesn't hold for some i. Then, there exists a segment $J_-^j \subset \overline{\Omega^-_{\zeta(l_{\sigma_i})}} \cap (G_- \setminus I)$. But this would imply the existence of a curve starting at a point on J_-^j and ending at a point on $J_-^{l_{\sigma_i}}$ whose interior is contained in $\Omega^-_{\zeta(l_{\sigma_i})}$. The existence of such a curve clearly contradicts the connectedness of $\omega^+_{j_0}$, and (2.17) follows.

Finally, we define the map $\tilde{u} = \tilde{u}_1 + i\tilde{u}_2$ on $B_r(0)$ as follows. Set $\tilde{u}_1 = u_1$ and

$$\tilde{u}_2(x) = \begin{cases} v_2(x) & x \in \bigcup_{i=1}^{2k+1} \Omega^-_{\zeta(l_{\sigma_i})}\,, \\ |v_2(x)| & \text{otherwise}\,. \end{cases}$$

From the above it follows that $\widetilde{U} = \frac{\tilde{u}}{|\tilde{u}|} = \widetilde{U}_1 + i\widetilde{U}_2$ satisfies

$$\int_{\partial B_r(0)} \widetilde{U} \wedge \widetilde{U}_\tau = 2\delta = \pm 2\pi\,.$$

Therefore, either $\tilde{u}$ or its complex conjugate $\tilde{u}_1 - i\tilde{u}_2$ has degree 1 as required.

Finally, we use the above construction to define a map $\tilde{u}$ on $\mathbb{R}^2$ possessing the property stated in the proposition. Choose a sequence $\{R_n\}_{n=1}^\infty$ with $R_n \to \infty$, so we may assume that $R_n > R_0$ for all n. For each n we may find $r_n \in (R_n, R_n + 1)$ satisfying (2.4) with $r = r_n$ and repeat the above construction to get a map $\tilde{u}^{(n)} \in W^{1,p}\left(B_{r_n}(0), \mathbb{R}^2\right)$ satisfying

$$\tilde{u}_1^{(n)}(x) = u_1(x) \text{ and } \left|\tilde{u}_2^{(n)}(x)\right| = |u_2(x)|,\ \forall x \in B_{r_n}(0)\,, \tag{2.18}$$

and $\deg\left(\tilde{u}^{(n)}, \partial B_{r_n}(0)\right) = 1$. Note that (2.18) implies

$$|\nabla \tilde{u}^{(n)}(x)| = |\nabla u(x)|\,,\ \text{a.e. in } B_{r_n}(0)\,. \tag{2.19}$$

By (2.18)–(2.19) the sequence $\{\tilde{u}^{(n)}\}$ is bounded in $W^{1,p}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$, and by passing to a subsequence we may assume that $\tilde{u}^{(n)}$ converges to a map $\tilde{u} \in W^{1,p}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$, weakly in $W^{1,p}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$, hence also in $C_{loc}(\mathbb{R}^2, \mathbb{R}^2)$. Clearly, $\tilde{u}$ satisfies the assertion of the proposition. □

By using exactly the same method, we can prove an analogous result for the half-plane.

Proposition 2. *Let $D \geq 2$ be an integer. Then, for each $u \in \mathcal{E}_D^+$ there exists $\tilde{u} \in \mathcal{E}_1^+$ such that $E_p(\tilde{u}) = E_p(u)$, where*

$$\tilde{u}_1(x) = u_1(x) \text{ and } |\tilde{u}_2(x)| = |u_2(x)|,\ \forall x \in \mathbb{R}^2\,.$$

3 Existence of minimizers in $\mathbb{R}^2$

In this section we study the existence of minimizers on $\mathbb{R}^2$ and prove Theorem 1. The main difficulty we face here is to show that the (weak) limit of a minimizing sequence must satisfy the degree condition. Our main tool in overcoming this difficulty is Proposition 1.

Proof of Theorem 1. Clearly, without any loss of generality, we can consider the case $d = 1$. Let $\{u_n\}_{n=1}^{\infty}$ be a minimizing sequence in $\mathcal{E}_1$ for $I_p(1)$, i.e.,

$$\lim_{n\to\infty} E_p(u_n) = I_p(1)\,.$$

By (2.2), there exists a constant $\lambda_0 > 0$ such that,

$$|u_n(x_0)| \le \frac{1}{2} \;\Rightarrow\; |u_n(x)| \le \frac{3}{4}\,,\ \forall x \in B_{\lambda_0}(x_0)\,,\, \forall n \in \mathbb{N}\,. \tag{3.1}$$

Consider the set

$$S_n = \left\{ x \in \mathbb{R}^2 \,:\, |u_n(x)| \le \frac{1}{2} \right\}\,. \tag{3.2}$$

