On the distance between homotopy classes of S^1-valued maps in multiply connected domains

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Abstract

Certain Sobolev spaces of S^1-valued functions can be written as a disjoint union of homotopy classes. The problem of finding the distance between different homotopy classes in such spaces is considered. In particular several types of one-dimensional and two-dimensional domains are studied. Lower bounds are derived for these distances. Furthermore, in many cases it is shown that the lower bounds are sharp but are not achieved.

1 Introduction

Let D be a multiply connected domain or an embedded compact multiply connected manifold in $\mathbb{R}^N$. Suppose that the space $H^1(D,S^1)$ can be written as a disjoint union of homotopy classes

$$H^1(D,S^1)=\bigcup_{\boldsymbol{d}}\mathcal{E}_{\boldsymbol{d}}, \tag{1.1}$$

so that the homotopy classes are indexed by vectors of integers $\boldsymbol{d}$. Such partitions of Sobolev spaces was first observed by White [7]. In particular partitions like (1.1) indeed exist when D is the circle S^1, a planar graph, a compact multiply connected two-dimensional domain, etc.. This partition found interesting applications in physics, where it was used [6] to explain persistent currents in superconductivity, and to predict [5] new structures in liquid crystals.

For $\boldsymbol{d}^{(1)}\neq\boldsymbol{d}^{(2)}$ set

$$\delta^2(\boldsymbol{d}^{(1)},\boldsymbol{d}^{(2)})=\inf\Big\{\int_D|\nabla(u_1-u_2)|^2 : u_1\in\mathcal{E}_{\boldsymbol{d}^{(1)}},u_2\in\mathcal{E}_{\boldsymbol{d}^{(2)}}\Big\}. \tag{1.2}$$

Two natural questions arise concerning this distance function:

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(i) What is the value of $\delta(\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)})$?

(ii) Is the infimum in (1.2) achieved?

In the next section we solve both questions for the case $D = S^1$. In section 3 we consider two-dimensional multiply connected domains D. We derive a general lower bound for $\delta(\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)})$, and prove that the bound is sharp under certain conditions (called property (C)) on the homotopy class vectors. Finally, in section 4 we demonstrate through an example that the general lower bound derived in section 3 may not be optimal if condition (C) is not satisfied. Although some of our results can be readily extended to some three-dimensional domains (such as the solid torus), we have not examined in detail more general three-dimensional multiply connected domains.

2 Maps from S^1 to S^1

Note that $H^1(S^1, S^1) \subset C^{1/2}(S^1, S^1)$, so that each $u \in H^1(S^1, S^1)$ has a well defined degree, and we may write:

$$H^1(S^1, S^1) = \bigcup_{d \in \mathbb{Z}} \mathcal{E}_d = \bigcup_{d \in \mathbb{Z}} \{u : \deg u = d\}.$$

The formula (1.2) for the distance between $\mathcal{E}_{d_1}$ and $\mathcal{E}_{d_2}$ reads in the current case:

$$\delta^2(d_1, d_2) = \inf \Big\{ \int_{S^1} |(u_1 - u_2)'|^2 : u_1 \in \mathcal{E}_{d_1}, u_2 \in \mathcal{E}_{d_2} \Big\}. \tag{2.1}$$

A simple lower-bound for $\delta(d_1, d_2)$ is given by the following lemma.

Lemma 2.1. *We have:*

$$\delta^2(d_1, d_2) \geq \frac{8(d_2 - d_1)^2}{\pi}.$$

Proof. Clearly it suffices to consider the case $m := d_2 - d_1 > 0$. For each pair $u_1 \in \mathcal{E}_{d_1}$, $u_2 \in \mathcal{E}_{d_2}$ we have $v = u_2/u_1 \in \mathcal{E}_m$; hence v covers S^1 (algebraically) m times. In particular, each of the values ± 1 is attained at least m times. It follows that there are points:

$$0 \leq s_1 < t_1 < s_2 < t_2 < s_3 < \cdots < s_m < t_m < s_{m+1} = s_1 + 2\pi$$

such that $v(e^{is_j}) = 1$ (i.e., $|u_2 - u_1|(e^{is_j}) = 0$) and $v(e^{it_j}) = -1$ (i.e., $|u_2 - u_1|(e^{it_j}) = 2$) for each j. A simple consideration and direct calculation gives

$$\int_{S^1} \big|(u_2 - u_1)'\big|^2 \geq \int_{S^1} \big||u_2 - u_1|'\big|^2 \geq \sum_{j=1}^{m} \Big(\frac{4}{t_j - s_j} + \frac{4}{s_{j+1} - t_j} \Big). \tag{2.2}$$

It is clear, for example from the arithmetic-harmonic means inequality, that the smallest value of the last expression is achieved when all the points $\{s_j, t_i\}$ are equally spaced. Therefore

$$\int_{S^1} \left|(u_2 - u_1)'\right|^2 \geq \int_{S^1} \left||u_2 - u_1|'\right|^2 \geq \frac{(4m)^2}{2\pi} = \frac{8m^2}{\pi}\,. \tag{2.3}$$

□

The next simple lemma shows that the distance between two homotopy classes depends on the difference of the degrees only.

