A Weighted Erdös-Mordell Inequality for Polygons

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1 Introduction

Let $\triangle A_1A_2A_3$ be a triangle, and let P be an interior point of $\triangle A_1A_2A_3$. We denote the distances from P to the vertices by $PA_i = r_i$, $i = 1, 2, 3$, and the distances from P to the sides A_1A_2, A_2A_3, A_3A_1, by $d_{1,2}$, $d_{2,3}$, $d_{3,1}$, respectively. The famous Erdös-Mordell (EM) inequality asserts that

$$r_1 + r_2 + r_3 \geq 2(d_{1,2} + d_{2,3} + d_{3,1}), \tag{1.1}$$

with equality if and only if the triangle is equilateral and the point P is its center.

The EM-inequality was conjectured by Erdös [5] in 1935. It was first proved in 1937 [8], and has drawn attention since. Some related results with historical comments on this problem can be found in [2, 9]. For different proofs of (1.1) see [1, 6, 7]. For variations of (1.1), such as

$$(r_1)^t + (r_2)^t + (r_3)^t \geq 2^t((d_{1,2})^t + (d_{2,3})^t + (d_{3,1})^t), \quad \forall t \in (0, 1), \tag{1.2}$$

see [10, 3].

A generalization of (1.1), involving weights, was recently proved in the MONTHLY [4]. Using the above notations the weighted EM-inequality states that for any $\lambda_1, \lambda_2, \lambda_3 > 0$:

$$\lambda_1 r_1 + \lambda_2 r_2 + \lambda_3 r_3 \geq 2(\sqrt{\lambda_1\lambda_2}\, d_{1,2} + \sqrt{\lambda_2\lambda_3}\, d_{2,3} + \sqrt{\lambda_3\lambda_1}\, d_{3,1})\,, \tag{1.3}$$

with equality if and only if $A_1A_2 : A_2A_3 : A_3A_1 = \sqrt{\lambda_1} : \sqrt{\lambda_2} : \sqrt{\lambda_3}$ and P is the circumcenter of $\triangle A_1A_2A_3$. Clearly, the Classical EM-inequality (1.1) is a special case of (1.3) where $\lambda_1 = \lambda_2 = \lambda_3$.

Our main objective is to present a generalization of the weighted EM-inequality (1.3) for a polygon with $n \geq 3$ sides, and study the cases where equality occurs. The special case of equal weights is a generalization of (1.1) to an n-gon.

Before stating our results, we first agree on a few notations that will be used throughout the paper. Let $\mathcal{P}_n$ be a polygon with $n \geq 3$ vertices, and let P be an interior point of $\mathcal{P}_n$ such that $\mathcal{P}_n$ is star-like with respect to P. The distances from P to the sides $A_1A_2, A_2A_3, \ldots, A_{n-1}A_n, A_nA_1$ are denoted, respectively, by $d_{12}, d_{23}, \ldots, d_{n-1\,n}, d_{n1}$. The distances PA_i are denoted by r_i, $i = 1, 2, \ldots, n$. We also denote $\theta_{i,i+1} = \angle A_iPA_{i+1}, i =$

$1, 2, \ldots, n$, where the index i is taken modulo n. With these notations, our weighted EM-inequality for $\mathcal{P}_n$ reads as follows

Theorem 1. *For any positive $\lambda_1, \lambda_2, \ldots, \lambda_n$ we have,*

$$\sum_{i=1}^{n} \lambda_i r_i \geq \frac{1}{\cos(\pi/n)} \sum_{i=1}^{n} \sqrt{\lambda_i \lambda_{i+1}}\, d_{i,i+1}\,. \tag{1.4}$$

The characterization of the equality case in (1.4) is given in the next theorem. It involves the following two dimensional subspace (over $\mathbb{C}$) of $\mathbb{C}^n$:

$$\mathcal{L} = \left\{ \left(a_1 e^{(\pi k/n)i} + a_2 e^{-(\pi k/n)i}\right)_{k=1}^{n} \, : \, a_1, a_2 \in \mathbb{C} \right\}.$$

In order to avoid degenerate configurations we also consider the following subset of $\mathcal{L}$,

$$\mathcal{B} = \left\{ \left(a_1 e^{(\pi k/n)i} + a_2 e^{-(\pi k/n)i}\right)_{k=1}^{n} \in \mathcal{L} \, : \, |a_1| \neq |a_2| \right\}.$$

Theorem 2.