Next, borrowing an argument from [5], we show that S_n can be covered by a finite number of "bad disks". Starting from a point $x_{1,n} \in S_n$, we choose a point $x_{2,n} \in S_n \setminus B_{5\lambda_0}(x_{1,n})$ (if this set is nonempty) and then, by recurrence, $x_{k,n} \in S_n \setminus \bigcup_{j=1}^{k-1} B_{5\lambda_0}(x_{j,n})$ (if this set is nonempty). This selection process must stop after a finite number of iterations (bounded uniformly in n), because $\int_{\mathbb{R}^2} \left(1 - |u_n|^2\right)^2 \le C$, and the disks $\{B_{\lambda_0}\left(x_{j,n}\right)\}_{j=1}^{k}$ are mutually disjoint at each step, while

$$\int_{B_{\lambda_0}(x_{j,n})} \left(1 - |u_n|^2\right)^2 \ge \frac{\pi\lambda_0^2}{16}\,, \tag{3.3}$$

by (3.1).

Passing to a further subsequence (if necessary), we find that the number of disks is independent of n, i.e.,

$$S_n \subset \bigcup_{j=1}^{m} B_{5\lambda_0}(x_{j,n})\,,$$

where $\{x_{j,n}\}_{j=1}^{m} \subset S_n$ and the disks $\{B_{\lambda_0}(x_{j,n})\}_{j=1}^{m}$ are mutually disjoint. By replacing $u_n(x)$ with $u_n(x - x_{1,n})$, we may assume that

$$x_{1,n} = 0\,,\ \forall n \in \mathbb{N}\,. \tag{3.4}$$

From (2.2) and (3.4) it follows that $\{u_n\}$ is bounded in $C^{\alpha}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$. Therefore, by passing to a subsequence and relabeling, we may assume that $\{u_n\}$ converges in $C_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ and weakly in $W^{1,p}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ to a map $u \in W^{1,p}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$. By weak lower semicontinuity and the local uniform convergence it follows that

$$E_p(u) \le \lim_{n\to\infty} E_p(u_n) = I_p(1)\,. \tag{3.5}$$

It remains to show that $u \in \mathcal{E}_1$.

Let

$$R := \sup_{n\ge 1} \max\{|x_{j,n}| \,:\, 1 \le j \le m\} \in (0, \infty]\,. \tag{3.6}$$

We distinguish two cases:

(i) $R < \infty$.
(ii) $R = \infty$.

In the case (i), we clearly have

$$|u_n(x)| \geq \frac{1}{2}, \; |x| \geq R + 5\lambda_0, \, \forall n \in \mathbb{N}.$$

By the local uniform convergence,

$$\deg(u, \partial B_r(0)) = \deg(u_n, \partial B_r(0)) = 1,$$

for each $r \geq R + 5\lambda_0$, i.e., $u \in \mathcal{E}_1$ and we conclude from (3.5) that u is a minimizer for (1.2).

Next, we show that the case (ii) is impossible. Assume by negation that the case (ii) holds. Then, by passing to a subsequence, we may assume the following: the index set $\mathcal{J} = \{1, \ldots, m\}$ is a union of $K \geq 2$ disjoint subsets, $\mathcal{J}_1, \ldots, \mathcal{J}_K$, such that the (generalized) limit $l_{j_1,j_2} := \lim_{n\to\infty} |x_{j_1,n} - x_{j_2,n}| \in (0, \infty]$ exists for every pair of distinct indices $j_1, j_2 \in \{1, \ldots, m\}$ and

$$l_{j_1,j_2} < \infty \iff \exists k \in \{1, \ldots, K\} \text{ s.t. } j_1, j_2 \in \mathcal{J}_k.$$

For every $k \in \{1, \ldots, K\}$ and each n we define

$$\delta_{k,n} = \max\{|x_{j_1,n} - x_{j_2,n}| \, : \, j_1, j_2 \in \mathcal{J}_k\}. \tag{3.7}$$

Note that $\Delta_k = \sup_n \delta_{k,n} < \infty$ for every $k \in \{1, \ldots, K\}$. For $j \in \{1, \ldots, m\}$ we denote by $\sigma(j)$ the index in $\{1, \ldots, K\}$ such that $j \in \mathcal{J}_{\sigma(j)}$. Defining

$$\rho_n = \inf\{|x_{j_1,n} - x_{j_2,n}| \, : \, j_1, j_2 \in \{1, \ldots, m\} \text{ s.t. } \sigma(j_1) \neq \sigma(j_2)\},$$

we have $\lim_{n\to\infty} \rho_n = \infty$.