Lemma 2.2. *We have* $\delta(d_1 + k, d_2 + k) = \delta(d_1, d_2),\ \forall k \in \mathbb{Z}$.

Proof. Take any $u_1 \in \mathcal{E}_{d_1}, u_2 \in \mathcal{E}_{d_2}$ with $m = d_2 - d_1 \neq 0$. With a slight abuse of notations we view each u_j also as a map from $[0, 2\pi]$ to S^1 satisfying $u_j(0) = u_j(2\pi)$. Since the image of u_2/u_1 is the whole circle S^1, the point 1 is in this image, and so we may assume w.l.o.g. that $u_1(0) = u_2(0)$. For a small $\varepsilon > 0$ define the "rescaled" maps $\tilde{u}_j = \tilde{u}_j^{(\varepsilon)}$ on $[0, 2\pi - \varepsilon]$ by

$$\tilde{u}_j(\theta) = u_j\Big(\frac{2\pi}{2\pi - \varepsilon}\theta\Big),\ j = 1, 2\,. \tag{2.4}$$

On the remaining interval $[2\pi - \varepsilon, 2\pi]$ complete the definition of $\tilde{u}_1, \tilde{u}_2$ by

$$\tilde{u}_1(\theta) = \tilde{u}_2(\theta) = u_1(0) \cdot \exp\Big(2\pi k i \frac{\theta - (2\pi - \varepsilon)}{\varepsilon}\Big)\,.$$

Clearly, $\tilde{u}_j \in \mathcal{E}_{d_j + k}$, $j = 1, 2$, and

$$\lim_{\varepsilon \to 0} \int_{S^1} \left|(\tilde{u}_2 - \tilde{u}_1)'\right|^2 = \int_{S^1} \left|(u_2 - u_1)'\right|^2.$$

The result follows since u_j can be chosen arbitrarily in $\mathcal{E}_{d_j}$. □

Next we give the main result of this section.

Theorem 1. *For every* $d_1, d_2 \in \mathbb{Z}$ *we have:*

(i) $\delta^2(d_1, d_2) = \frac{8(d_2 - d_1)^2}{\pi}$.

(ii) *For* $d_1 \neq d_2$, $\delta(d_1, d_2)$ *is not attained.*

Proof. (i) In view of Lemma 2.2 it suffices to consider two cases:

(1) $d_2 = d > 0$, $d_1 = -d$,

(2) $d_2 = d > 0$, $d_1 = -d + 1$.

In each of these cases we put $m = d_2 - d_1$. Note that the equality

$$\int_{S^1} \left||u_2 - u_1|'\right|^2 = \frac{8m^2}{\pi} \tag{2.5}$$

is achieved in (2.2) if and only if the $2m+1$ points:

$$s_1, t_1, s_2, t_2, \dots, s_m, t_m, s_{m+1}$$

are equidistant and the graph of the function $|u_2 - u_1|$ is piecewise linear with vertices at the points $\{(s_j, 0), (t_j, 2)\}_{j=1}^m$. Motivated by the above, we set for $j = 1, \dots, m+1$:

$$\tilde{s}_j = (j-1)\frac{2\pi}{m} \quad \text{and} \quad \tilde{t}_j = (j-1)\frac{2\pi}{m} + \frac{\pi}{m},$$

and define the function ρ by

$$\rho(\theta) = \begin{cases} (\theta - \tilde{s}_j)\frac{2m}{\pi} & \theta \in [\tilde{s}_j, \tilde{t}_j], \\ 2 - (\theta - \tilde{t}_j)\frac{2m}{\pi} & \theta \in (\tilde{t}_j, \tilde{s}_{j+1}], \end{cases} \tag{2.6}$$

for $j = 1, \dots, m$. For any small $\varepsilon > 0$ consider the following approximation ρ_ε of ρ:

$$\rho^{(\varepsilon)}(\theta) = 2J_\varepsilon(\frac{\rho}{2}),$$

where the map $J_\varepsilon : [-1, 1] \to [-1, 1]$ is an odd C^2-map enjoying the following properties:

