The logarithmic HLS inequality for systems on compact manifolds

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1 Abstract

We prove an optimal logarithmic Hardy-Littlewood-Sobolev inequality for systems on compact m-dimensional Riemannian manifolds, for any $m \geq 2$. We show that a special case of the inequality, involving only two functions, implies the general case by using an argument from the theory of Linear Programing.

2 Introduction

Let $\mathcal{M}$ be a compact m-dimensional Riemannian manifold equipped with the geodesic distance d. Let a vector $\boldsymbol{M} = (M_1, \ldots, M_n)$ of positive masses and an $n \times n$ symmetric matrix $A = (a_{i,j})$ with nonnegative elements be given. Consider the functional

$$\Psi(\boldsymbol{\rho}) = \sum_{i=1}^{n} \int_{\mathcal{M}} \rho_i \ln \rho_i + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} \int_{\mathcal{M}} \int_{\mathcal{M}} \rho_i(x) \ln d(x,y)\ \rho_j(y)\, dx\, dy, \tag{2.1}$$

on the set

$$\Gamma_{\boldsymbol{M}}(\mathcal{M}) = \big\{ \boldsymbol{\rho} \in L_1(\mathcal{M}, \mathbb{R}^n) : \rho_i \geq 0 \text{ a.e.}, \int_{\mathcal{M}} \rho_i = M_i, \int_{\mathcal{M}} \rho_i |\ln \rho_i| < \infty, i = 1, \ldots n \big\}. \tag{2.2}$$

The main object of this paper is to find necessary and sufficient conditions for the boundedness from below of Ψ on $\Gamma_{\boldsymbol{M}}(\mathcal{M})$, in terms of $\boldsymbol{M}$ and A. Let $I = \{1, 2, \ldots, n\}$ and for any $J \subseteq I$ and $\boldsymbol{M} \in \mathbb{R}^n_+$ set

$$\Lambda_J(\boldsymbol{M}) = m \sum_{i \in J} M_i - \sum_{i \in J} \sum_{j \in J} a_{i,j} M_i M_j\,. \tag{2.3}$$

Our main result is the following:

Main Theorem. *The functional* Ψ *is bounded from below on* $\Gamma_{\boldsymbol{M}}(\mathcal{M})$ *if and only if the following two conditions hold:*

$$\begin{aligned} &\text{(i) } \Lambda_J(\boldsymbol{M}) \geq 0 \textit{ for all } J \subseteq I, \\ &\text{(ii) } \textit{if } \Lambda_J(\boldsymbol{M}) = 0 \textit{ for some } \emptyset \neq J \subseteq I, \textit{ then } a_{i,i} + \Lambda_{J\setminus\{i\}}(\boldsymbol{M}) > 0,\ \forall i \in J. \end{aligned} \tag{2.4}$$

Note that in the special case of a single component (i.e. $n = 1$), when we set without loss of generality $a_{1,1} = 1$, our result yields the condition $M_1 \leq m$, and we recover a version of the logarithmic Hardy-Littlewood-Sobolev inequality of Beckner [1], see also Carlen and Loss [3].

To our knowledge, the only previously known results for the *system case* ($n > 1$) were in dimension two. In [2] a version of the Main Theorem was proved for a functional defined on a bounded domain Ω in $\mathbb{R}^2$ in the *subcritical case*, i.e., when

$$\Lambda_J(\boldsymbol{M}) > 0 \text{ for all } J \subseteq I\,. \tag{2.5}$$

In this version the logarithmic potential takes the form $\ln G(x, y)$, where G stands for the Green function of the operator $-\Delta$ on Ω with Dirichlet boundary condition. The motivation there was mainly a dual formulation which yields a Moser-Trudinger type inequality for systems, and as a byproduct, existence of solutions for Liouville-type systems in $\mathbb{R}^2$. This latter subject was first studied by Chanillo and Kiessling in [4] where Λ_I was first introduced (with a slightly different definition). Wang [10] established the boundedness of the dual functional to Ψ (this requires positive definite A) on a compact two dimensional Riemannian manifold in the *subcritical case* (2.5), and in a very special case of the critical case. With our current notations, this latter result treats the case of a positive definite double-stochastic matrix A with $M_i = m$, $\forall i$. Jost and Wang [5] proved an optimal Moser-Trudinger inequality for the special case of the Toda system. Previously (see [7, 8]) we proved (among other things) the optimality of the conditions (2.4) for the boundedness of Ψ in the case of the two dimensional sphere S^2, and for the fore-mentioned version on a bounded domain in $\mathbb{R}^2$. In the present paper we establish a generalization of the result from [8] by allowing for arbitrary manifold in arbitrary dimension.

The main new ingredient of the proof is our observation that the inequality for a general system actually follows from a particular case which involves only two masses. A special case of this inequality takes the form

$$m \int_{\mathcal{M}} \int_{\mathcal{M}} F(x) \ln \frac{1}{d(x,y)} G(y)\, dx\, dy \leq \\ (1-\alpha) \int_{\mathcal{M}} F \ln F + \alpha \int_{\mathcal{M}} G \ln G + C(\alpha), \tag{2.6}$$

for all $\alpha \in (0,1)$ and all F and G satisfying

$$F, G \in \Gamma_1(\mathcal{M}) = \{f \in L^1(\mathcal{M}) : f \geq 0 \text{ a.e.}, \int_{\mathcal{M}} f|\ln f| < \infty, \int_{\mathcal{M}} f = 1\}. \tag{2.7}$$

The case $\alpha = \frac{1}{2}$ in (2.6) is due to Beckner [1]. In fact, this special case enjoys the *conformal invariance* property, when $\mathcal{M} = S^m$, which does not hold for $\alpha \neq \frac{1}{2}$. Interestingly, the inequality (2.6) is false in the limiting cases $\alpha = 0, 1$. A reduction process, which uses an argument from the theory of Linear Programing (LP), see the Appendix for statements and proofs, allows us to deduce the general case from the two masses inequality. A weaker subcritical version of (2.6), where m on the left-hand side is replaced by $m - \varepsilon$, for some $\varepsilon > 0$, turns out to be an easy consequence of the elementary inequality

$$st \leq e^{s-1} + t \ln t, \quad \forall s \in \mathbb{R}, \ \forall t > 0. \tag{2.8}$$

Therefore, combining this with the LP reduction argument, we are able to give in Section 3 a very short proof of the subcritical case of the Main Theorem (generalizing the result of Wang [10] which treated the two dimensional case and positive definite A).

The proof of the inequality (2.6) is much more involved than in the weaker version and is the subject of Section 4. Actually, we prove a slightly more general variant which is needed for the LP reduction process. The proof relies on two basic notions: *localization* and *symmetrization*. Indeed it turns out that the validity of the inequality for masses supported in a small ball implies its validity for the general case. It suffices thus to consider the case of two masses supported in a ball in $\mathbb{R}^m$ with d denoting the Euclidean distance. In this case we can apply Schwarz symmetrization and restrict ourselves to radially symmetric functions. Now we can apply an argument similar to the one used in [8] for the case $m = 2$, although a new ingredient is needed to deal with general m. Finally, in Section 5 we give the proof of the Main Theorem using the two-masses inequality and the LP reduction argument.

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3 The subcritical case

In this section we shall prove the Main Theorem in the subcritical case, which is considerably easier than the general case. We start with a special and simple case of the two-masses inequality (2.6) (the general and more difficult version of this inequality is the subject of Section 4).

Lemma 3.1. *For each $\varepsilon > 0$ there exists a constant $C = C(\varepsilon)$ such that for every $\alpha \in [0,1]$ we have*

$$(m-\varepsilon)\int_{\mathcal{M}}\int_{\mathcal{M}} F(x)\ln\frac{1}{d(x,y)}\,G(y)\,dx\,dy \leq (1-\alpha)\int_{\mathcal{M}} F\ln F + \alpha\int_{\mathcal{M}} G\ln G + C, \quad (3.1)$$

for all F and G satisfying (2.7)*.*

Proof. Applying (2.8) with $s = (m-\varepsilon)\ln\frac{1}{d(x,y)}$ and $t = G(y)$ yields

$$(m-\varepsilon)\int_{\mathcal{M}} \ln\frac{1}{d(x,y)}\,G(y)\,dy \leq \frac{1}{e}\int_{\mathcal{M}}\frac{dy}{d(x,y)^{m-\varepsilon}} + \int_{\mathcal{M}} G\ln G \leq C(\varepsilon) + \int_{\mathcal{M}} G\ln G.$$

Multiplying the last inequality by $F(x)$ and integrating over x in $\mathcal{M}$ gives

$$(m-\varepsilon)\int_{\mathcal{M}}\int_{\mathcal{M}} F(x)\ln\frac{1}{d(x,y)}\,G(y)\,dx\,dy \leq C(\varepsilon) + \int_{\mathcal{M}} G\ln G. \quad (3.2)$$

Similarly,

$$(m-\varepsilon)\int_{\mathcal{M}}\int_{\mathcal{M}} F(x)\ln\frac{1}{d(x,y)}\,G(y)\,dx\,dy \leq C(\varepsilon) + \int_{\mathcal{M}} F\ln F. \quad (3.3)$$

Multiplying inequalities (3.2) and (3.3) by α and $1-\alpha$ respectively, and adding the results leads to (3.1). □

Now we are in position to prove the Main Theorem in the subcritical case.