Fix any $k \in \{1, \ldots, K\}$ and any $j_k \in \sigma^{-1}(k)$. Define the sequence $\left\{v_n^{(k)}\right\}$ by $v_n^{(k)}(x) = u_n(x + x_{j_k,n})$. For any $r_1 > 0$ we have $r_1 < \rho_n/2$ for a sufficiently large $n \in \mathbb{N}$. If we take $r_1 > \Delta_k$, then the degree $d_{k,n} = \deg\left(v_n^{(k)}, \partial B_r(0)\right)$ does not depend on r for $r_1 < r < \rho_n/2$. Passing to a further subsequence, we may assume that $d_{k,n} = d_k$ for all $n \in \mathbb{N}$ and further, that $v_n^{(k)} \to v_k$ in $C_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ and weakly in $W^{1,p}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$, for some $v_k \in \mathcal{E}_{d_k}$. In case $d_k \neq 0$ we have

$$I_p(d_k) \leq E_p(v_k). \tag{3.8}$$

Note that (3.8) is obviously true when $d_k = 0$ because $I_p(0) = 0$. However, thanks to (3.3), we have that

$$\frac{\pi\lambda_0^2}{32} \leq E_p(v_k). \tag{3.9}$$

Set

$$\mathcal{K} = \{k \in \{1, \dots, K\} \, : \, d_k \neq 0\} \, ,$$

and denote its complement in $\{1, \dots, K\}$ by $\mathcal{K}^c$. Note that by the properties of the degree

$$1 = \sum_{k=1}^{K} d_k = \sum_{k \in \mathcal{K}} d_k \, ,$$

so that, in particular, $\mathcal{K} \neq \emptyset$. By weak lower semi-continuity, the aforementioned convergence, and (3.8)–(3.9) we obtain

$$|\mathcal{K}^c| \frac{\pi \lambda_0^2}{32} + \sum_{k \in \mathcal{K}} I_p(d_k) \leq \sum_{k \in K} E_p(v_k) \leq \lim_{n \to \infty} E_p(u_n) = I_p(1) \, . \tag{3.10}$$

Using Proposition 1 in (3.10) yields

$$|\mathcal{K}^c| \frac{\pi \lambda_0^2}{32} + |\mathcal{K}| I_p(1) \leq I_p(1) \, ,$$

from which it is clear that $\mathcal{K}^c = \emptyset$ and thus $\mathcal{K}$ must be a singleton, i.e., $K = 1$—a contradiction. □

4 Limiting behaviour of global minimizers when $p \to 2$

Throughout this section we denote by u_p a global minimizer realizing $I_p(1)$ for $p > 2$ (the existence is guaranteed by Theorem 1) satisfying

$$u_p(0) = 0 \, . \tag{4.1}$$

The condition (4.1) can always be fulfilled by an appropriate translation. The following proposition is needed in Section 5 where we study the existence problem for minimizers on $\mathbb{R}^2_+$.

Proposition 3. *Let $\{u_p\}_{p>2}$ be a family of minimizers satisfying* (4.1)*. Then, for every sequence $p_n \to 2^+$ we have, up to a subsequence,*

$$u_{p_n} \rightharpoonup \tilde{u} \quad \text{weakly in } H^1_{loc}(\mathbb{R}^2) \, , \tag{4.2}$$

where $\tilde{u}$ is a degree-one solution of the classical Ginzburg-Landau equation

$$-\Delta \tilde{u} = (1 - |\tilde{u}|^2) \tilde{u} \, . \tag{4.3}$$

on $\mathbb{R}^2$. Furthermore,

$$\lim_{p \to 2^+} \int_{\mathbb{R}^2} \left(1 - |u_p|^2\right)^2 = 2\pi \, . \tag{4.4}$$

To prove this proposition we need the following Pohozaev-type identity that will also be used later on in § 5.

Lemma 4.1. *For every $p > 2$ we have*

$$\int_{\mathbb{R}^2} \left(1-|u_p|^2\right)^2 = \frac{2(p-2)}{p} I_p(1)\,. \tag{4.5}$$

Proof. Let $\lambda > 0$ and set $w_\lambda(x) = u_p(\lambda x)$ and

$$F(\lambda) := E_p(w_\lambda) = \lambda^{p-2} \int_{\mathbb{R}^2} |\nabla u_p|^p + \frac{1}{2\lambda^2} \int_{\mathbb{R}^2} (1-|u_p|^2)^2\,.$$

Since F has a local minimum at $\lambda = 1$, we must have $F'(1) = 0$. Thus

$$(p-2) \int_{\mathbb{R}^2} |\nabla u_p|^p = \int_{\mathbb{R}^2} \left(1-|u_p|^2\right)^2\,,$$

and (4.5) follows. □

An upper bound for $I_p(1)$ is given by the next lemma.

Lemma 4.2. *We have*

$$I_p(1) \le \frac{2\pi}{p-2} + 3\pi\,, \quad \forall p > 2\,. \tag{4.6}$$

Proof. Define a function $f(r)$ by

$$f(r) = \begin{cases} \frac{r}{\sqrt{2}}, & 0 \le r \le \sqrt{2}\,, \\ 1, & \sqrt{2} < r\,. \end{cases}$$

A direct computation gives

$$I_p(1) \le E_p(fe^{i\theta}) \le 3\pi + 2\pi \int_{\sqrt{2}}^{\infty} r^{1-p}\,dr = 3\pi + 2\pi \frac{2^{1-p/2}}{p-2}\,,$$

and (4.6) follows. □

Remark 4.1. Although our main interest is in the limit $p \to 2$, we note that the result of Lemma 4.2 provides a uniform bound in the limit $p \to \infty$ as well.