$$\begin{aligned} J_\varepsilon(\pm 1) = \pm 1, J'_\varepsilon(\pm 1) = 0, \\ J_\varepsilon(t) = t, \quad |t| \le 1 - \varepsilon, \\ 0 < J'_\varepsilon(t) < c_0, \quad |t| < 1, \\ \frac{c_1}{\varepsilon} \le |J''_\varepsilon(t)| \le \frac{c_2}{\varepsilon}, \quad 1 - \frac{\varepsilon}{2} \le |t| \le 1, \end{aligned} \tag{2.7}$$

for some positive constants c_0, c_1, c_2 (independent of ε). Set $u_2^{(\varepsilon)}(\theta) = e^{i\alpha(\theta)}$ and then $u_1^{(\varepsilon)}(\theta) = \bar{u}_2^{(\varepsilon)}(\theta) = e^{-i\alpha(\theta)}$ where

$$\alpha(\theta) = \alpha^{(\varepsilon)}(\theta) = \begin{cases} \sin^{-1}(\rho^{(\varepsilon)}/2) - (1 + (-1)^j)\frac{\pi}{2} & \theta \in [\tilde{s}_j, \tilde{t}_j], \\ \pi - \sin^{-1}(\rho^{(\varepsilon)}/2) - (1 + (-1)^j)\frac{\pi}{2} & \theta \in (\tilde{t}_j, \tilde{s}_{j+1}], \end{cases} \tag{2.8}$$

for $j = 1, \dots, m$. Thanks to (2.7) we have $e^{i\alpha} \in Lip[0, 2\pi]$.

Consider first case (1). Then, $u_2^{(\varepsilon)} \in \mathcal{E}_d, u_1^{(\varepsilon)} \in \mathcal{E}_{-d}$ and

$$(u_2^{(\varepsilon)} - u_1^{(\varepsilon)})(\theta) = \pm i |(u_2^{(\varepsilon)} - u_1^{(\varepsilon)})(\theta)| = \pm i \rho^{(\varepsilon)}(\theta), \quad \text{for all } \theta. \tag{2.9}$$

Therefore,

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{S^1} \left|(u_2^{(\varepsilon)} - u_1^{(\varepsilon)})'\right|^2 &= \lim_{\varepsilon \to 0} \int_{S^1} \left||u_2^{(\varepsilon)} - u_1^{(\varepsilon)}|'\right|^2 \\ &= \lim_{\varepsilon \to 0} \int_{S^1} \left|(\rho^{(\varepsilon)})'\right|^2 = \int_{S^1} (\rho')^2 = \frac{8(2d)^2}{\pi}. \end{aligned}$$

In case (2), the maps $u_1^{(\varepsilon)}, u_2^{(\varepsilon)}$ are well defined as maps from $[0, 2\pi]$ to S^1, but are not well defined as maps on S^1, since their changes of phase on $[0, 2\pi]$ equal $-(2d-1)\pi$ and $(2d-1)\pi$, respectively. Note that in particular we have, $u_1^{(\varepsilon)}(2\pi) = u_2^{(\varepsilon)}(2\pi) = -1$. Therefore, we modify $u_1^{(\varepsilon)}, u_2^{(\varepsilon)}$ slightly to maps $\tilde{u}_j^{(\varepsilon)}, j = 1, 2$, in a similar manner to the argument used in the proof of Lemma 2.2. First, we use the rescaling (2.4) to define $\tilde{u}_j^{(\varepsilon)}, j = 1, 2$, on $[0, 2\pi - \varepsilon]$. Then, on $(2\pi - \varepsilon, 2\pi]$ we set

$$\tilde{u}_1^{(\varepsilon)}(\theta) = \tilde{u}_2^{(\varepsilon)}(\theta) = -\exp\left(i\pi\frac{\theta - (2\pi - \varepsilon)}{\varepsilon}\right).$$

Evidently, $\tilde{u}_2^{(\varepsilon)} \in \mathcal{E}_d, \tilde{u}_1^{(\varepsilon)} \in \mathcal{E}_{1-d}$ and a simple computation yields

$$\lim_{\varepsilon\to 0}\int_{S^1}\left|(u_2^{(\varepsilon)} - u_1^{(\varepsilon)})'\right|^2 = \lim_{\varepsilon\to 0}\int_{S^1}\left||u_2^{(\varepsilon)} - u_1^{(\varepsilon)}|'\right|^2$$
$$= \lim_{\varepsilon\to 0}\int_{S^1}\left|(\rho^{(\varepsilon)})'\right|^2 = \int_{S^1}(\rho')^2 = \frac{8(2d-1)^2}{\pi}.$$