Proposition 3.1. *In the subcritical case* (2.5)*, the functional Ψ is bounded from below on $\Gamma_{\boldsymbol{M}}(\mathcal{M})$.*

Proof. Fix $\varepsilon \in (0,m)$ and any set of weights $(\alpha_{i,j})_{i,j\in I}$ satisfying

$$\alpha_{i,j} \in [0,1] \text{ and } \alpha_{i,j} + \alpha_{j,i} = 1,\ \forall i,j \in I, \quad (3.4)$$

to be determined later. Applying Lemma 3.1 with $F = \rho_i/M_i, G = \rho_j/M_j$ and $\alpha = \alpha_{i,j}$ gives for every i and j:

$$\int_{\mathcal{M}}\int_{\mathcal{M}} \rho_i(x)\ln\frac{1}{d(x,y)}\,\rho_j(y)\,dx\,dy \leq \frac{1}{m-\varepsilon}\Big(\alpha_{i,j}M_j\int_{\mathcal{M}}\rho_i\ln\rho_i + \alpha_{j,i}M_i\int_{\mathcal{M}}\rho_j\ln\rho_j\Big) + C_{i,j}(\varepsilon).$$

Plugging it in the definition of Ψ (see (2.1)) leads to

$$\Psi(\boldsymbol{\rho}) \geq \sum_{i\in I}\Big(1-\frac{2}{m-\varepsilon}\sum_{j\in I}a_{i,j}\alpha_{i,j}M_j\Big)\int_{\mathcal{M}}\rho_i\ln\rho_i - C(\varepsilon).$$

Since

$$\int_{\mathcal{M}}\rho_i\ln\rho_i \geq M_i\ln(M_i/|\mathcal{M}|),$$

the proof of the proposition will follow once we show that ε and the $(\alpha_{i,j})$ can be chosen to satisfy

$$\frac{2}{m-\varepsilon}\sum_{j\in I}a_{i,j}\alpha_{i,j}M_j \leq 1,\ \forall i\in I,$$

or equivalently,

$$2\sum_{j\in I}a_{i,j}\alpha_{i,j}M_iM_j \leq (m-\varepsilon)M_i,\ \forall i\in I. \tag{3.5}$$

Introducing a new set of unknowns $x_{i,j} = 2a_{i,j}\alpha_{i,j}M_iM_j$, $i,j\in I$, and denoting

$$b_{i,j} = a_{i,j}M_iM_j,\ \forall i,j\in I \ \text{ and } \ a_i = (m-\varepsilon)M_i,\ \forall i\in I,$$

we see that a solution $(\alpha_{i,j})_{i,j\in I}$ to (3.4)–(3.5) exists if and only if there is a solution $(x_{i,j})_{i,j\in I}$ to the problem (P) in the Appendix for these specific values of (a_i) and $(b_{i,j})$. The necessary and sufficient condition (A.1) of Proposition A.1 reads

$$(m-\varepsilon)\sum_{i\in J}M_i - \sum_{i,j\in J}a_{i,j}M_iM_j \geq 0,\ \forall J\subseteq I. \tag{3.6}$$

Since the inequalities in (2.5) are strict, we can choose $\varepsilon > 0$ small enough to satisfy (3.6) and the result follows. □

4 Proof of the two-masses inequality

We start with a version of the two-masses inequality in the case of an Euclidean ball $B_R = B_R(0)$ in $\mathbb{R}^m$.

Lemma 4.1. *For any $\alpha\in(0,1)$ and $\gamma\in[0,m)$ there exists a constant $C(\alpha,\gamma)$ such that*

$$\begin{aligned}\Phi^{(\alpha,\gamma)}(F,G) := \alpha\int_{B_R}F\ln\big(|x|^\gamma F\big) + (1-\alpha)\int_{B_R}G\ln\big(|x|^\gamma G\big)\\ + (m-\gamma)\int_{B_R}\int_{B_R}F(x)\ln|x-y|\,G(y)\,dx\,dy \geq -C(\alpha,\gamma),\\ \forall F,G\in\mathcal{L}\ln\mathcal{L}(B_R)\ s.t.\ \int_{B_R}F = \int_{B_R}G = 1\,.\end{aligned} \tag{4.1}$$

Proof. Let $F^*(r)$ and $G^*(r)$ denote the non-increasing Schwarz symmetrization of F and G, respectively. By Riesz rearrangement inequality we have $\Phi^{(\alpha,\gamma)}(F,G) \geq \Phi^{(\alpha,\gamma)}(F^*,G^*)$, where F^* and G^* are the Schwarz symmetrization of F and G, respectively. Therefore, we may assume in advance that F and G are radially symmetric and non-increasing with respect to the origin.

For $x, y \in B_R$ we write

$$x = r\zeta, y = \tilde{r}\eta, \quad \text{with } r, \tilde{r} \in \mathbb{R}^+ \text{ and } \zeta, \eta \in S^{m-1},$$

where S^{m-1} denotes the $(m-1)$-dimensional unit sphere. Then,

$$\int_{B_R}\int_{B_R} F(x)\ln|x-y|\, G(y)\, dx\, dy = I_1 + I_2, \tag{4.2}$$

where

$$\begin{aligned} I_1 &= \int_0^R\int_0^{\tilde{r}} \Big(\int_{S^{m-1}}\int_{S^{m-1}} r^{m-1}F(r)\ln|r\zeta-\tilde{r}\eta|\, \tilde{r}^{m-1}G(\tilde{r})\, d\eta\, d\zeta\Big)\, dr\, d\tilde{r}\,, \\ I_2 &= \int_0^R\int_{\tilde{r}}^R \Big(\int_{S^{m-1}}\int_{S^{m-1}} r^{m-1}F(r)\ln|r\zeta-\tilde{r}\eta|\, \tilde{r}^{m-1}G(\tilde{r})\, d\eta\, d\zeta\Big)\, dr\, d\tilde{r}\,. \end{aligned}$$

Define

$$h(\beta) := \int_{S^{m-1}}\int_{S^{m-1}} \ln|\beta\zeta-\eta|\, d\zeta\, d\eta, \quad \text{for } \beta\in[0,1].$$

It is easy to see that

$$|h(\beta)| \leq C, \ \forall\beta\in[0,1], \quad \text{for some constant } C. \tag{4.3}$$

We can rewrite I_1 and I_2 as

$$\begin{aligned} I_1 &= \int_0^R \tilde{r}^{m-1}G(\tilde{r})\Big(\int_0^{\tilde{r}} r^{m-1}F(r)h(\frac{r}{\tilde{r}})\, dr\Big)d\tilde{r} \\ &\quad + |S^{m-1}|^2\int_0^R \tilde{r}^{m-1}G(\tilde{r})\ln\tilde{r}\Big(\int_0^{\tilde{r}} r^{m-1}F(r)dr\Big)\, d\tilde{r}\,, \\ I_2 &= \int_0^R \tilde{r}^{m-1}G(\tilde{r})\Big(\int_{\tilde{r}}^R r^{m-1}F(r)h(\frac{\tilde{r}}{r})\, dr\Big)d\tilde{r} \\ &\quad + |S^{m-1}|^2\int_0^R \tilde{r}^{m-1}G(\tilde{r})\Big(\int_{\tilde{r}}^R r^{m-1}F(r)\ln r\, dr\Big)\, d\tilde{r}\,. \end{aligned}$$

Using our assumption that $\int_{B_R} F = \int_{B_R} G = 1$ and (4.3) we obtain that

$$I_1 + I_2 \geq |S^{m-1}|^2 \int_0^R\int_0^R r^{m-1}\tilde{r}^{m-1}F(r)G(\tilde{r})\ln(\max\{r,\tilde{r}\})\, dr\, d\tilde{r} - C\,. \tag{4.4}$$

Define next the mass functions corresponding to the functions F and G by

$$m_F(r) = |S^{m-1}| \int_0^r s^{m-1} F(s)\, ds \quad \text{and} \quad m_G(r) = |S^{m-1}| \int_0^r s^{m-1} G(s)\, ds\,.$$

Since $F, G, F\ln|x|, G\ln|x| \in L_1(B_R)$ we obtain that

$$\begin{gathered} m_F(0) = m_G(0) = 0,\; m_F(R) = m_G(R) = 1, \\ \lim_{r\to 0} m_F(r)\ln r = \lim_{r\to 0} m_G(r)\ln r = 0\,. \end{gathered} \tag{4.5}$$