Proof of Proposition 3. First, we show that the maps $\{u_p\}_{p>2}$ are uniformly bounded in $H^1_{\mathrm{loc}}(\mathbb{R}^2)$. The Euler-Lagrange equation associated with (1.1) is

$$\frac{p}{2} \nabla \cdot (|\nabla u_p|^{p-2} \nabla u_p) + u_p(1-|u_p|^2) = 0\,. \tag{4.7}$$

Let $\eta \in C_0^\infty(\mathbb{R}_+, [0,1])$ be a cutoff function satisfying

$$\eta(r) = \begin{cases} 1 & r < \frac{1}{2} \\ 0 & r > 1 \end{cases} \quad |\eta'| \le 4\,. \tag{4.8}$$

Fix any $x_0 \in \mathbb{R}^2$. Using the identity

$$\nabla u_p \nabla(\eta^2 u_p) = |\nabla(\eta u_p)|^2 - |u_p|^2 |\nabla \eta|^2 \,,$$

we obtain, upon multiplying (4.7) by $\eta^2(|x-x_0|)u_p(x)$ and integrating over $\mathbb{R}^2$,

$$\begin{aligned}\int_{B_{1/2}(x_0)} |\nabla u_p|^p &\leq \int_{B_1(x_0)} |\nabla u_p|^{p-2} |\nabla(\eta u_p)|^2 \\ &= \int_{B_1(x_0)} |\nabla \eta|^2 |u_p|^2 |\nabla u_p|^{p-2} + \frac{2}{p} \int_{B_1(x_0)} \eta^2 |u_p|^2 (1-|u_p|^2)\,.\end{aligned}$$

Since we have $\|u_p\|_\infty \leq 1$ for every p (otherwise, replacing $u_p(x)$ by $\frac{u_p(x)}{|u_p(x)|}$ on the set $\{x\,:\,|u_p(x)|>1\}$ would yield a map with a lower energy), we conclude, using the Hölder inequality, that

$$\int_{B_{1/2}(x_0)} |\nabla u_p|^p \leq C\left(1 + \Big(\int_{B_1(x_0)} |\nabla u_p|^p\Big)^{(p-2)/p}\right). \tag{4.9}$$

Here and for the remainder of the proof, C denotes a constant independent of $p > 2$. Inserting (4.6) into (4.9) yields

$$\int_{B_{1/2}(x_0)} |\nabla u_p|^p \leq C\left((p-2)^{-(p-2)/p} + 1\right),$$

hence

$$\int_{B_{1/2}(x_0)} |\nabla u_p|^p \leq C,$$

uniformly in $p > 2$. Applying the Hölder inequality once again and using a covering argument we find that

$$\int_{B_R(0)} |\nabla u_p|^2 \leq C(R)\,, \quad \forall p > 2\,, \forall R > 0\,. \tag{4.10}$$

Thanks to (4.10), there exists a sequence $p_n \to 2^+$ such that $u_{p_n} \rightharpoonup \tilde{u}$ weakly in $H^1_{\text{loc}}(\mathbb{R}^2)$. We now verify that $\tilde{u}$ satisfies (4.3). To this end, choose an arbitrary test function $\phi \in C_c^\infty(\mathbb{R}^2)$. By (4.7) we have for each n,

$$\int_{\mathbb{R}^2} \frac{p_n}{2} |\nabla u_{p_n}|^{p_n-2} \nabla u_{p_n} \nabla \phi = \int_{\mathbb{R}^2} u_{p_n}(1-|u_{p_n}|^2)\phi\,. \tag{4.11}$$

Using (4.10) and the Rellich-Kondrachov compact embedding theorem, we deduce that $\{u_{p_n}\}$ is relatively compact in $L^q_{\text{loc}}(\mathbb{R}^2)$ for every $q > 2$. By passing, if necessary, to a further subsequence, we then deduce that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} u_{p_n}(1-|u_{p_n}|^2)\phi = \int_{\mathbb{R}^2} \tilde{u}(1-|\tilde{u}|^2)\phi\,. \tag{4.12}$$

Next, we claim that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\frac{p_n}{2}|\nabla u_{p_n}|^{p_n-2}\nabla u_{p_n}\nabla\phi=\int_{\mathbb{R}^2}\nabla\tilde{u}\nabla\phi\,. \tag{4.13}$$

Clearly, (4.13) would follow if we can show that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\Big(|\nabla u_{p_n}|^{p_n-2}-1\Big)\nabla u_{p_n}\nabla\phi=0\,. \tag{4.14}$$