(ii) Assume by negation that there exist $u_1 \in \mathcal{E}_{d_1}$ and $u_2 \in \mathcal{E}_{d_2}$ such that

$$\int_{S^1}|(u_1 - u_2)'|^2 = \delta^2(d_1, d_2). \tag{2.10}$$

We may assume w.l.o.g. that $u_2(0) = u_1(0)$. It then follows from (2.2)–(2.3) that the function $|u_2 - u_1|$ must be equal to the function ρ given by (2.6). On the interval $K = [\frac{\pi}{2m}, \frac{3\pi}{2m}]$ we may write $u_2 - u_1 = \rho e^{i\phi}$, so that

$$\int_K|(u_2 - u_1)'|^2 = \int_K(\rho')^2 + \rho^2(\phi')^2.$$

Hence, using (2.10) and (2.3) we infer that ϕ is identically equal to a constant on K. W.l.o.g. we may assume that the constant is equal to $\pi/2$. Therefore, $u_1 = \bar{u}_2$ on K, where we may write $u_2 = e^{i\psi}$, $u_1 = e^{-i\psi}$. It follows that $\rho = 2\sin\psi$, i.e., $\psi = \sin^{-1}\frac{\rho}{2}$ on K. Because of the nature of the singularity of ψ' at $\theta = \frac{\pi}{m}$, the function ψ does not belong to $H^1(K)$. Hence also $u_2 = e^{\psi} \notin H^1(K, S^1)$, contradicting our starting assumption. $\square$

Remark 2.1. It is of course possible to consider the distance between homotopy classes with respect to other norms. For example, one may work in $W^{1,p}(S^1, S^1)$, $p \in [1, \infty]$, and consider the distance

$$\delta_{(p)}(d_1, d_2) = \inf\left\{\|(u_1 - u_2)'\|_{L^p(S^1)} : u_1 \in \mathcal{E}_{d_1}, u_2 \in \mathcal{E}_{d_2}\right\}.$$

By the above techniques we get:

$$\delta_{(p)}(d_1, d_2) = \begin{cases} \frac{2^{1+1/p}m}{\pi^{1-1/p}} & 1 \le p < \infty, \\ \frac{2m}{\pi} & p = \infty. \end{cases}$$

However, the situation may be different when working with weaker norms. For example, although maps in $H^{1/2}(S^1,S^1)$ have a well defined degree, it was shown by Brezis and Nirenberg [3, Lemma 6 and Remark 6] that for all d_1 and d_2:

$$\inf\left\{\|u_1-u_2\|_{H^{1/2}} : u_1\in\mathcal{E}_{d_1}, u_2\in\mathcal{E}_{d_2}\right\}=0,$$

where $\|\cdot\|_{H^{1/2}}$ denotes the $H^{1/2}$-semi-norm.

It is also straightforward to extend the statements and arguments presented above to the case where the entire Sobolev norm is used as a distance function between homotopy classes.

3 S^1-valued maps on multiply connected domains in $\mathbb{R}^2$

Let $G,\omega_1,\ldots,\omega_n$ be smooth bounded simply connected domains in $\mathbb{R}^2$ with $\omega_j\subset\subset G$ for all j, and let $\Omega=G\setminus\bigcup_{j=1}^{n}\omega_j$ (see a sketch in Figure 1).

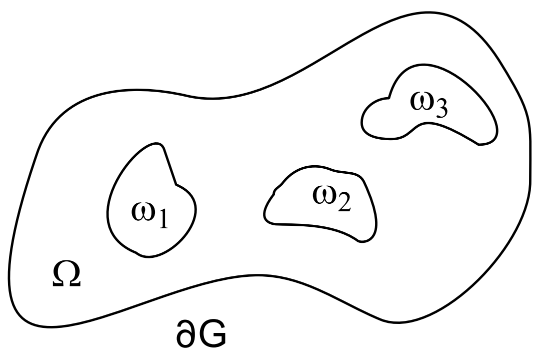

Figure 1: The domain Ω

The orientation with respect to which we shall define degrees in the sequel is: positive (i.e., counter-clockwise) on $\partial\omega_j$, $j=1,\ldots,n$ and clockwise on $\partial\omega_0:=\partial G$. Set

$$\mathcal{V}_0=\{\boldsymbol{d}=(d_0,d_1,\ldots,d_n)\in\mathbb{Z}^{n+1}\text{ s.t. }\sum_{j=0}^{n}d_j=0\}\,. \tag{3.1}$$

For $\boldsymbol{d}\in\mathcal{V}_0$ define

$$\mathcal{E}_{\boldsymbol{d}}=\{v\in H^1(\Omega,S^1)\,:\,\deg(v,\partial\omega_j)=d_j,\ j=0,1,\ldots,n\}\,.$$