We can then rewrite (4.4) as

$$I_1 + I_2 \geq \int_0^R m_G'(\tilde r) m_F(\tilde r) \ln \tilde r\, d\tilde r + \int_0^R \Big(\int_{\tilde r}^R m_F'(r) \ln r\, dr \Big) m_G'(\tilde r)\, d\tilde r - C\,,$$

and using integration by parts in conjunction with (4.5) we get that

$$I_1 + I_2 \geq -\int_0^R \frac{m_F(r) m_G(r)}{r}\, dr - C\,. \tag{4.6}$$

A simple computation gives

$$\begin{cases} \displaystyle\int_{B_R} F \ln(|x|^\gamma F)\, dx = \int_0^R m_F' \ln\Big(\frac{m_F'}{|S^{m-1}| r^{m-1-\gamma}} \Big)\, dr\,, \\ \displaystyle\int_{B_R} G \ln(|x|^\gamma G)\, dx = \int_0^R m_G' \ln\Big(\frac{m_G'}{|S^{m-1}| r^{m-1-\gamma}} \Big)\, dr\,. \end{cases} \tag{4.7}$$

Using (4.7), (4.6) and (4.2) in the definition of $\Phi^{(\alpha,\gamma)}(F,G)$ (see (4.1)) yields

$$\begin{aligned} \Phi^{(\alpha,\gamma)}(F,G) \geq \alpha \int_0^R m_F' \ln\left(\frac{m_F'}{r^{m-1-\gamma}} \right) dr + (1-\alpha) \int_0^R m_G' \ln\left(\frac{m_G'}{r^{m-1-\gamma}} \right) dr \\ - (m-\gamma) \int_0^R \frac{m_F m_G}{r}\, dr - C\,. \end{aligned}$$

An additional integration by parts (using (4.5) again) gives

$$\begin{aligned} \Phi^{(\alpha,\gamma)}(F,G) \geq \alpha \int_0^R m_F' \ln m_F'\, dr + (1-\alpha) \int_0^R m_G' \ln m_G'\, dr - (m-\gamma) \int_0^R \frac{m_F m_G}{r}\, dr \\ + (m-1-\gamma) \big[\alpha \int_0^R \frac{m_F}{r}\, dr + (1-\alpha) \int_0^R \frac{m_G}{r}\, dr \big] - C\,. \end{aligned} \tag{4.8}$$

Next we introduce a new variable s by $r = Re^s$ and define

$$\omega_1(s) = m_F(r) \text{ and } \omega_2(s) = m_G(r), \quad \text{for } s \in (-\infty, 0]\,.$$

By (4.5) it follows that the function ω_i is absolutely continuous, monotone non-decreasing on $(-\infty, 0]$ and satisfies

$$\lim_{s\to-\infty} \omega_i(s) = \lim_{s\to-\infty} s\omega_i(s) = 0 \text{ and } \omega_i(0) = 1 \ (i = 1, 2). \tag{4.9}$$

Using the identities

$$\int_0^R \frac{m_F}{r}\, dr = \int_{-\infty}^0 \omega_1(s)\, ds$$

and

$$\begin{aligned}\int_0^R m_F' \ln m_F'\, dr &= \int_{-\infty}^0 \omega_1'(s) \ln \omega_1'(s)\, ds - \int_{-\infty}^0 s\omega_1'(s)\, ds - \ln R \\ &= \int_{-\infty}^0 \omega_1'(s) \ln \omega_1'(s)\, ds + \int_{-\infty}^0 \omega_1(s)\, ds - \ln R,\end{aligned}$$

and the analogous ones for m_G and ω_2, in (4.8) we obtain

$$\Phi^{(\alpha,\gamma)}(F,G) \geq \alpha \int_{-\infty}^0 \omega_1' \ln \omega_1'\, ds + (1-\alpha) \int_{-\infty}^0 \omega_2' \ln \omega_2'\, ds + (m-\gamma) \int_{-\infty}^0 \Omega(\omega_1, \omega_2)\, ds - C, \tag{4.10}$$

where

$$\Omega(\omega_1, \omega_2) = \alpha\omega_1 + (1-\alpha)\omega_2 - \omega_1\omega_2\, .$$

Finally, the proof that the r.h.s. of (4.10) is bounded from below on all functions ω_1, ω_2 satisfying (4.9) follows the same lines as the one in [8, Prop. 4.1]. For the convenience of the reader we give below the proof for our particularly simple case. Applying inequality (2.8) yields for each $s \in (-\infty, 0)$,

$$\begin{aligned}\omega_1' \ln \omega_1' + \frac{m-\gamma}{2\alpha}\, \Omega(\omega_1, \omega_2) &\geq \omega_1' \Big(\ln \Omega(\omega_1, \omega_2) + \ln \big(\frac{(m-\gamma)e}{2\alpha}\big)\Big) \\ \omega_2' \ln \omega_2' + \frac{m-\gamma}{2(1-\alpha)}\, \Omega(\omega_1, \omega_2) &\geq \omega_2' \Big(\ln \Omega(\omega_1, \omega_2) + \ln \big(\frac{(m-\gamma)e}{2(1-\alpha)}\big)\Big)\, .\end{aligned} \tag{4.11}$$

Plugging (4.11) in (4.10) gives

$$\Phi^{(\alpha,\gamma)}(F,G) \geq \int_{-\infty}^0 (\alpha\omega_1' + (1-\alpha)\omega_2') \ln \Omega(\omega_1, \omega_2) - C. \tag{4.12}$$

Next, it is easy to verify that

$$\Omega(\omega_1, \omega_2) \geq \alpha(1-\alpha)\omega_i(1-\omega_i) \quad \text{for all } \omega_1, \omega_2 \in [0,1],\ i = 1, 2.$$

Thus, for $i = 1, 2$, we have

$$\int_{-\infty}^{0} \omega_i' \ln \Omega(\omega_1, \omega_2)\, ds \geq \int_{-\infty}^{0} \omega_i' \ln \big(\alpha(1-\alpha)\omega_i(1-\omega_i)\big)\, ds$$
$$= \int_{0}^{1} \ln \big(\alpha(1-\alpha)\omega(1-\omega)\big)\, d\omega \geq -C\,. \quad (4.13)$$

Using (4.13) in (4.12) gives the desired result. □

Next, we obtain an equivalent inequality in terms of G and α alone. We denote the logarithmic potential of G by

$$u_G(x) = \int_{B_R} G(y) \ln \frac{1}{|x-y|}\, dy\,.$$

We begin with the simple

Lemma 4.2. *For every $\alpha \in (0,1), \gamma \in [0,m)$ and every $G \in \mathcal{L}\ln\mathcal{L}(B_R)$ satisfying $\int_{B_R} G = 1$ we have*

$$\inf\{\Phi^{(\alpha,\gamma)}(F,G) \,:\, F \in \mathcal{L}\ln\mathcal{L}(B_R), \int_{B_R} F = 1\}$$
$$= (1-\alpha)\int_{B_R} G \ln[G|x|^\gamma] - \alpha \ln\big(\int_{B_R} |x|^{-\gamma} e^{(m-\gamma)u_G/\alpha}\big) := H^{(\alpha,\gamma)}(G) = H(G)\,.$$

Proof. Using Lagrange multipliers it is easy to see that the (coercive) functional

$$Q(F) := \alpha \int_{B_R} F \ln(F|x|^\gamma) - (m-\gamma)\int_{B_R} F u_G$$

attains its minimum over the closed convex set $\{F \in \mathcal{L}\ln\mathcal{L},\, \int_{B_R} F = 1\}$ at the function

$$F_0 = \mu |x|^{-\gamma} e^{(m-\gamma)u_G/\alpha}, \text{ with } \mu = \big(\int_{B_R} |x|^{-\gamma} e^{(m-\gamma)u_G/\alpha}\big)^{-1}. \quad (4.14)$$

The result follows since

$$Q(F_0) = \alpha \ln \mu = -\alpha \ln\big(\int_{B_R} |x|^{-\gamma} e^{(m-\gamma)u_G/\alpha}\big)\,.$$

□

Next we prove

Corollary 4.1. *For every $\alpha \in (0,1)$ and $\gamma \in [0,m)$ there exists a constant $C(\alpha,\gamma)$ such that*

$$H(G) \geq -C(\alpha,\gamma), \quad \forall G \in \mathcal{L}\ln\mathcal{L}(B_R) \text{ with } \int_{B_R} G \leq 1.$$

Proof. If $\int_{B_R} G = 1$ then the result follows from Lemma 4.1 and Lemma 4.2. When $\int_{B_R} G = \lambda < 1$ write $G = \lambda\widetilde{G}$, so that $u_G = \lambda u_{\widetilde{G}}$, with $\int_{B_R}\widetilde{G} = 1$. Note that

$$H(G) = (1-\alpha)\int_{B_R}\lambda\widetilde{G}\ln\big(\lambda\widetilde{G}|x|^\gamma\big) - \alpha\ln\Big(\int_{B_R}|x|^{-\gamma}e^{\lambda(m-\gamma)u_{\widetilde{G}}/\alpha}\Big).$$

By Hölder inequality

$$\int_{B_R}|x|^{-\gamma}e^{\lambda(m-\gamma)u_{\widetilde{G}}/\alpha} \leq \Big(\int_{B_R}|x|^{-\gamma}e^{(m-\gamma)u_{\widetilde{G}}/\alpha}\Big)^\lambda\Big(\int_{B_R}|x|^{-\gamma}\Big)^{1-\lambda}.$$

Therefore

$$H(G) \geq \lambda H(\widetilde{G}) + (1-\alpha)\lambda\ln\lambda - \alpha(1-\lambda)\ln\Big(\int_{B_R}|x|^{-\gamma}\Big),$$

and the result follows from the first case. □

We turn now to the general case of functions defined on a compact m-dimensional Riemannian manifold $\mathcal{M}$. The advantage of the formulation of Corollary 4.1 is that it allows a localization argument which will make the inequality valid, independent of the support of the functions involved, see Proposition 4.1 below.