(i) For a set of positive $\lambda_1, \lambda_2, \ldots, \lambda_n$, equality in (1.4) *is possible if and only if there exists a vector $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in \mathcal{B}$ such that*

$$|w_j|^2 = \lambda_j, \; j = 1, 2, \ldots, n. \tag{1.5}$$

(ii) For a set of weights for which the condition in (i) is satisfied, equality in (1.4) *is attained for the (cyclic) polygon whose vertices are given by*

$$P_j = w_j^2/\lambda_j, \; j = 1, 2, \ldots, n, \tag{1.6}$$

and for the point $P = 0$ (which is the circumcenter of this polygon). Up to translations, rotations and dilations, this is the only case of equality in (1.4).

Remark 1. In the special case of equal weights, $\lambda_1 = \lambda_2 = \ldots = \lambda_n$, Theorem 2 implies that equality in (1.4) occurs if and only if $\mathcal{P}_n$ is a regular polygon, and P is its center.

Our proofs of Theorem 1 and Theorem 2 are quite elementary. Using a simple geometric inequality (Lemma 1) we reduce the problem to the analysis of a certain quadratic form. This analysis is carried out in Proposition 1.

2 Proof of the main results

We start with a simple lemma.

Lemma 1. *Let $\triangle ABC$ be a triangle with angles α, β, γ and sides' lengths a, b, c (in standard notation). Let h_A be the length of the altitude dropped from the vertex A. Then,*

$$h_A \le \sqrt{bc}\cos(\alpha/2), \tag{2.1}$$

with equality if and only if $b = c$.

Proof. By the cosine theorem we have

$$\begin{aligned} a = \sqrt{b^2 + c^2 - 2bc\cos\alpha} &= \sqrt{(b-c)^2 + 2bc(1-\cos\alpha)} \\ &\ge \sqrt{2bc(1-\cos\alpha)} = 2\sqrt{bc}\sin(\alpha/2)). \end{aligned}$$

Therefore,

$$h_A = \frac{2}{a}S_{\triangle ABC} = \frac{bc}{a}\sin\alpha \le \sqrt{bc}\,\frac{\sin\alpha}{2\sin(\alpha/2)} = \sqrt{bc}\cos(\alpha/2).$$

□

The following result is an immediate consequence of Lemma 1.

Lemma 2. *For a polygon $\mathcal{P}_n$ and an internal point P such that $\mathcal{P}_n$ is star-like with respect to P, we have*

$$\sum_{i=1}^n \lambda_i r_i - \frac{1}{\cos(\pi/n)}\sum_{i=1}^n \sqrt{\lambda_i\lambda_{i+1}}d_{i,i+1} \ge \sum_{i=1}^n \lambda_i r_i - \frac{1}{\cos(\pi/n)}\sqrt{\lambda_i\lambda_{i+1}}\sqrt{r_i r_{i+1}}\,\cos(\theta_{i,i+1}/2). \tag{2.2}$$

Equality in (2.2) *holds if and only if the n vertices of $\mathcal{P}_n$ lie on a circle centered at P.*

Proof. It suffices to apply Lemma 1 to each of the triangles $\triangle PA_iA_{i+1}$ to get $d_{i,i+1} \le \sqrt{r_i r_{i+1}}\cos(\theta_{i,i+1}/2)$. The equality case follows directly from Lemma 1. □

Motivated by Lemma 2, we define a quadratic form $\mathcal{Q}$ on $\mathbb{R}^n$, which is associated with the polygon $\mathcal{P}_n$ and the point P.

$$\mathcal{Q}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^n x_i^2 - \frac{1}{\cos(\pi/n)}\sum_{i=1}^n x_i x_{i+1}\cos(\theta_{i,i+1}/2)\,. \tag{2.3}$$

The next proposition is the crucial step in the proof of Theorem 1.