We therefore obtain a disjoint decomposition

$$H^1(\Omega,S^1)=\bigcup_{\boldsymbol{d}\in\mathcal{V}_0}\mathcal{E}_{\boldsymbol{d}}\,.$$

We shall investigate the distance $\delta(\boldsymbol{d}^{(1)},\boldsymbol{d}^{(2)})$ as defined in (1.2) (for $D=\Omega$) for any $\boldsymbol{d}^{(1)}\neq\boldsymbol{d}^{(2)}$ in $\mathcal{V}_0$. We first recall some known results on the minimum of the energy in each homotopy class. Let

$$I(\boldsymbol{d})=\inf_{v\in\mathcal{E}_{\boldsymbol{d}}}\int_\Omega|\nabla v|^2\,. \tag{3.2}$$

By [1, Th. I.1] we have

$$I(\boldsymbol{d})=\int_\Omega|\nabla\Phi|^2\,,$$

where $\Phi=\Phi_{\boldsymbol{d}}$ satisfies, for some unprescribed constants $C_1,\dots,C_n$,

$$\begin{cases}\Delta\Phi=0 \text{ in } \Omega,\\ \quad\Phi=C_j \text{ on } \partial\omega_j,\ j=1,\dots,n,\\ \quad\Phi=0 \text{ on } \partial G,\\ \displaystyle\int_{\partial\omega_j}\frac{\partial\Phi}{\partial\nu}=2\pi d_j,\ \ j=1,\dots,n.\end{cases}$$

Moreover, the infimum in (3.2) is attained by a unique u, up to a constant rotation, which satisfies

$$\begin{cases}u\times\dfrac{\partial u}{\partial x_1}=-\dfrac{\partial\Phi}{\partial x_2}\,,\\ u\times\dfrac{\partial u}{\partial x_2}=\dfrac{\partial\Phi}{\partial x_1}\,.\end{cases}$$

We also have $\Phi=\sum_{j=1}^n d_j\Phi_j$, where for each $j=1,\dots,n$, Φ_j satisfies

$$\begin{cases}\Delta\Phi_j=0 \text{ in } \Omega,\\ \quad\Phi_j=\text{const on } \partial\omega_j,\ j=1,\dots,n,\\ \quad\Phi_j=0 \text{ on } \partial G,\\ \displaystyle\int_{\partial\omega_j}\frac{\partial\Phi_j}{\partial\nu}=2\pi\delta_{i,j},\ \ i=1,\dots,n.\end{cases}$$

Then,

$$I(\boldsymbol{d})=\int_\Omega|\nabla\Phi|^2=\sum_{i,j=1}^n c_{i,j}d_id_j\,,$$

where $c_{i,j}:=\int_\Omega\nabla\Phi_i\nabla\Phi_j$.

Next, for any $u_j\in\mathcal{E}_{\boldsymbol{d}^{(j)}}$, $j=1,2$, we have $v:=u_2/u_1\in\mathcal{E}_{\boldsymbol{d}}$ with $\boldsymbol{d}=\boldsymbol{d}^{(2)}-\boldsymbol{d}^{(1)}$. Clearly

$$\int_\Omega|\nabla(u_2-u_1)|^2\geq\int_\Omega\big|\nabla|u_2-u_1|\big|^2=\int_\Omega\big|\nabla|v-1|\big|^2\,. \tag{3.3}$$

Defining

$$H(\boldsymbol{d}):=\inf\Big\{\int_\Omega\big|\nabla|w-1|\big|^2\ :\ w\in\mathcal{E}_{\boldsymbol{d}}\Big\}\,, \tag{3.4}$$

we therefore proved the following lower bound:

Lemma 3.1. *For every* $\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)} \in \mathcal{V}_0$ *we have* $\delta^2(\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)}) \geq H(\boldsymbol{d}^{(2)} - \boldsymbol{d}^{(1)})$.

The next proposition provides an explicit relation between $H(\boldsymbol{d})$ and $I(\boldsymbol{d})$.