For $x \in \mathcal{M}$ and $r > 0$ we define the geodesic ball $B_r(x) = \{y \in \mathcal{M} : d(x,y) < r\}$. There exists an $\varepsilon_0 > 0$ such that:

$$\text{the balls } \{B_{\varepsilon_0}(x)\}_{x\in\mathcal{M}} \text{ are (uniformly) diffeomorphic to an Euclidean ball } B_R. \tag{4.15}$$

An immediate consequence of Lemma 4.1 is:

Lemma 4.3. *For any $z \in \mathcal{M}$, $\alpha \in (0,1)$ and $\gamma \in [0,m)$ there exists a constant $C(\alpha,\gamma)$ such that*

$$\begin{aligned}\Phi^{(\alpha,\gamma)}_{\mathcal{M}}(F,G) := \alpha\int_{\mathcal{M}}F(x)\ln(d^\gamma(z,x)F(x))\,dx + (1-\alpha)\int_{\mathcal{M}}G(x)\ln(d^\gamma(z,x)G(x))\,dx \\ + (m-\gamma)\int_{\mathcal{M}}\int_{\mathcal{M}}F(x)\ln d(x,y)\,G(y)\,dx\,dy \geq -C(\alpha,\gamma), \\ \forall F,G \in \mathcal{L}\ln\mathcal{L}(\mathcal{M}) \text{ s.t. } \int_{\mathcal{M}}F = \int_{\mathcal{M}}G = 1 \text{ and } \operatorname{supp}(F),\operatorname{supp}(G) \subset B_{\varepsilon_0}(z).\end{aligned} \tag{4.16}$$

Our next proposition, which is the main result of this section, shows that the conclusion of Lemma 4.3 remains valid if we drop the smallness assumption on the supports of F and G.

Proposition 4.1. *For any $z \in \mathcal{M}$, $\alpha \in (0,1)$ and $\gamma \in [0,m)$ there exists a constant $C(\alpha,\gamma) < \infty$ such that*

$$\Phi_{\mathcal{M}}^{(\alpha,\gamma)}(F,G) \geq -C(\alpha,\gamma), \quad \forall F, G \in \mathcal{L}\ln\mathcal{L}(\mathcal{M}) \ s.t. \int_{\mathcal{M}} F = \int_{\mathcal{M}} G = 1. \tag{4.17}$$

Proof. Define, again, the logarithmic potential u_G with respect to $\mathcal{M}$ by

$$u_G(x) = \int_{\mathcal{M}} G(y) \ln \frac{1}{d(x,y)}\, dy$$

and the functional

$$H_{\mathcal{M}}^{(\alpha,\gamma)}(G) = H_{\mathcal{M}}(G) = (1-\alpha)\int_{\mathcal{M}} G(x)\ln(G(x)d^{\gamma}(x,z))\, dx - \alpha \ln\Big(\int_{\mathcal{M}} d^{-\gamma}(x,z) e^{(m-\gamma)u_G(x)/\alpha}\, dx\Big).$$

The same argument as in the proof of Lemma 4.2 gives that

$$\inf\{\Phi_{\mathcal{M}}^{(\alpha,\gamma)}(F,G) \,:\, F \in \mathcal{L}\ln\mathcal{L}(\mathcal{M}), \int_{\mathcal{M}} F = 1\} = H_{\mathcal{M}}(G).$$

Therefore, we shall prove the following equivalent formulation of (4.17):

$$H_{\mathcal{M}}(G) \geq -C(\alpha,\gamma), \quad \forall G \in \mathcal{L}\ln\mathcal{L}(\mathcal{M}) \text{ s.t. } \int_{\mathcal{M}} G = 1. \tag{4.18}$$

Fix any $0 < \delta < \varepsilon_0/2$ and choose a finite number of points $x_1, \ldots, x_N \in \mathcal{M}$ with the following properties:
(i) $x_1 = z$,
(ii) $d(x_i, x_j) > \delta, \quad \forall i \neq j$,
(iii) $\bigcup_{j=1}^{N} B_\delta(x_j) = \mathcal{M}$.

For each $i = 1, \ldots, N$, put

$$G_i = \chi_{B_{2\delta}(x_i)} G \quad \text{and} \quad u_i(x) := u_{G_i}(x) = \int_{\mathcal{M}} G_i(y) \ln\frac{1}{d(x,y)}\, dy\,.$$

Then, for every $x \in B_\delta(x_i)$ we have:

$$u_G(x) = \int_{B_{2\delta}(x_i)} G(y)\ln\frac{1}{d(x,y)}\,dy + \int_{\mathcal{M}\setminus B_{2\delta}(x_i)} G(y)\ln\frac{1}{d(x,y)}\,dy \leq u_i(x) + \ln\frac{1}{\delta}\,. \tag{4.19}$$

Since $\mathrm{supp}(G_1) \subset B_{\varepsilon_0}(x_1) = B_{\varepsilon_0}(z)$ and $\int_{\mathcal{M}} G_1 \leq 1$ we deduce from Lemma 4.3, as in the proof of Corollary 4.1, that

$$(1-\alpha)\int_{B_{2\delta}(x_1)} G_1 \ln(G_1 d^\gamma(x,x_1)) - \alpha \ln\Big(\int_{B_{2\delta}(x_1)} d^{-\gamma}(x,x_1) e^{(m-\gamma)u_1/\alpha}\Big) \geq -C\,,$$

or equivalently:

$$\int_{B_{2\delta}(x_1)} d^{-\gamma}(x,x_1) e^{(m-\gamma)u_1(x)/\alpha}\,dx \leq C \exp\Big(\frac{1-\alpha}{\alpha}\int_{B_{2\delta}(x_1)} G_1(x)\ln(G_1(x) d^\gamma(x,x_1))\,dx\Big)\,. \tag{4.20}$$

Here and in the sequel we denote by C different positive constants depending only on δ, α and γ. Combining (4.20) with (4.19) we are led to,

$$\begin{aligned}
&\int_{B_\delta(x_1)} d^{-\gamma}(x,x_1) e^{(m-\gamma)u_G(x)/\alpha}\,dx \leq C\int_{B_\delta(x_1)} d^{-\gamma}(x,x_1) e^{(m-\gamma)u_1(x)/\alpha}\,dx \\
&\leq C\int_{B_{2\delta}(x_1)} d^{-\gamma}(x,x_1) e^{(m-\gamma)u_1(x)/\alpha}\,dx \leq C\exp\Big(\frac{1-\alpha}{\alpha}\int_{\mathcal{M}} G_1(x)\ln(G_1(x)d^\gamma(x,x_1))\,dx\Big) \\
&\qquad \leq C\exp\Big(\frac{1-\alpha}{\alpha}\int_{\mathcal{M}} G(x)\ln(G(x)d^\gamma(x,x_1))\,dx\Big)\,.
\end{aligned} \tag{4.21}$$

For every $i \geq 2$ we can use the case $\gamma = 0$ of Lemma 4.3 as above to obtain that

$$\int_{B_{2\delta}(x_i)} e^{mu_i(x)/\alpha}\,dx \leq C\exp\Big(\frac{1-\alpha}{\alpha}\int_{B_{2\delta}(x_i)} G_i(x)\ln G_i(x)\,dx\Big)\,.$$

But since $d(x,z) > \delta$ on $B_\delta(x_i)$ by property (ii) above, it follows that, for $i \geq 2$,

$$\int_{B_\delta(x_i)} d^{-\gamma}(x,x_1) e^{(m-\gamma)u_i(x)/\alpha}\,dx \leq C\int_{B_\delta(x_i)} e^{mu_i(x)/\alpha}\,dx\,.$$

Therefore, we may conclude as in (4.21) that, for $i \geq 2$,

$$\int_{B_\delta(x_i)} d^{-\gamma}(x,z) e^{(m-\gamma)u_G(x)/\alpha}\,dx \leq C\exp\Big(\frac{1-\alpha}{\alpha}\int_{\mathcal{M}} G(x)\ln(G(x)d^\gamma(x,z))\,dx\Big)\,. \tag{4.22}$$

Combining (4.21) with (4.22), for $i = 2,\ldots,N$, we get that

$$\begin{aligned}
\int_{\mathcal{M}} d^{-\gamma}(x,z) e^{(m-\gamma)u_G(x)/\alpha}\,dx &\leq \sum_{i=1}^N \int_{B_\delta(x_i)} d^{-\gamma}(x,z) e^{(m-\gamma)u_G(x)/\alpha}\,dx \leq \\
&\qquad NC\exp\Big(\frac{1-\alpha}{\alpha}\int_{\mathcal{M}} G(x)\ln(G(x)d^\gamma(x,z))\,dx\Big)\,,
\end{aligned}$$

and (4.18) follows. □

We summarize in Corollary 4.2 below two easy consequences of Proposition 4.1 and Lemma 3.1 which are needed in the proof of the Main Theorem in Section 5.