Proposition 1. *The quadratic form $\mathcal{Q}$ is positive semi-definite.*

Proof. We denote $\phi_{i,i+1} = \theta_{i,i+1}/2$, so that

$$\sum_{i=1}^n \phi_{i,i+1} = \pi, \tag{2.4}$$

and set

$$I = \max\{\sum_{i=1}^{n} x_i x_{i+1} \cos\phi_{i,i+1} \ : \ \sum_{i=1}^{n} x_i^2 = 1\}\,. \tag{2.5}$$

The result of Proposition 1 will follow once we prove that

$$I \le \cos\frac{\pi}{n}. \tag{2.6}$$

We fix some $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ such that $\sum_{i=1}^n x_i^2 = 1$. Since $\cos\phi_{i,i+1} \ge 0$ for all i (as $\theta_{i,i+1} < \pi$), we can assume, with no loss of generality, that $x_i \ge 0$, $i = 1, 2, \ldots, n$. We now construct the $2n$ dimensional vector $\mathbf{z} = (z_1, z_2, \ldots, z_{2n}) \in \mathbb{C}^{2n}$ as follows:

$$\begin{cases} z_1 = x_1, \\ z_k = x_k \prod_{j=1}^{k-1} e^{i\phi_{j,j+1}} \quad k = 2, \ldots, n, \\ z_{n+k} = -z_k \quad k = 1, 2, \ldots, n. \end{cases} \tag{2.7}$$

We denote by $(z, w) \equiv \mathrm{Re}(z\bar{w})$ the (real) scalar product of two complex numbers z and w. By (2.4) we have $(z_n, z_{n+1}) = x_1 x_n \cos\phi_{n,1}$, and thus, using (2.7), we conclude that

$$\sum_{j=1}^{2n} (z_j, z_{j+1}) = 2\sum_{j=1}^{n} x_j x_{j+1} \cos\phi_{j,j+1}\,, \tag{2.8}$$

where in (2.8), the index j is taken modulo $2n$ in the LHS, and modulo n in the RHS. Since $\sum_{j=1}^{2n} z_j = 0$ and $\sum_{j=1}^{2n} |z_j|^2 = 2$, it follows that

$$I \le J \equiv \frac{1}{2}\max\{\sum_{j=1}^{2n}(w_j, w_{j+1}) \ : \ \mathbf{w} \in \mathbb{C}^{2n} \text{ s.t. } \sum_{j=1}^{2n} w_j = 0 \text{ and } \sum_{j=1}^{2n} |w_j|^2 = 2\}\,. \tag{2.9}$$

To compute J, we apply the method of Lagrange multipliers and set

$$\mathcal{F}(\mathbf{w}, \nu, \mu) = \frac{1}{2}\sum_{j=1}^{2n-1}(w_j, w_{j+1}) + \mu\Big|\sum_{j=1}^{2n} w_j\Big|^2 - \frac{\nu}{2}\Big(\sum_{j=1}^{2n}|w_j|^2 - 2\Big), \quad \nu, \mu \in \mathbb{R}, \mathbf{w} \in \mathbb{C}^{2n}. \tag{2.10}$$

The conditions $\frac{\partial\mathcal{F}}{\partial w_j} = 0$, $\forall j$, and the constraint $\sum_{j=1}^{2n} w_j = 0$ yield

$$w_{j+1} = 2\nu w_j - w_{j-1}, \quad j = 1, 2, \ldots, 2n, \tag{2.11}$$

where the indices are taken modulo $2n$. Taking scalar product of (2.11) with w_j, and summing over j, $j = 1, 2, \ldots, 2n$, yields

$$J = \nu. \tag{2.12}$$

It is convenient to extend the sequence $\{w_j\}_{j=1}^{2n}$ to an infinite sequence of period $2n$, $\{w_j\}_{j=-\infty}^{\infty}$, which satisfies then (2.11) for all $j \in \mathbb{Z}$. The characteristic polynomial of the recurrence relation (2.11) is $p(t) = t^2 - 2\nu t + 1$, and we denote its roots by ρ_1, ρ_2. The general solution of this recurrence relation is then given by

$$w_j = a_1\rho_1^j + a_2\rho_2^j, \quad j \in \mathbb{Z} \quad \text{for some } a_1, a_2 \in \mathbb{C}.$$