Proposition 3.1. *For every* $\boldsymbol{d} \in \mathcal{V}_0$ *we have*

$$H(\boldsymbol{d}) = \Big(\frac{2}{\pi}\Big)^2 I(\boldsymbol{d})\,. \tag{3.5}$$

Furthermore, the infimum in (3.4) *is never attained (for* $\boldsymbol{d} \neq \mathbf{0}$*).*

Proof. Consider the map $T : S^1 \to S^1$ defined as follows. For any $e^{i\phi} \in S^1, \phi \in (-\pi, \pi]$, let $T(e^{i\phi}) = e^{i\theta}$ with $\theta = \pi \sin\phi/2$. Note that

$$|e^{i\phi} - 1| = 2|\sin\frac{\phi}{2}| = \Big(\frac{2}{\pi}\Big)|\theta|\,. \tag{3.6}$$

Let the operator $\mathcal{T} : H^1(\Omega, S^1) \to H^1(\Omega, S^1)$ be defined for any $w \in H^1(\Omega, S^1)$ by $(\mathcal{T}w)(x) = T(w(x))$, $\forall x \in \Omega$. Since T is a bijective C^1-map from S^1 to S^1, $\mathcal{T}$ sends each $\mathcal{E}_{\boldsymbol{d}}$ to itself. By (3.6)

$$\int_\Omega \big|\nabla|w - 1|\big|^2 = \Big(\frac{2}{\pi}\Big)^2 \int_\Omega |\nabla(\mathcal{T}w)|^2,\ \forall w \in \mathcal{E}_{\boldsymbol{d}}\,, \tag{3.7}$$

and it follows that $H(\boldsymbol{d}) \geq \big(\frac{2}{\pi}\big)^2 I(\boldsymbol{d})$.

Next we turn to the proof of the reverse inequality. The inverse $S = T^{-1}$ of T is given by: $S(e^{i\theta}) = e^{i\phi}$, with $\phi = 2\sin^{-1}\theta/\pi$, $\forall\theta \in (-\pi, \pi]$. This map is continuous but not Lipschitz. We therefore define, for each small $\varepsilon > 0$, an approximation S_ε by:

$$S_\varepsilon(e^{i\theta}) = e^{i\phi} \ \text{ with } \phi = 2\sin^{-1}\Big(J_\varepsilon\big(\frac{\theta}{\pi}\big)\Big),\ \forall\theta \in (-\pi, \pi], \tag{3.8}$$

where J_ε is defined in (2.7). Since $|S_\varepsilon(e^{i\theta}) - 1| = 2|J_\varepsilon(\theta/\pi)|$ it follows from (2.7) that

$$\Big|\frac{d}{d\theta}\big(|S_\varepsilon(e^{i\theta}) - 1|\big)\Big| \leq C,\ \forall\theta, \forall\varepsilon\,.$$

Therefore, defining for each $W \in \mathcal{E}_{\boldsymbol{d}}$,

$$(\mathcal{S}_\varepsilon W)(x) = S_\varepsilon(W(x)),\ \forall x \in \Omega, \tag{3.9}$$

we have $\mathcal{S}_\varepsilon W \in \mathcal{E}_{\boldsymbol{d}}$ and

$$\lim_{\varepsilon\to 0} \int_\Omega \big|\nabla|\mathcal{S}_\varepsilon W - 1|\big|^2 = \int_\Omega \big|\nabla|S(W) - 1|\big|^2\,. \tag{3.10}$$

From (3.10) and (3.6) we finally infer that

$$\lim_{\varepsilon\to 0} \int_\Omega \big|\nabla|\mathcal{S}_\varepsilon W - 1|\big|^2 = \Big(\frac{2}{\pi}\Big)^2 \int_\Omega |\nabla W|^2\,, \tag{3.11}$$

which yields $H(\boldsymbol{d}) \le (\frac{2}{\pi})^2 I(\boldsymbol{d})$.

Finally, we show that the infimum in (3.4) is not attained. Looking for a contradiction, assume that it is attained by some $w \in \mathcal{E}_{\boldsymbol{d}}$. From (3.5) and (3.7) it then follows that $W := \mathcal{T}w$ must be a minimizer in (3.2). We recall that by a result of [1] (see the beginning of this section), W is a C^∞ map that can be written locally in Ω as $W = e^{i\psi}$ with ψ a smooth harmonic function. It is a standard fact that the critical points of a non-constant harmonic function are isolated and that the level-set through a critical point z_0 consists locally of two or more analytic curves intersecting at z_0 with equal angles (c.f. [4, pages 18–19]). Therefore, regardless of the property of -1 being a critical or a regular value of W, there exists a subsegment of an analytic curve, $S \subset \{W = -1\}$, on which $\psi \equiv \pi$ and $|\nabla W| = |\nabla \psi| \neq 0$. We can then choose a narrow enough "tube", $D = \{x \in \Omega : \operatorname{dist}(x, S) < \varepsilon\}$, such that $|\nabla \psi| \ge \eta > 0$ on D. But then it follows easily that the function $\phi = 2\sin^{-1} \psi/\pi$ does not belong to $H^1(D)$. Therefore, also $w = e^{i\phi} \notin H^1(D, S^1)$, which is a contradiction. □

The next result is a direct consequence of Lemma 3.1 and Proposition 3.1.