Corollary 4.2.
(i) Let $\alpha \in (0,1)$ and $\boldsymbol{M} = (M_1, M_2) \in \mathbb{R}_+^2$ be given. Then, there exists a constant $C > 0$ such that

$$\int_{\mathcal{M}}\int_{\mathcal{M}} \rho_1(x) \ln d^{-m}(x,y)\; \rho_2(y)\, dx\, dy \le \alpha M_2 \int_{\mathcal{M}} \rho_1 \ln \rho_1 + (1-\alpha) M_1 \int_{\mathcal{M}} \rho_2 \ln \rho_2 + C\,,$$

for all $(\rho_1, \rho_2) \in \Gamma_{\boldsymbol{M}}(\mathcal{M})$ (see (2.2)).
(ii) Suppose that $a \ge 0$, $b \ge 0$ and $M > 0$ satisfy $b \le m - aM$. Assume, in addition, that if $a = 0$ then $b < m$. Then, there exists $C > 0$ such that for every $\rho \in \Gamma_M(\mathcal{M})$ and every $z \in \mathcal{M}$ we have,

$$\int_{\mathcal{M}} \rho \ln \rho + a \int_{\mathcal{M}}\int_{\mathcal{M}} \rho(x) \ln d(x,y)\; \rho(y)\, dx\, dy - b u_\rho(z) \ge -C\,. \tag{4.23}$$

Proof. (i) It suffices to apply Proposition 4.1 with $\gamma = 0$, $F = \rho_1/M_1$ and $G = \rho_2/M_2$.
(ii) Consider first the case $a > 0$. We may assume that $b = m - aM$, since it implies the case $b < m - aM$. Setting $F = \rho/M$ gives:

$$\begin{aligned}
&\int_{\mathcal{M}} \rho \ln \rho + a \int_{\mathcal{M}}\int_{\mathcal{M}} \rho(x) \ln d(x,y)\; \rho(y)\, dx\, dy - b u_\rho(z) \\
&\quad = M\Big(\int_{\mathcal{M}} F \ln F + aM \int_{\mathcal{M}}\int_{\mathcal{M}} F(x) \ln d(x,y)\; F(y)\, dx\, dy - b u_F(z) + \ln M\Big)\,.
\end{aligned}$$

The result then follows from Proposition 4.1 by taking $G = F$, any $\alpha \in (0,1)$ and $\gamma = b$.

It remains to consider the case $a = 0$, $b < m$. Here it suffices to apply Lemma 3.1 with $\alpha = 1, G = \rho/M,\ \varepsilon = m - b$, where we replace F by a sequence of nonnegative, normalized L^1-functions, converging weak* to a Dirac delta distribution supported at z. □

5 Proof of the Main Theorem; General case

The following technical result will be used in the proof of the Main Theorem. Roughly speaking, it says that up to an additive constant, the logarithmic kernel $-\ln d(x,y)$ is positive definite on integrable functions with zero integral on $\mathcal{M}$.

Lemma 5.1. *There is a constant $C = C(\mathcal{M})$ such that*

$$\int_{\mathcal{M}}\int_{\mathcal{M}} (f_1(x) - f_2(x)) \ln \frac{1}{d(x,y)}\, (f_1(y) - f_2(y))\, dx\, dy \ge -C\,,$$

$$\forall f_1, f_2 \in \mathcal{L}\ln\mathcal{L}(\mathcal{M}) \text{ s.t. } \int_{\mathcal{M}} f_1 = \int_{\mathcal{M}} f_2 = 1\,. \tag{5.1}$$

Proof. Thanks to Nash's imbedding theorem we may assume in the sequel that $(\mathcal{M}, g)$ is an m-dimensional submanifold of $(\mathbb{R}^N, \mathcal{E})$ for some N ($\mathcal{E}$ stands for the Euclidean metric on $\mathbb{R}^N$). With respect to this realization of $\mathcal{M}$, $d(x, y)$ is the geodesic distance between x and y on the submanifold $\mathcal{M}$ while dx and dy indicate m-dimensional area elements in $\mathcal{M}$. Clearly we have

$$c_1|x-y| \le d(x,y) \le c_2|x-y|, \quad \forall x, y \in \mathcal{M},$$

for some positive constants c_1, c_2, where $|x-y|$ is the Euclidean distance in $\mathbb{R}^N$. Therefore,

$$\begin{aligned} &\int_{\mathcal{M}}\int_{\mathcal{M}} (f_1(x)-f_2(x)) \ln \frac{1}{d(x,y)} \, (f_1(y)-f_2(y)) \, dx\, dy \\ &\qquad \ge \int_{\mathcal{M}}\int_{\mathcal{M}} (f_1(x)-f_2(x)) \ln \frac{1}{|x-y|} \, (f_1(y)-f_2(y)) \, dx\, dy - C\,. \end{aligned} \tag{5.2}$$

It is well known that for each $\lambda \in (0, N)$ the kernel $|x-y|^{-\lambda}$ is positive definite on $\mathbb{R}^N$. This fact follows easily from the identity $|x|^{-(N+\lambda)/2} * |x|^{-(N+\lambda)/2} = c_{\lambda,N}|x|^{-\lambda}$ (see [9, p. 118]), for some $c_{\lambda,N} > 0$. Using $\int_{\mathcal{M}}(f_1 - f_2) = 0$ we infer that

$$\int_{\mathcal{M}}\int_{\mathcal{M}} (f_1(x)-f_2(x)) \, \frac{|x-y|^{-\lambda}-1}{\lambda} \, (f_1(y)-f_2(y)) \, dx\, dy \ge 0\,. \tag{5.3}$$

Passing to the limit $\lambda \to 0^+$ in (5.3) yields

$$\int_{\mathcal{M}}\int_{\mathcal{M}} (f_1(x)-f_2(x)) \ln \frac{1}{|x-y|} \, (f_1(y)-f_2(y)) \, dx\, dy \ge 0\,. \tag{5.4}$$

More precisely, we can establish (5.4) first for $f_1, f_2 \in L^\infty(\mathcal{M})$, in which case passing to the limit is justified by bounded convergence, and the general case then follows by an approximation argument. Combining (5.4) with (5.2) leads to (5.1). □

We are now in a position to present the proof of our main result.

Proof of the Main Theorem. The necessity of condition (2.4) follows from a simple adaptation to dimension m of the argument of [8, Proposition 4.1] in dimension 2. Therefore we shall only give the proof of the sufficiency assertion.

We first consider the special case where $\boldsymbol{M}$ satisfies

$$\Lambda_J(\boldsymbol{M}) > 0, \ \forall J \subsetneq I \text{ and } \Lambda_I(\boldsymbol{M}) \ge 0\,. \tag{5.5}$$

For each $i \le j$ and $\alpha_{i,j} \in (0,1)$ we have by Corollary 4.2(i),

$$\int_{\mathcal{M}}\int_{\mathcal{M}} \rho_i(x) \ln d^{-m}(x,y)\, \rho_j(y)\, dx\, dy \le \alpha_{i,j} M_j \int_{\mathcal{M}} \rho_i \ln \rho_i + (1-\alpha_{i,j}) M_i \int_{\mathcal{M}} \rho_j \ln \rho_j + C\,. \tag{5.6}$$

Setting $\alpha_{i,j} = 1 - \alpha_{j,i}$ for $i > j$, we get by summing over all pairs i, j (as in the proof of Proposition 3.1) that

$$\Psi(\boldsymbol{\rho}) \geq \sum_{i \in I} \Big(1 - \frac{2}{m} \sum_{j \in I} a_{i,j}\alpha_{i,j} M_j\Big) \int_{\mathcal{M}} \rho_i \ln \rho_i - C\,. \tag{5.7}$$

Thus, we need to prove existence of a nonnegative solution to the problem

$$\alpha_{i,j} + \alpha_{j,i} = 1 \quad \text{and} \quad mM_i - 2\sum_{j \in I} a_{i,j}\alpha_{i,j} M_i M_j \geq 0,\ \forall i, j \in I,$$

but now we must satisfy also the additional condition:

$$\alpha_{i,j} > 0 \text{ whenever } i \neq j \text{ and } a_{i,j} > 0\,.$$

Similarly to the proof of Proposition 3.1 we set:

$$a_i = mM_i,\ \forall i \in I \quad \text{and} \quad b_{i,j} = a_{i,j} M_i M_j,\ \forall i, j \in I\,.$$

With respect to the new unknowns $x_{i,j} = 2a_{i,j}\alpha_{i,j} M_i M_j$, $\forall i, j \in I$, we obtain a problem of type (P) (see in the Appendix) in which we are looking for a solution satisfying $x_{i,j} > 0$ whenever $b_{i,j} > 0$. But now thanks to (5.5) we have the *strict inequalities*, $\sum_{i,j \in J} b_{i,j} < \sum_{i \in J} a_i$, $\forall J \subsetneq I$, so that we can apply Corollary A.1 to conclude.