For any $p>2$ define the function

$$g_p(t)=\Big|t^{p-2}-1\Big|t \text{ on } t\in[0,\infty)\,. \tag{4.15}$$

An elementary computation shows that, for any $\beta>1$,

$$\max_{t\in[0,\beta]}g_p(t)=\max\left\{g_p(\beta),\,g_p\left(\Big(\frac{1}{p-1}\Big)^{\frac{1}{p-2}}\right)\right\}\to 0,\quad\text{as } p\to 2\,.$$

It follows that

$$\lim_{n\to\infty}\int_{\{|\nabla u_{p_n}(x)|\le\beta\}}\Big(|\nabla u_{p_n}|^{p_n-2}-1\Big)\nabla u_{p_n}\nabla\phi=0\,. \tag{4.16}$$

Let $R>0$ be such that

$$\text{supp}(\phi)\subset B_R(0)$$

and set

$$A_{n,\beta}=\{x\in B_R(0)\,:\,|\nabla u_{p_n}(x)|>\beta\}\,.$$

By (4.10), we have

$$\mu(A_{n,\beta})\le\frac{C(R)}{\beta^2}\,.$$

Therefore,

$$\begin{aligned}\Big|\int_{A_{n,\beta}}\Big(|\nabla u_{p_n}|^{p_n-2}-1\Big)\nabla u_{p_n}\nabla\phi\Big|&\le C(R)\int_{A_{n,\beta}}\big(|\nabla u_{p_n}|^{p_n-1}+|\nabla u_{p_n}|\big)\\ &\le C(R)\Big(\int_{B_R(0)}|\nabla u_{p_n}|^2\Big)^{\frac{p_n-1}{2}}\mu(A_{n,\beta})^{\frac{3-p_n}{2}}\le C(R)\beta^{p_n-3}\,.\end{aligned} \tag{4.17}$$

Since we may choose β to be arbitrary large, we deduce (4.14) from (4.16)-(4.17) and (4.13) follows. Consequently, (4.3) follows from (4.12)–(4.13).

Finally, we need to identify the degree of $\tilde{u}$. Combining Lemma 4.1 with Lemma 4.2 we get that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^2}(1-|u_{p_n}|^2)^2\le 2\pi\,.$$

Since $u_{p_n}\to\tilde{u}$ in $L^4_{\text{loc}}(\mathbb{R}^2)$, we obtain

$$\int_{\mathbb{R}^2}(1-|\tilde{u}|^2)^2\le 2\pi\,.$$

From the quantization result of [7] it follows that there are only two possibilities

$$\int_{\mathbb{R}^2} (1 - |\tilde{u}|^2)^2 = 2\pi \text{ or } 0\,, \tag{4.18}$$

corresponding to the degrees ± 1 or 0, respectively.

We now establish an improved regularity result for $\{u_p\}$. We make use of Theorem 4.1 in [9]. While it is not clearly stated there, it is possible to verify by examining the proof provided in [9] that all of the estimates in [9] are uniform in p when $p \to 2^+$. It follows that there exists a $q > 2$ and a constant $C > 0$ such that

$$\int_{B_1(y)} |\nabla u_{p_n}|^q \leq C\,, \tag{4.19}$$

for each n and for each disk $B_1(y)$ in $\mathbb{R}^2$. From (4.19) and Morrey's lemma we deduce that

$$|u_{p_n}(x) - u_{p_n}(y)| \leq C|x-y|^{1-2/q}\,, \quad \forall x, y \in \mathbb{R}^2\,, \forall n \in \mathbb{N}\,, \tag{4.20}$$

i.e., the family $\{u_{p_n}\}$ is equicontinuous on $\mathbb{R}^2$. Therefore, $\tilde{u}(0) = 0$, the integral in (4.18) cannot vanish, and (4.4) follows. Finally, using the equicontinuity again, $\deg(\tilde{u}) = 1$. □

Combining (4.4) with (4.5) we obtain the following result.

Corollary 4.1. *We have*

$$\lim_{p\to 2^+} (p-2)\, I_p(1) = 2\pi$$

and

$$\lim_{p\to 2^+} I_p(1) = \infty\,.$$

5 Existence of minimizers in $\mathbb{R}^2_+$

In this section we study the problem of existence of minimizers in $\mathbb{R}^2_+$ under the degree condition at infinity. In contrast with the case of the entire plane, here we are only able to prove the existence of minimizers of degree ± 1 when p is restricted to some right semi-neighborhood $(2, p_0)$ of $p = 2$. A major difference between the two cases is due to the different asymptotic behaviour of the energies when $p \to 2^+$. While in the $\mathbb{R}^2$-case the energy blows up in that limit, i.e., $\lim_{p\to 2^+} I_p(1) = +\infty$ (Corollary 4.1), in the $\mathbb{R}^2_+$-case the energy $I^+_p(1)$ remains bounded when $p \to 2^+$. The latter result is demonstrated in the following lemma.