Theorem 2. *For every* $\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)} \in \mathcal{V}_0$ *we have*

$$\delta^2(\mathcal{E}_{\boldsymbol{d}^{(1)}}, \mathcal{E}_{\boldsymbol{d}^{(2)}}) \ge \left(\frac{2}{\pi}\right)^2 I(\boldsymbol{d}^{(1)} - \boldsymbol{d}^{(2)})\,. \tag{3.12}$$

Another consequence is a (partial) analogue to assertion (ii) of Theorem 1 (see open problem 1 at the end of this section).

Corollary 3.1. *If* $\boldsymbol{d}^{(1)} \neq \boldsymbol{d}^{(2)}$ *are such that equality holds in* (3.12) *then* $\delta(\mathcal{E}_{\boldsymbol{d}^{(1)}}, \mathcal{E}_{\boldsymbol{d}^{(2)}})$ *is not attained.*

Proof. Assume by negation that $\delta(\mathcal{E}_{\boldsymbol{d}^{(1)}}, \mathcal{E}_{\boldsymbol{d}^{(2)}})$ is attained by the two maps $u_j \in \mathcal{E}_{\boldsymbol{d}^{(j)}}$, $j = 1, 2$. Then, it follows from (3.3) that $v = u_2/u_1 \in \mathcal{E}_{\boldsymbol{d}}$, with $\boldsymbol{d} = \boldsymbol{d}^{(2)} - \boldsymbol{d}^{(1)}$, realizes the infimum in (3.4), and thus contradicting Proposition 3.1. □

Next we look for conditions that guarantee that the inequality in (3.12) is actually an equality. The following theorem shows that this is the case when $\boldsymbol{d}^{(2)} = -\boldsymbol{d}^{(1)}$.

Theorem 3. *If* $\boldsymbol{d}^{(2)} = -\boldsymbol{d}^{(1)}$ *then the lower bound of Theorem 2 is sharp.*

Proof. Put $\boldsymbol{d} = \boldsymbol{d}^{(2)} - \boldsymbol{d}^{(1)} = 2\boldsymbol{d}^{(2)}$. By Proposition 3.1 and the density of $C^1(\Omega, S^1)$ in $H^1(\Omega, S^1)$ (see [2]), for every $\varepsilon > 0$ there exists $w_\varepsilon \in C^1(\Omega, S^1) \cap \mathcal{E}_{\boldsymbol{d}}$ such that

$$\int_\Omega \big|\nabla |w_\varepsilon - 1|\big|^2 \le \left(\frac{2}{\pi}\right)^2 I(\boldsymbol{d}) + \varepsilon\,.$$

Since d_j is even for all j it follows that there exists $u_\varepsilon \in \mathcal{E}_{\boldsymbol{d}^{(2)}} \cap C^1(\Omega, S^1)$ such that $u_\varepsilon^2 = w_\varepsilon$. Put $v_\varepsilon = 1/u_\varepsilon = \bar{u}_\varepsilon$ which belongs to $\mathcal{E}_{\boldsymbol{d}^{(1)}} \cap C^1(\Omega, S^1)$. Clearly, $|u_\varepsilon - v_\varepsilon| = |w_\varepsilon - 1|$ and $\operatorname{Re}(u_\varepsilon - v_\varepsilon) = \operatorname{Re}(u_\varepsilon - \bar{u}_\varepsilon) = 0$. Therefore,

$$\int_\Omega |\nabla(u_\varepsilon - v_\varepsilon)|^2 = \int_\Omega \big|\nabla |u_\varepsilon - v_\varepsilon|\big|^2 = \int_\Omega \big|\nabla |w_\varepsilon - 1|\big|^2 \le \left(\frac{2}{\pi}\right)^2 I(\boldsymbol{d}) + \varepsilon\,.$$