The general case is proved by induction. The case $n = 1$ follows from Corollary 4.2(i) by taking $M_2 = M_1$. Assume that the result holds for all k-components systems with $k \leq n - 1$. If $\Lambda_J(\boldsymbol{M}) > 0$ for all $J \subsetneq I$ then the result follows from the above. Otherwise, let $K \subsetneq I$ be a *maximal* subset with respect to the property $\Lambda_K(\boldsymbol{M}) = 0$. By this we mean that $\Lambda_K(\boldsymbol{M}) = 0$, and if $\Lambda_{K_1}(\boldsymbol{M}) = 0$ for some K_1 satisfying $K \subseteq K_1 \subseteq I$, then either $K_1 = K$ or $K_1 = I$. For any subset $J \subseteq I$ we set

$$\Psi_J(\boldsymbol{\rho}) = \sum_{i \in J} \int_{\mathcal{M}} \rho_i \ln \rho_i + \sum_{i,j \in J} a_{i,j} \int_{\mathcal{M}}\int_{\mathcal{M}} \rho_i(x) \ln d(x,y)\ \rho_j(y)\, dx\, dy.$$

For $J := I \setminus K$ we may write then

$$\begin{aligned}\Psi(\boldsymbol{\rho}) = {} & \sum_{i \in J} \int_{\mathcal{M}} \rho_i \ln \rho_i + \sum_{i,j \in J} a_{i,j} \int_{\mathcal{M}}\int_{\mathcal{M}} \rho_i(x) \ln d(x,y)\ \rho_j(y)\, dx\, dy \\ & + \sum_{i \in K} \int_{\mathcal{M}} \rho_i \ln \rho_i + \sum_{i,j \in K} a_{i,j} \int_{\mathcal{M}}\int_{\mathcal{M}} \rho_i(x) \ln d(x,y)\ \rho_j(y)\, dx\, dy \\ & - \sum_{\{i \in J, k \in K\}} 2a_{i,k} \int_{\mathcal{M}} u_i \rho_k = \Psi_J(\boldsymbol{\rho}) + \Psi_K(\boldsymbol{\rho}) - \sum_{\{i \in J, k \in K\}} 2a_{i,k} \int_{\mathcal{M}} u_i \rho_k\,,\end{aligned} \tag{5.8}$$

with $u_i(x) := u_{\rho_i}(x) = \int_{\mathcal{M}} \rho_i(y) \ln \frac{1}{d(x,y)}\, dy$. Applying the induction hypothesis to Ψ_K in (5.8) yields

$$\Psi(\boldsymbol{\rho}) \geq \Psi_J(\boldsymbol{\rho}) - 2\sum_{i\in J}(\sum_{k\in K} a_{i,k}M_k)u_i(\bar{x}_i) - C\,, \tag{5.9}$$

where $\bar{x}_i$ is a maximum point of u_i (the maximum of u_i is attained since it is an upper semi-continuous function). From the maximality of K we deduce that

$$0 < \Lambda_{\tilde{J}\cup K}(\boldsymbol{M}) = \Lambda_{\tilde{J}\cup K}(\boldsymbol{M}) - \Lambda_K(\boldsymbol{M}) = m\sum_{i\in\tilde{J}} M_i - 2\sum_{i\in\tilde{J},k\in K} a_{i,k}M_iM_k - \sum_{i,j\in\tilde{J}} a_{i,j}M_iM_j\,, \quad \text{for all } \tilde{J} \subsetneq J. \tag{5.10}$$

Write $J = J_1 \cup (J \setminus J_1)$ where

$$J_1 = \{i \in J \,:\, \exists j \neq i \text{ in } J \text{ s.t. } a_{i,j} > 0\}.$$

Note that

$$\Psi_J(\boldsymbol{\rho}) - 2\sum_{i\in J}(\sum_{k\in K} a_{i,k}M_k)u_i(\bar{x}_i) = \Psi_{J_1}(\boldsymbol{\rho}) - 2\sum_{i\in J_1}(\sum_{k\in K} a_{i,k}M_k)u_i(\bar{x}_i)$$
$$+ \sum_{i\in J\setminus J_1}\Big[\int_{\mathcal{M}} \rho_i \ln\rho_i + a_{i,i}\int_{\mathcal{M}}\int_{\mathcal{M}} \rho_i(x)\ln d(x,y)\; \rho_i(y)\,dx\,dy - 2(\sum_{k\in K} a_{i,k}M_k)u_i(\bar{x}_i)\Big] \tag{5.11}$$

(in case $J_1 = \emptyset$ we use the convention $\Psi_{J_1}(\boldsymbol{\rho}) = 0$).

Next we claim that

$$\int_{\mathcal{M}} \rho_i \ln\rho_i + a_{i,i}\int_{\mathcal{M}}\int_{\mathcal{M}} \rho_i(x)\ln d(x,y)\; \rho_i(y)\,dx\,dy - 2(\sum_{k\in K} a_{i,k}M_k)u_i(\bar{x}_i) \geq -C, \;\; \forall i \in J. \tag{5.12}$$

Put $a = a_{i,i}$ and $b = 2\sum_{k\in K} a_{i,k}M_k$, and note that (like in (5.10)),

$$mM_i - aM_i^2 - bM_i = mM_i - a_{i,i}M_i^2 - 2\sum_{k\in K} a_{i,k}M_kM_i = \Lambda_{K\cup\{i\}}(\boldsymbol{M}).$$

If $K \cup \{i\} \subsetneq I$ then $\Lambda_{K\cup\{i\}}(\boldsymbol{M}) > 0$ and we deduce (5.12) from Corollary 4.2(ii). If $K \cup \{i\} = I$ then by (2.4), either $\Lambda_{K\cup\{i\}}(\boldsymbol{M}) > 0$, or $\Lambda_{K\cup\{i\}}(\boldsymbol{M}) = 0$ and $a_{i,i} > 0$. In either case (5.12) follows again from Corollary 4.2(ii).

Combining (5.9) with (5.11) and (5.12) we find that

$$\Psi(\boldsymbol{\rho}) \geq \Psi_{J_1}(\boldsymbol{\rho}) - 2\sum_{i\in J_1}(\sum_{k\in K} a_{i,k}M_k)u_i(\bar{x}_i) - C. \tag{5.13}$$

Next we distinguish four cases:

Case 1: J_1 is empty.
In this case the lower bound for Ψ follows immediately from (5.13).

Case 2: J_1 is a singleton.
Let $J_1 = \{i_0\}$. In view of (5.13) we only need to prove that

$$\int_{\mathcal{M}} \rho_{i_0} \ln \rho_{i_0} + a_{i_0,i_0} \int_{\mathcal{M}} \int_{\mathcal{M}} \rho_{i_0}(x) \ln d(x,y)\ \rho_{i_0}(y)\, dx\, dy - 2(\sum_{k \in K} a_{i_0,k} M_k) u_{i_0}(\bar{x}_{i_0}) \geq -C.$$

But this is just (5.12) for $i = i_0$.

Case 3: $|J_1| \geq 2$ and $a_{i,i} > 0,\ \forall i \in J_1$.
By (5.13) we need to prove that

$$\sum_{i \in J_1} \int_{\mathcal{M}} \rho_i \ln \rho_i + \sum_{i,j \in J_1} a_{i,j} \int_{\mathcal{M}} \int_{\mathcal{M}} \rho_i(x) \ln d(x,y)\ \rho_j(y)\, dx\, dy$$
$$- 2 \sum_{i \in J} (\sum_{k \in K} a_{i,k} M_k) u_i(\bar{x}_i) \geq -C. \quad (5.14)$$

Let $\{\alpha_{i,j}\}_{i,j \in J_1}$ satisfy:

$$\begin{cases} \alpha_{i,j} \in (0,1) \text{ and } \alpha_{i,j} + \alpha_{j,i} = 1,\ \forall i,j \in J_1,\ i \neq j, \\ \alpha_{i,i} = 0,\ \forall i \in J_1. \end{cases} \quad (5.15)$$

Applying (5.6) to each pair $i \neq j$ on the l.h.s. of (5.14) yields:

$$\Psi_{J_1}(\boldsymbol{\rho}) - 2 \sum_{i \in J_1} (\sum_{k \in K} a_{i,k} M_k) u_i(\bar{x}_i) \geq$$
$$\sum_{i \in J_1} \Big\{ \big(1 - \frac{2}{m} \sum_{j \in J_1} a_{i,j} \alpha_{i,j} M_j\big) \int_{\mathcal{M}} \rho_i \ln \rho_i + a_{i,i} \int_{\mathcal{M}} \int_{\mathcal{M}} \rho_i(x) \ln d(x,y)\ \rho_i(y)\, dx\, dy$$
$$- 2(\sum_{k \in K} a_{i,k} M_k) u_i(\bar{x}_i) \Big\}. \quad (5.16)$$

Denote for each $i \in J_1$:

$$t_i = 1 - \frac{2}{m} \sum_{j \in J_1} a_{i,j} \alpha_{i,j} M_j, \quad \tilde{a}_i = \frac{a_{i,i}}{t_i} \quad \text{and} \quad \tilde{b}_i = \frac{2}{t_i} \sum_{k \in K} a_{i,k} M_k\, .$$

If the $\{\alpha_{i,j}\}$ can be chosen such that $t_i > 0$, $\forall i \in J_1$, and

$$m M_i - \tilde{a}_i M_i^2 - \tilde{b}_i M_i \geq 0,\ \forall i \in J_1,$$

i.e., after multiplying by t_i,

$$m M_i - 2 \sum_{j \in J_1} a_{i,j} \alpha_{i,j} M_i M_j - a_{i,i} M_i^2 - 2 \sum_{k \in K} a_{i,k} M_k M_i \geq 0,\ \forall i \in J_1, \quad (5.17)$$

then, applying Corollary 4.2(ii) to (5.16), we would get (5.14) as required (recall that $a_{i,i} > 0$, $\forall i \in J_1$ in our case). Note that if (5.17) holds, then automatically $t_i > 0$.