Since $\{w_j\}$ has a period of $2n$, ρ_2, ρ_1 are necessarily unity roots of order $2n$. Therefore,

$$\rho_{1,2} = \nu \pm i\sqrt{1-\nu^2} \quad \text{with } |\nu| \leq 1, \text{ i.e., } \nu = \text{Re } \rho_1.$$

It follows from (2.12) that the maximum of J is attained by the largest possible value for $\nu = \text{Re } \rho_1$, as ρ_1 ranges over all the unity roots of order $2n$. The value $\nu = 1$ is excluded since it would lead to $\rho_1 = \rho_2 = 1$. In this case $\{w_j\}$ is a constant sequence, and therefore cannot satisfy the two constraints in (2.9). The maximum is thus achieved for $\rho_1 = e^{i(2\pi/2n)} = e^{i(\pi/n)}$, which gives $\nu = \cos\frac{\pi}{n}$. Finally, we have $I \leq J = \cos\frac{\pi}{n}$, implying (2.6), and thus completing the proof. □

We are now in a position to present the proofs of our main theorems.

Proof of Theorem 1. Put $y_i = \sqrt{\lambda_i r_i}$, $i = 1, 2, \ldots, n$. By Lemma 2 and Proposition 1 we obtain

$$\sum_{i=1}^{n} \lambda_i r_i - \frac{1}{\cos(\pi/n)} \sum_{i=1}^{n} \sqrt{\lambda_i \lambda_{i+1}} d_{i,i+1} \geq \mathcal{Q}(y_1, y_2, \ldots, y_n) \geq 0,$$

and the conclusion follows. □

Proof of Theorem 2. Suppose that equality in (1.4) is attained for some $\mathcal{P}_n$, $P \in \mathcal{P}_n$ and $\boldsymbol{\lambda} \in \mathbb{R}_+^n$. In particular, equality must hold in (2.2), so by Lemma 2, $\mathcal{P}_n$ is inscribed in a circle centered at P. Without loss of generality, we may assume that $P = \mathbf{0}$ and that the radius of the circumcircle equals 1. Identifying each vertex P_j with a point $z_j \in \mathbb{C}$ we set

$$w_j = (\lambda_j z_j)^{1/2}, \ j = 1, 2, \ldots, n,$$

for some branch of the complex square root. We complete this sequence to a sequence in $\mathbb{C}^{2n}$ by setting

$$w_{n+j} = -w_j \quad j = 1, 2, \ldots, n.$$

Using our assumption on equality in (1.4), and Lemma 1, we infer that

$$\begin{aligned} 0 &= \sum_{j=1}^{n} \lambda_j r_j - \frac{1}{\cos(\pi/n)} \sum_{j=1}^{n} \sqrt{\lambda_j \lambda_{j+1}}\, d_{j,j+1} = \sum_{j=1}^{n} \lambda_j - \frac{1}{\cos(\pi/n)} \sum_{j=1}^{n} \sqrt{\lambda_j \lambda_{j+1}} \cos(\theta_{j,j+1}/2) \\ &= \sum_{j=1}^{n} |w_j|^2 - \frac{1}{\cos(\pi/n)} \sum_{j=1}^{n} (w_j, w_{j+1}) = \frac{1}{2}\Big(\sum_{j=1}^{2n} |w_j|^2 - \frac{1}{\cos(\pi/n)} \sum_{j=1}^{2n} (w_j, w_{j+1}) \Big). \end{aligned}$$

Therefore, the sequence $\mathbf{v} \in \mathbb{C}^{2n}$, defined by

$$v_j = \frac{2^{1/2} w_j}{\left(\sum_{j=1}^{2n} |w_j|^2\right)^{1/2}}, \quad j = 1, 2, \ldots, 2n,$$

realizes the maximum for the quantity J defined in (2.9). From the proof of Proposition 1 it follows that $\mathbf{v}$ is a nontrivial linear combination of $(e^{(\pi j/n)i})_{j=1}^{2n}$ and $(e^{-(\pi j/n)i})_{j=1}^{2n}$, so the same is true for $\mathbf{w}$. Therefore,