Lemma 5.1. *We have* $\lim_{p\to 2+} I^+_p(1) = 2\pi$.

Proof. Let

$$u_\lambda(z) = \frac{z - \lambda i}{z + \lambda i}\,,$$

where $0 < \lambda \le 1/2$ and $z = x_1 + i x_2$. We obtain an upper bound for $\limsup_{p\to 2^+} I_p^+(1)$ by introducing a smooth test function satisfying

$$\tilde{u}_\lambda(z) = \begin{cases} u_\lambda, & |z| \le 1, \\ 1, & |z| \ge 2, \end{cases} \qquad \text{and} \quad |\nabla \tilde{u}_\lambda(z)| \le C\lambda\,,\ 1 \le |z| \le 2\,,$$

for $\lambda < 1/2$. As u_λ is a conformal mapping of $\mathbb{R}^2_+$ on $B_1(0)$, we have

$$\int_{\mathbb{R}^2_+} |\nabla u_\lambda|^2 = 2\pi\,.$$

Hence,

$$\lim_{\lambda\to 0} \int_{\mathbb{R}^2_+} |\nabla \tilde{u}_\lambda|^2 = 2\pi\,.$$

As $1 - \tilde{u}_\lambda$ is compactly supported, we have

$$\lim_{\lambda\to 0^+} \lim_{p\to 2^+} E_p(\tilde{u}_\lambda) = 2\pi\,,$$

from which we obtain that $\limsup_{p\to 2^+} I_p^+(1) \le 2\pi$.

Next, we prove the lower bound. Fix any $u \in \mathcal{E}_1^+$ (see (1.4)) and for each $\beta \in (0,1)$ set

$$\Omega_\beta = \{x \in \mathbb{R}^2_+ \,:\, |u(x)| < \beta\}\,.$$

Clearly,

$$E_p(u) \ge \frac{1}{2} \int_{\Omega_\beta} \left(1 - |u|^2\right)^2 \ge \frac{1}{2} \left(1 - \beta^2\right)^2 \mu(\Omega_\beta)\,,$$

and hence

$$\mu(\Omega_\beta) \le \frac{2E_p(u)}{\left(1 - \beta^2\right)^2}\,. \tag{5.1}$$

Consider any connected component ω of Ω_β. If ω contains a point x_0 where $u(x_0) = 0$ then $B_r(x_0) \subset \omega$ for some $r > 0$, which depends only on the modulus of continuity of u. It follows in particular that the number of the components ω with $\deg(u, \partial\omega) \ne 0$ is *finite*. Denoting the union of these components by A, we obtain that the image of A under u is the disk $B_\beta(0)$, hence

$$\int_{\Omega_\beta} |\nabla u|^2 \ge \int_A |\nabla u|^2 \ge 2 \int_A u_{x_1} \wedge u_{x_2} \ge 2\pi\beta^2\,. \tag{5.2}$$

The Hölder inequality implies that

$$\int_{\Omega_\beta} |\nabla u|^p \ge \frac{\left(\int_{\Omega_\beta} |\nabla u|^2\right)^{p/2}}{\mu(\Omega_\beta)^{(p-2)/2}}\,. \tag{5.3}$$

Combining (5.1), (5.2), and (5.3) we obtain

$$E_p(u) \geq \int_{\Omega_\beta} |\nabla u|^p \geq \frac{(2\pi)^{p/2}(1-\beta^2)^{p-2}\beta^p}{(2E_p(u))^{(p-2)/2}}\,.$$

Consequently,

$$E_p(u) \geq 2^{\frac{2}{p}}\,\pi\beta^2(1-\beta^2)^{\frac{2(p-2)}{p}}\,.$$

Letting $p \to 2$, we obtain

$$\varliminf_{p\to 2^+} E_p(u) \geq 2\pi\beta^2\,, \quad \forall \beta < 1\,,$$

and the desired lower bound follows. □

Proof of Theorem 2. By Corollary 4.1 and Lemma 5.1 there exists a $p_0 > 2$ such that

$$I_p^+(1) < I_p(1)\,, \quad \forall p \in (2, p_0)\,. \tag{5.4}$$

Next, we show that the theorem holds with this value of p_0, thus we assume in the sequel that $p \in (2, p_0)$. As in the proof of Theorem 1, we consider a minimizing sequence $\{u_n\}$ for $I_p^+(1)$. Our argument is very similar to the one used in the proof of Theorem 1, with the only new difficulty related to the possibility of a "vortex" whose distance to $\partial\mathbb{R}^2_+$ goes to infinity with n. The equation (5.4) is needed precisely in order to exclude this possibility.