Set

$$b_{i,j} = \begin{cases} a_{i,j} M_i M_j & \text{for } i \neq j, \\ 0 & \text{for } i = j, \end{cases} \quad \forall i,j \in J_1,$$

and

$$a_i = m M_i - a_{i,i} M_i^2 - 2 \sum_{k \in K} a_{i,k} M_i M_k, \ \forall i \in J_1.$$

We claim that

$$\sum_{i,j \in \tilde{J}} b_{i,j} < \sum_{i \in \tilde{J}} a_i, \ \forall \tilde{J} \subsetneq J_1 \quad \text{and} \quad \sum_{i,j \in J_1} b_{i,j} \leq \sum_{i \in J_1} a_i \,. \tag{5.18}$$

Indeed, by (5.10),

$$\begin{aligned} \sum_{i \in \tilde{J}} a_i - \sum_{i,j \in \tilde{J}} b_{i,j} &= \sum_{i \in \tilde{J}} \Big(m M_i - 2 \sum_{k \in K} a_{i,k} M_i M_k \Big) - \sum_{i,j \in \tilde{J}} a_{i,j} M_i M_j \\ &= \Lambda_{\tilde{J} \cup K}(\boldsymbol{M}) - \Lambda_K(\boldsymbol{M}) = \Lambda_{\tilde{J} \cup K}(\boldsymbol{M}) \,. \end{aligned}$$

For $\tilde{J} \subsetneq J_1$ we have $\Lambda_{\tilde{J} \cup K}(\boldsymbol{M}) > 0$ by the maximality of K, while for $\tilde{J} = J_1$ we have $\Lambda_{\tilde{J} \cup K}(\boldsymbol{M}) \geq 0$ by assumption (we may have equality if $J = J_1$ and $\Lambda_I(\boldsymbol{M}) = 0$), and (5.18) follows in either case.

Introducing the new unknowns $x_{i,j} = 2 a_{i,j} \alpha_{i,j} M_i M_j$, $\forall i,j \in J_1$, we see that (5.17) is equivalent to $\sum_{j \in J_1} x_{i,j} \leq a_i$, $\forall i \in J_1$. However, by Corollary A.1 and (5.18) there exists a solution to the problem:

$$\begin{cases} x_{i,j} \geq 0 \text{ and } x_{i,j} + x_{j,i} = 2 b_{i,j}, \ \forall i,j \in J_1, \\ x_{i,j} > 0 \text{ whenever } b_{i,j} > 0, \\ \displaystyle\sum_{j \in J_1} x_{i,j} \leq a_i, \ \forall i \in J_1. \end{cases}$$

<u>Case 4:</u> $|J_1| \geq 2$ and $\{i \in J_1 \, : \, a_{i,i} = 0\} \neq \emptyset$.
Our strategy will be to reduce this case to Case 3. Consider any $i_0 \in J_1$. By the definition of J_1 it follows that there exists at least one $i_1 \in J_1 \setminus \{i_0\}$ with $a_{i_0,i_1} > 0$. By Lemma 5.1 we have, for any $\varepsilon > 0$,

$$\begin{aligned} &\varepsilon \int_{\mathcal{M}} \int_{\mathcal{M}} \rho_{i_0}(x) \ln \frac{1}{d(x,y)} \, \rho_{i_1}(y) \, dx \, dy = \\ &\quad \varepsilon M_{i_0} M_{i_1} \int_{\mathcal{M}} \int_{\mathcal{M}} (\rho_{i_0}(x)/M_{i_0}) \ln \frac{1}{d(x,y)} \, (\rho_{i_1}(y)/M_{i_1}) \, dx \, dy \leq \\ &\quad \frac{\varepsilon M_{i_1}}{2 M_{i_0}} \int_{\mathcal{M}} \int_{\mathcal{M}} \rho_{i_0}(x) \ln \frac{1}{d(x,y)} \, \rho_{i_0}(y) \, dx \, dy \\ &\qquad + \frac{\varepsilon M_{i_0}}{2 M_{i_1}} \int_{\mathcal{M}} \int_{\mathcal{M}} \rho_{i_1}(x) \ln \frac{1}{d(x,y)} \, \rho_{i_1}(y) \, dx \, dy + C\varepsilon. \end{aligned}$$

Therefore,

$$
\begin{aligned}
\int_{\mathcal{M}}\int_{\mathcal{M}} & \rho_{i_0}(x)\ln\frac{1}{d(x,y)}\,\rho_{i_1}(y)\,dx\,dy \le \\
& (1-\varepsilon)\int_{\mathcal{M}}\int_{\mathcal{M}}\rho_{i_0}(x)\ln\frac{1}{d(x,y)}\,\rho_{i_1}(y)\,dx\,dy \\
& + \frac{\varepsilon M_{i_1}}{2M_{i_0}}\int_{\mathcal{M}}\int_{\mathcal{M}}\rho_{i_0}(x)\ln\frac{1}{d(x,y)}\,\rho_{i_0}(y)\,dx\,dy \\
& + \frac{\varepsilon M_{i_0}}{2M_{i_1}}\int_{\mathcal{M}}\int_{\mathcal{M}}\rho_{i_1}(x)\ln\frac{1}{d(x,y)}\,\rho_{i_1}(y)\,dx\,dy + C\varepsilon.
\end{aligned}
\tag{5.19}
$$

Applying the above for *each* $i_0 \in J_1$ we see that

$$
\Psi_{J_1}(\boldsymbol{\rho}) \ge \widetilde{\Psi}_{J_1}(\boldsymbol{\rho}) - C\varepsilon, \tag{5.20}
$$

where $\widetilde{\Psi}_{J_1}$ corresponds to a new matrix $\{\tilde{a}_{i,j}\}_{i,j\in I}$ which differs from the original $\{a_{i,j}\}_{i,j\in I}$ only in some elements $a_{i,j}$ with $i,j \in J_1$. Moreover, we have $\tilde{a}_{i,i} > 0,\ \forall i \in J_1$ and $\tilde{a}_{i,j} \ge 0,\ \forall i,j \in J_1$, provided that we choose $\varepsilon > 0$ small enough. By (5.20) and (5.13) it follows that it is enough to prove that, for an appropriate choice of ε,

$$
\widetilde{\Psi}_{J_1}(\boldsymbol{\rho}) - 2\sum_{i\in J_1}\Big(\sum_{k\in K}\tilde{a}_{i,k}M_k\Big)u_i(\bar{x}_i) \ge -C. \tag{5.21}
$$

The proof of (5.21) will follow from the argument of Case 3 once we verify the inequalities:

$$
\widetilde{\Lambda}_{\tilde{J}\cup K}(\boldsymbol{M}) > 0,\ \forall \tilde{J} \subsetneq J_1, \tag{5.22}
$$

$$
\widetilde{\Lambda}_{J_1\cup K}(\boldsymbol{M}) \ge 0, \tag{5.23}
$$

where, analogously to (2.3), we denote for each $\tilde{J} \subseteq I$,

$$
\widetilde{\Lambda}_{\tilde{J}}(\boldsymbol{M}) = m\sum_{i\in\tilde{J}} M_i - \sum_{i\in\tilde{J}}\sum_{j\in\tilde{J}}\tilde{a}_{i,j}M_iM_j\ .
$$

Evidently, it suffices to consider the situation after applying (5.19) for one pair $i_0 \ne i_1$. Since

$$
\tilde{a}_{i_0,i_0} = a_{i_0,i_0} + \Big(\frac{M_{i_1}}{M_{i_0}}\Big)a_{i_0,i_1}\varepsilon \quad , \quad \tilde{a}_{i_1,i_1} = a_{i_1,i_1} + \Big(\frac{M_{i_0}}{M_{i_1}}\Big)a_{i_0,i_1}\varepsilon
$$

and

$$
\tilde{a}_{i_0,i_1} = \tilde{a}_{i_1,i_0} = (1-\varepsilon)a_{i_0,i_1},
$$

it follows that,

$$\tilde{a}_{i_0,i_0} M_{i_0}^2 + 2\tilde{a}_{i_0,i_1} M_{i_0} M_{i_1} + \tilde{a}_{i_1,i_1} M_{i_1}^2 = a_{i_0,i_0} M_{i_0}^2 + 2a_{i_0,i_1} M_{i_0} M_{i_1} + a_{i_1,i_1} M_{i_1}^2 \,.$$

Hence,

$$\widetilde{\Lambda}_{\tilde{J}\cup K}(\boldsymbol{M}) = \Lambda_{\tilde{J}\cup K}(\boldsymbol{M}),\ \forall \tilde{J} \subseteq J_1 \text{ s.t. } i_0, i_1 \in \tilde{J}.$$

This implies in particular (5.23), and we only need to verify (5.22) for $\tilde{J} \subsetneq J_1$ such that $i_0 \in \tilde{J}$ while $i_1 \notin \tilde{J}$ (or vice versa). But thanks to (5.10) this can be guaranteed by choosing $\varepsilon < a_{i_0,i_1}$ small enough. □

A An elementary Linear Programing problem

This appendix is devoted to the study of the following Linear Programing problem that was used in the proofs of Proposition 3.1 and the Main Theorem.