$$w_j = a e^{(\pi j/n)i} + b e^{-(\pi j/n)i}, \quad j = 1, 2, \ldots, 2n,$$

for some $a, b \in \mathbb{C}$, i.e. the first n components of $\mathbf{w}$ form a vector in $\mathcal{L}$. We further want to establish that this vector belongs to $\mathcal{B}$, i.e. that $|a| \neq |b|$. Note that the arguments of $z_1, z_2, \ldots, z_n$ are distinct, and therefore the difference of argument between any consecutive pair z_j, z_{j+1} is a number in $(0, \pi)$. Therefore, the difference of arguments between any consecutive pair w_j, w_{j+1} is a number in $(0, \pi/2)$ (or $(-\pi/2, 0)$, according to the chosen branch of the square root). In particular, for each $1 \leq j \leq n$ we have

$$0 \neq \operatorname{Im} \frac{w_{j+1}}{w_j} = \frac{1}{|w_j|^2} \cdot \operatorname{Im}\Big((a e^{i\pi(j+1)/n} + b e^{-i\pi(j+1)/n})(\bar{a} e^{-i\pi j/n} + \bar{b} e^{i\pi j/n}) \Big) = \frac{\sin(\pi/n)}{|w_j|^2} (|a|^2 - |b|^2),$$

and it follows that $|a| \neq |b|$.

On the other hand, it is clear from the above argument that if for some $\mathbf{w} \in \mathcal{B}$ we define $\boldsymbol{\lambda} \in \mathbb{R}_+^n$ by (1.5), and then define a polygon $\mathcal{P}_n$ by (1.6) and set $P = \mathbf{0}$, then equality occurs in (1.4). The uniqueness of the polygon realizing the minimum (up to similarity transformations in the plane) is clear from the above argument. □

3 Geometric interpretation of the equality conditions in the weighted EM-inequality for a triangle

Reference [4] discusses the conditions for equality in the weighted EM-inequality for a triangle $\triangle A_1 A_2 A_3$. These conditions are formulated in terms of the sides of $\triangle A_1 A_2 A_3$, namely: equality occurs if and only if P is the circumcenter of the triangle, and the weights satisfy $A_1A_2 : A_2A_3 : A_3A_1 = \sqrt{\lambda_1} : \sqrt{\lambda_2} : \sqrt{\lambda_3}$.

In general, the relation between the conditions for equality, stated in Theorem 2, and the sides of the polygon, is not clear. However, for $n = 3$ we can attribute a geometrical interpretation to these conditions, and show how Theorem 2 implies the result of [4].

Indeed, suppose that for a triangle $\triangle A_1 A_2 A_3$, an interior point P and a triplet of weights $\lambda_1, \lambda_2, \lambda_3$, equality is obtained in (1.3). By Theorem 2, P must be the circumcenter of the triangle, and we choose a coordinate system for which P is the origin. We assume, with no loss of generality, that the circumradius of the triangle equals 1. We also denote the angles of $\triangle A_1A_2A_3$ by α_1, α_2, α_3 (in standard notation). Let the vertices of $\triangle A_1A_2A_3$ correspond to the complex numbers z_1, z_2, z_3. By our assumption, $|z_1| = |z_2| = |z_3| = 1$. Considering the central angles between z_i, z_{i+1} from the circumcenter P, we have

$$\arg(z_{i+1}/z_i) = 2\alpha_{i+2},\ i = 1, 2, 3 \quad \text{(the index notation is cyclic modulo 3).} \tag{3.1}$$

Setting $w_i = (\lambda_i z_i)^{1/2},\ i = 1, 2, 3$, we obtain, using (3.1) that

$$\arg(w_{i+1}/w_i) = \frac{1}{2}\arg(z_{i+1}/z_i) = \alpha_{i+2},\ i = 1, 2, 3. \tag{3.2}$$

From Theorem 2 we infer the necessary and sufficient condition $\mathbf{w} \in \mathcal{B}$, which for $n = 3$ is equivalent to the relation $w_2 = w_1 + w_3$. An equivalent formulation of that relation is that the quadrilateral with vertices at $0, w_1, w_2, w_3$ is a parallelogram. Combining it with (3.2) yields that the angles of the triangle with vertices at $0, w_1, w_2$ are, respectively, α_3, α_2 and α_1. In other words, this triangle is similar to the original triangle $\triangle A_1A_2A_3$ and we get that $A_2A_3 : A_1A_3 : A_1A_2 = |w_1| : |w_2| : |w_3| = \sqrt{\lambda_1} : \sqrt{\lambda_2} : \sqrt{\lambda_3}$, which is the desired equality condition.