As in (3.2) we set for each n

$$S_n = \left\{ x \in \mathbb{R}^2_+ \,:\, |u_n(x)| \leq \frac{1}{2} \right\}\,.$$

With λ_0 defined as in (3.1), we can find (along the lines of the proof of Theorem 1) a collection of mutually disjoint disks $\{B_{\lambda_0/5}(x_{j,n})\}_{j=1}^m$ such that

$$\{x_{j,n}\}_{j=1}^m \subset S_n \quad \text{and} \quad S_n \subset \bigcup_{j=1}^m B_{\lambda_0}(x_{j,n})\,,$$

where m is independent of n (upon passing to a further subsequence, if necessary). In what follows, the coordinates of $x_{j,n}$ are denoted by $(x_{j,n})_1$ and $(x_{j,n})_2$. Note that $B_{\lambda_0}(x_{j,n}) \subset \mathbb{R}^2_+$ because of (3.1) and the boundary condition (1.3).

Next, we divide the index set $\mathcal{J} = \{1,\dots,m\}$ into $K \geq 1$ disjoint subsets $\mathcal{J}_1,\dots,\mathcal{J}_K$ so that the distance $|x_{j_1,n} - x_{j_2,n}|$ remains bounded as n goes to ∞ if and only if j_1 and j_2 belong to the same $\mathcal{J}_i$ (cf. Theorem 1). Now we can subdivide the index set $\{1,\dots,K\}$ into two disjoint subsets:

$$\mathcal{K}_1 = \{k \,:\, (x_{j,n})_2 \to \infty\,,\ j \in \mathcal{J}_k\}\,,$$
$$\mathcal{K}_2 = \{k \,:\, \{(x_{j,n})_2\}_{n=1}^\infty \text{ is bounded}\,,\ j \in \mathcal{J}_k\}\,.$$

Note that one of the sets $\mathcal{K}_1, \mathcal{K}_2$ may be empty. By passing to a subsequence we may further assume that $\lim_{n\to\infty}(x_{j,n})_2$ exists for every $k \in \mathcal{K}_2$ and $j \in \mathcal{J}_k$. For each $k \in \{1, \dots, K\}$ we fix an arbitrary $j_k \in \mathcal{J}_k$ and define

$$v_n^{(k)}(x) = u_n(x + x_{j_k,n}) \ \text{ on } A_{j_k,n} := \mathbb{R}^2_+ - x_{j_k,n}\,.$$

Consider first the case $k \in \mathcal{K}_1$. Then, the limit of the sets $\{A_{j_k,n}\}_{n\geq 1}$, as $n \to \infty$, is $\mathbb{R}^2$. Further, there exists an $R > 0$ such that, for every $r \geq R$, the degree $d_{k,n} = \deg(v_n^{(k)}, \partial B_r(0))$ does not depend on r and n and may be denoted by d_k. This statement follows (for a subsequence) from the equicontinuity of $\{v_n^{(k)}\}_{n=1}^\infty$ on $\partial B_R(0)$. Passing to a further subsequence, we obtain that $v_n^{(k)} \to v_k$ in $C_{loc}(\mathbb{R}^2, \mathbb{R}^2)$, and weakly in $W^{1,p}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$, where the map $v_k \in \mathcal{E}_{d_k}$. Clearly

$$\begin{aligned}\sup_{R>0} \varliminf_{n\to\infty} E_p\Big(v_n^{(k)}; B_R(0)\Big) &\geq \sup_{R>0} E_p(v_k; B_R(0)) = E_p(v_k)\\ &\geq \min\left(I_p(d_k), \frac{\pi\lambda_0^2}{32}\right)\,.\end{aligned} \tag{5.5}$$

Here

$$E_p(u; D) := \int_D |\nabla u|^p + \frac{1}{2}(1 - |u|^2)^2\,,$$

for every $D \subset \mathbb{R}^2$.

When $k \in \mathcal{K}_2$ the limit of $\{A_{j_k,n}\}_{n\geq 1}$, as $n \to \infty$, is the half-plane

$$\mathcal{H}_t = \{y \in \mathbb{R}^2 \,:\, y_2 > t\} \quad \text{with} \quad t = -\lim_{n\to\infty}(x_{j,n})_2\,.$$

Similar to the previous case, for $r \geq R$, we find that

$$d_{n,k} = \deg\left(v_n^{(k)}, \partial\left(B_r(0) \cap A_{j_k,n}\right)\right) = d_k,$$

is independent of r and n. As in (5.5) we obtain

$$\begin{aligned}\sup_{R>0} \varliminf_{n\to\infty} E_p\left(v_n^{(k)}; B_R(0) \cap A_{j_k,n}\right) &\geq \sup_{R>0} E_p\left(v_k; B_R(0) \cap A_{j_k,n}\right)\\ &\geq \min\left(I_p^+(d_k), \frac{\pi\lambda_0^2}{32}\right)\,.\end{aligned} \tag{5.6}$$