Problem (P): Given a symmetric matrix $B = (b_{i,j})_{i,j\in I}$ with $b_{i,j} \geq 0, \forall i,j \in I$, and a vector $(a_i)_{i\in I}$ with $a_i \geq 0$, $\forall i \in I$, find a matrix $(x_{i,j})_{i,j\in I}$ satisfying the three conditions:

(i) $x_{i,j} \geq 0,\ \forall i,j \in I$,
(ii) $\sum_{j\in I} x_{i,j} \leq a_i,\ \forall i \in I$,
(iii) $x_{i,j} + x_{j,i} = 2b_{i,j},\ \forall i,j \in I$.

Problem (P) is a feasibility problem in Linear Programing and we shall use a duality principle from this theory to resolve it.

Proposition A.1. *There exists a solution to problem (P) if and only if the following condition holds:*

$$\sum_{i,j\in J} b_{i,j} \leq \sum_{i\in J} a_i, \quad \forall J \subseteq I. \tag{A.1}$$

Proof. Necessity of (A.1) is obvious. Indeed, if a solution $(x_{i,j})$ to problem (P) exists then we have for every $J \subseteq I$:

$$\sum_{i,j\in J} b_{i,j} = \sum_{i,j\in J} x_{i,j} \leq \sum_{i\in J} a_i \,.$$

Next we turn to the proof of sufficiency of (A.1). From the standard theory of Linear Programing (see [6]) it follows that the following two problems are dual to each other:

$$\max\{y'\cdot 0 \,|\, y'D \leq d',\, y'E = c',\, y' \geq 0\}, \tag{A.2}$$

and

$$\min\{d'\cdot x^{(1)} + c'\cdot x^{(2)} \,|\, Dx^{(1)} + Ex^{(2)} \geq 0,\, x^{(1)} \geq 0\}. \tag{A.3}$$

Here $d, x^{(1)} \in \mathbb{R}^{m_1}$, $c, x^{(2)} \in \mathbb{R}^{m_2}$, $y \in \mathbb{R}^k$, D is a $k \times m_1$ matrix and E is a $k \times m_2$ matrix. Note that (A.2) is just a feasibility problem of existence of a solution to

$$y'D \leq d',\ y'E = c',\ y' \geq 0. \tag{A.4}$$

By duality, there exists a solution to (A.4) iff 0 is the minimum of (A.3). The latter can be restated as:

$$Dx^{(1)} + Ex^{(2)} \geq 0 \text{ and } x^{(1)} \geq 0 \Longrightarrow d' \cdot x^{(1)} + c' \cdot x^{(2)} \geq 0. \tag{A.5}$$

Our problem (P) is clearly a feasibility problem of the type (A.4). Setting $x^{(1)} = (z_i)_{i \in I}$ and $x^{(2)} = (w_{i,j})_{i,j \in I}$ condition (A.5) reads in our case:

$$z_i + w_{i,j} + w_{j,i} \geq 0,\ \forall i, j \in I \text{ and } z_i \geq 0, \forall i \in I \Longrightarrow \sum_{i \in I} z_i a_i + 2 \sum_{i,j \in I} w_{i,j} b_{i,j} \geq 0\,. \tag{A.6}$$

Assume w.l.o.g. that $z_1 \leq z_2 \leq \cdots \leq z_n$ and compute:

$$\begin{aligned} 2\sum_{i,j \in I} w_{i,j} b_{i,j} &= \sum_{i,j \in I} (w_{i,j} + w_{j,i}) b_{i,j} \geq \sum_{i,j \in I} \max(-z_i, -z_j) b_{i,j} = -\sum_{i,j \in I} \min(z_i, z_j) b_{i,j} \\ &= -\sum_{i \in I} \Big(\sum_{j \leq i} z_j b_{i,j} + \sum_{j > i} z_i b_{i,j} \Big) = -\sum_{i \in I} z_i (b_{i,i} + 2 \sum_{j > i} b_{i,j})\,. \end{aligned}$$

Hence,

$$\sum_{i \in i} z_i a_i + 2 \sum_{i,j \in I} w_{i,j} b_{i,j} \geq \sum_{i \in I} z_i (a_i - b_{i,i} - 2 \sum_{j > i} b_{i,j}) := \sum_{i \in I} z_i t_i\,.$$

Note that by (A.1), with $J = \{n\}$, $t_n = a_n - b_{n,n} \geq 0$. Also, for any $1 \leq k \leq n-1$,

$$\sum_{i=k}^{n} t_i = \sum_{i=k}^{n} (a_i - b_{i,i} - 2 \sum_{j > i} b_{i,j}) = \sum_{i=k}^{n} a_i - \sum_{i,j=k}^{n} b_{i,j} \geq 0,$$

where we used again (A.1), with $J = \{k, k+1, \ldots, n\}$. Thus, setting for convenience $z_0 = 0$, we obtain finally that

$$\sum_{i=1}^{n} z_i t_i = \sum_{i=1}^{n} (z_i - z_{i-1}) \sum_{j=i}^{n} t_j \geq 0.$$

□

The following variant of Proposition A.1 was also used in the course of the proof of the Main Theorem.

Corollary A.1. *Under the assumption*

$$\sum_{i,j\in J} b_{i,j} < \sum_{i\in J} a_i,\ \forall J \subsetneq I \text{ and } \sum_{i,j\in I} b_{i,j} \le \sum_{i\in I} a_i, \tag{A.7}$$

there exists a solution to problem (P) with $x_{i,j} > 0,\ \forall i \neq j$ *such that* $b_{i,j} > 0$.

Proof. Let ε be a small positive number that will be fixed later. Introduce the new unknowns

$$y_{i,j} = \begin{cases} x_{i,j} - \varepsilon & \text{for } i \neq j \text{ with } b_{i,j} > 0, \\ x_{i,j} & \text{otherwise.} \end{cases}$$

Then, in terms of the new variables, problem (P) consists of finding $\{y_{i,j}\}_{i,j\in I}$ satisfying:

(i) $\sum_{j\in I} y_{i,j} \le \tilde{a}_i := a_i - L_i\varepsilon,\ \forall i \in I$, where $L_i = \#\{j\ :\ j \neq i \text{ and } b_{i,j} > 0\}$,

(ii) $y_{i,j} \ge 0,\ \forall i, j \in I$,

(iii) $y_{i,j} + y_{j,i} = 2\tilde{b}_{i,j},\ \forall i, j \in I$,

where

$$\tilde{b}_{i,j} = \begin{cases} b_{i,j} - \varepsilon & \text{for } i \neq j \text{ with } b_{i,j} > 0, \\ b_{i,j} & \text{otherwise.} \end{cases}$$

In order to apply Proposition A.1 we should verify that $\{\tilde{a}_i\}$ and $\{\tilde{b}_{i,j}\}$ satisfy condition (A.1). Note that $\tilde{a}_i, \tilde{b}_{i,j} > 0$ for ε small enough and that

$$\sum_{i,j\in J} \tilde{b}_{i,j} = \sum_{i,j\in J} b_{i,j} - \varepsilon \cdot \#\{i, j \in J \text{ s.t. } b_{i,j} > 0\} \quad \text{and} \quad \sum_{i\in J} \tilde{a}_i = \sum_{i\in J} a_i - \varepsilon \sum_{i\in J} L_i\,. \tag{A.8}$$

For $J \subsetneq I$ we deduce from (A.7) that for ε small enough we have indeed $\sum_{i,j\in J} \tilde{b}_{i,j} < \sum_{i\in J} \tilde{a}_i$. Finally, for $J = I$ we have by construction

$$\sum_{i\in I} \tilde{a}_i - \sum_{i,j\in I} \tilde{b}_{i,j} = \sum_{i\in I} a_i - \sum_{i,j\in I} b_{i,j} \ge 0,$$

and the result follows. □

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