4 Discussion and open problems

We first mention that the methods described in this paper can also be used for generalizing other Erdös-Mordell type triangle inequalities, like (1.2) (see [10, 3]), to their weighted n-gon analogs.

As shown in the paper, the addition of weights to the EM-inequality problem complicates the study of the cases where equality in (1.4) can occur. We discuss here some aspects of this difficulty.

The cases $n = 3$ and $n > 3$ are inherently different. As we saw in Section 3, each triangle realizes equality in (1.3) for some triplet of weights $\boldsymbol{\lambda} \in \mathbb{R}^3_+$ (actually this triplet is unique, up to a multiplicative factor). On the other hand, for $n > 3$, “most” of the polygons, and even most of the cyclic polygons, do not allow for equality in (1.4), for *any* set of weights $\lambda_1, \lambda_2, \ldots, \lambda_n$. To illustrated, note that the set of cyclic polygons, with center at the origin and vertices on the unit circle, depends on $n-1$ real parameters (when two polygons related by a rotation are considered as identical). However, by Theorem 2, the subset of these polygons which allow for equality in (1.4) depends on 2 real parameters only.

Theorem 2 completely characterizes the set of cyclic polygons for which equality may occur in (1.4) (for some weights). However, this characterization is algebraic and does not relate directly to the geometry (e.g., sides' lengths) of the polygon. The only case where the geometric investigation was carried out here was for $n = 3$. We therefore propose the following open problem.

Problem 1. *For $n > 3$, give a geometric characterization of the set of cyclic polygons that allow for equality in the weighted EM-inequality* (1.4).

The second open problem deals with the situation in which the weights $\lambda_1, \lambda_2, \dots, \lambda_n$ do not allow for equality in (1.4).

Problem 2. *Consider a set of positive weights $\lambda_1, \lambda_2, \dots, \lambda_n$, $n \geq 3$, that does not satisfy the condition for possible equality, as stated in part (i) of Theorem 2. What is the value of*

$$\inf \frac{\sum_{i=1}^{n} \lambda_i r_i}{\sum_{i=1}^{n} \sqrt{\lambda_i \lambda_{i+1}}\, d_{i,i+1}}, \tag{4.1}$$

where the infimum is taken over all polygons $\mathcal{P}_n$, and interior points P for which $\mathcal{P}_n$ is star-like?

We do not know the answer to Problem 2 even in the case $n = 3$, where the condition on $\lambda_1, \lambda_2, \lambda_3$ is simply that the numbers $\{\sqrt{\lambda_j}\}_{j=1}^{3}$ are not the lengths of the sides of a triangle.

References

[1] A. Avez, *A short proof of a theorem of Erdös and Mordell*, Amer. Math. Monthly **100** (1993), 60–62.

[2] L. Bankoff, *An elementary proof of the Erdös-Mordell theorem*, Amer. Math. Monthly **65** (1958), 521.

[3] L. Carlitz, *Some inequalities for a triangle*, Amer. Math. Monthly **71** (1964), 881–885.

[4] S. Dar and S. Gueron, *A Weighted Erdos-Mordell Inequality*, Amer. Math. Monthly **108** (2001), 165–168.

[5] P. Erdös, *Problem 3740*, Amer. Math. Monthly **42** (1935), 396.

[6] V. Komornik, *A short proof of the Erdös-Mordell theorem*, Amer. Math. Monthly **104** (1997), 57–60.

[7] H. Lee, *Another Proof of the Erdös-Mordell Theorem*, Forum Geometricorum **1** (2001), 7–8.

[8] L. J. Mordell and D. F. Barrow, *Solution 3740*, Amer. Math. Monthly **44** (1937), 252–254.

[9] L. J. Mordell, *On geometric problems of Erdös and Oppenheim*, Math. Gazette **46** (1962), 213–215.

[10] A. Oppenheim, *The Erdös inequality and other inequalities for a triangle*, Amer. Math. Monthly **68** (1961), 226–230.

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