Obviously, by construction,

$$\sum_{k=1}^K d_k = 1\,. \tag{5.7}$$

Using (5.5)–(5.6) we deduce that

$$I_p^+(1) = \lim_{n\to\infty} E_p(u_n) \geq \sum_{k\in\mathcal{K}_1} \min\left(I_p(d_k), \frac{\pi\lambda_0^2}{32}\right) + \sum_{k\in\mathcal{K}_2} \min\left(I_p^+(d_k), \frac{\pi\lambda_0^2}{32}\right)\,. \tag{5.8}$$

By Proposition 1 and (5.4) we have

$$I_p(d) \geq I_p(1) > I_p^+(1)\,,\ \forall d \neq 0\,,$$

which together with (5.8), (5.7), and Proposition 2 gives

$$\mathcal{K}_1 = \emptyset \text{ and } \mathcal{K}_2 = \{k_0\}\,.$$

Furthermore, it follows that $d_{k_0} = 1$. Choosing $j_0 \in \mathcal{J}_{k_0}$ and defining a new sequence by

$$\tilde{u}_n(x) = u_n(x + (x_{j_0,n})_1)\,,\ n \geq 1\,,$$

we conclude (again, after passing to a subsequence) that

$$\tilde{u}_n \to u \text{ in } C_{loc}(\overline{\mathbb{R}}^2_+, \mathbb{R}^2) \ \text{ and } \ \tilde{u}_n \rightharpoonup u \text{ weakly in } W^{1,p}_{loc}(\mathbb{R}^2_+, \mathbb{R}^2)\,.$$

It follows that $u \in \mathcal{E}_1^+$ and $E_p(u) = I_p^+(1)$. □

We conclude this section by providing an upper bound for the distance of the zeros of a minimizer from $\partial\mathbb{R}^2_+$.

Proposition 4. *For $p \in (2, p_0)$, let v_p denote a minimizer realizing the minimum in* (1.5)*. Let $v_p(x_p) = 0$ and assume w.l.o.g. that $x_p = (0, r_p)$. Then, there exists a positive constant C such that*

$$r_p < C(p-2)^{1/2}\,, \quad \forall p \in (2, p_0)\,. \tag{5.9}$$

Proof. For each $p \in (2, p_0)$ we set $\tilde{r}_p = \min(r_p, 1)$ and define a rescaled map $\tilde{v}_p(x)$ on $B_1(0)$ by

$$\tilde{v}_p(x) = v_p(\tilde{r}_p x + x_p)\,.$$

From the identity

$$\tilde{r}_p^{2-p} \int_{B_1(0)} |\nabla \tilde{v}_p|^p + \tilde{r}_p^2 \int_{B_1(0)} \frac{1}{2}(1 - |\tilde{v}_p|^2)^2 = E_p(v_p; B_{\tilde{r}_p}(x_p))\,,$$

it follows that $\tilde{v}_p$ is a minimizer for the energy

$$\widetilde{E}_p(v) = \int_{B_1(0)} |\nabla v|^p + \tilde{r}_p^p \int_{B_1(0)} \frac{1}{2}(1 - |v|^2)^2\,,$$

over the maps $v \in W^{1,p}(B_1(0), \mathbb{R}^2)$ satisfying $v = \tilde{v}_p$ on $\partial B_1(0)$. By Lemma 5.1 we have

$$\int_{B_1(0)} |\nabla \tilde{v}_p|^p = \tilde{r}_p^{p-2} \int_{B_{\tilde{r}_p}(x_p)} |\nabla v_p|^p \leq \int_{B_{\tilde{r}_p}(x_p)} |\nabla v_p|^p \leq C\,,$$

so we can again apply the same method as in the proof of the Giaquinta-Giusti regularity result from [9] in order to deduce a uniform bound for the Hölder semi-norm

$$[\tilde{v}_p]_{C^\beta(B_{1/2}(0))} \leq c_0\,,$$

with $\beta = 1 - 2/q$ for some $q > 2$. Rescaling back we get

$$[v_p]_{C^\beta(B_{\tilde{r}_p/2}(x_p))} \le \frac{c_0}{\tilde{r}_p^\beta}\,. \tag{5.10}$$

It follows from (5.10) that

$$|v_p(x)| \le \frac{c_0}{\tilde{r}_p^\beta}|x - x_p|^\beta\,, \quad x \in B_{\tilde{r}_p/2}(x_p)\,,$$

and we deduce easily that

$$\int_{\mathbb{R}^2_+}(1 - |v_p|^2)^2 \ge \int_{B_{\tilde{r}_p/2}(x_p)}(1 - |v_p|^2)^2 \ge c_1\tilde{r}_p^2\,, \tag{5.11}$$

for some positive constant c_1.

Finally, we note that the Pohozaev identity (4.5) also holds for minimizers on $\mathbb{R}^2_+$, i.e.,

$$\int_{\mathbb{R}^2_+}(1 - |v_p|^2)^2 = \frac{2(p-2)}{p}I_p^+(1)\,. \tag{5.12}$$

Combining (5.11)–(5.12) with Lemma 5.1 yields (5.9). □

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