The result follows since ε is arbitrary. $\square$

A second case where equality holds in (3.12) is given by the next theorem. In order to describe it we introduce a property of certain vectors $\boldsymbol{d} \in \mathcal{V}_0$. For such $\boldsymbol{d}$ we fix a minimizer W for $I(\boldsymbol{d})$ over $\mathcal{E}_{\boldsymbol{d}}$ (see (3.2)). Recall that all the minimizers in (3.2) are given by $\{e^{i\alpha}W \,:\, \alpha \in (-\pi,\pi]\}$. Note that by Sard's theorem, for a.e. $\alpha \in (-\pi,\pi]$, $e^{i\alpha}$ is a regular value of W. By this we mean that $e^{i\alpha}$ is a regular value of both $W|_\Omega$ and of $W|_{\partial\Omega}$. Therefore, denoting $g := W|_{\partial\Omega}$, we have: $g^{-1}(e^{i\alpha})$ consists of a finite number of points $x_1,\dots,x_m \in \partial\Omega$ and $W^{-1}(e^{i\alpha})$ is a union of smooth curves, each connecting some x_i to an x_j. In fact, $W^{-1}(e^{i\alpha})\cap\Omega$ cannot include closed loops since this would violate the minimizing property of W (we would redefine $W \equiv e^{i\alpha}$ inside the loop, hence decreasing its energy). For each such α consider a graph $\mathcal{G}_\alpha$ with vertices $y_0, y_1, \dots, y_n$ such that y_j corresponds to $\partial\omega_j$, for $j = 0,1,\dots,n$. For $i \neq j$ there is an edge in $\mathcal{G}_\alpha$ between y_i and y_j if and only if $W^{-1}(e^{i\alpha})$ contains a curve joining two points, one in $\partial\omega_i$ and the other one in $\partial\omega_j$.

Definition 3.1. *We shall say that* $\boldsymbol{d}$ *has property (C) if for a minimizer* W *in* (3.2) *there exists* $\alpha \in (-\pi,\pi]$ *for which the graph* $\mathcal{G}_\alpha$ *is connected.*

Theorem 4. *Assume that* $\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)} \in \mathcal{V}_0$ *are such that* $\boldsymbol{d} = \boldsymbol{d}^{(2)} - \boldsymbol{d}^{(1)}$ *has property (C). Then,*

$$\delta^2(\mathcal{E}_{\boldsymbol{d}^{(1)}}, \mathcal{E}_{\boldsymbol{d}^{(2)}}) = \Big(\frac{2}{\pi}\Big)^2 I(\boldsymbol{d})\,. \tag{3.13}$$

Proof. We may assume w.l.o.g. that property (C) is satisfied for W with $\alpha = 0$. Let $g := W|_{\partial\Omega}$ and $g^{-1}(1) = \{x_1,\dots,x_m\} \subset \partial\Omega$. To each x_j we associate a sign $s_j \in \{\pm 1\}$ as follows. Let $x_j \in \partial\omega_k$ for some $k \in \{0,1,\dots,n\}$. In a boundary interval around x_j in $\partial\omega_k$ we may write $g = e^{i\phi}$. We set $s_j = 1$ if ϕ is increasing w.r.t. the orientation that we fixed above on $\partial\Omega$ and $s_j = -1$ otherwise. Put

$$P^+ = \{x_j \,:\, s_j = 1\} \quad \text{and} \quad P^- = \{x_j \,:\, s_j = -1\}\,.$$

It is easy to see that $|P^+| = |P^-| = m/2$ (i.e., m is even), and then $W^{-1}(1)$ consists of $m/2$ disjoint curves, each connecting a point from P^+ to a point from P^-. By the definition of the degree we have

$$\sum_{x_j \in \partial\omega_i} s_j = d_i, \quad i = 0,1,\dots,n\,.$$

By assumption (C), $\mathcal{G}_0$ is connected, so there exists a spanning tree with n edges: $e_1,\dots,e_n$. To each edge e_j we associate two indices $i_+(j), i_-(j)$ such that e_j connects $y_{i_+(j)}$ (corresponding to a positive point on $\partial\omega_{i_+(j)}$) to $y_{i_-(j)}$. To e_j corresponds a curve $\gamma_j \subset W^{-1}(1)$ connecting $\partial\omega_{i_+(j)}$ to $\partial\omega_{i_-(j)}$.

For any small $\varepsilon > 0$ define a map $w_\varepsilon \in \mathcal{E}_{\boldsymbol{d}}$ by $w_\varepsilon = \mathcal{S}_\varepsilon W$ (see (3.8)–(3.9)). Note that by definition $w_\varepsilon^{-1}(1) = W^{-1}(1)$. Moreover, by (3.11)

$$\lim_{\varepsilon\to 0} \int_\Omega \big|\nabla|w_\varepsilon - 1|\big|^2 = \Big(\frac{2}{\pi}\Big)^2 \int_\Omega |\nabla W|^2 = \Big(\frac{2}{\pi}\Big)^2 I(\boldsymbol{d})\,. \tag{3.14}$$

Fix some $\varepsilon > 0$ and for small $\eta > 0$ consider a "tube" of width η around each γ_j:

$$T_\eta^{(j)} = \{x \in \overline{\Omega} \, : \, \operatorname{dist}(x, \gamma_j) < \eta\} \, .$$

One can readily check that there exists a C^1 map $z_{\varepsilon,\eta} \in \mathcal{E}_{\boldsymbol{d}}$ with the following properties:

(i) $z_{\varepsilon,\eta} \equiv 1$ on $T_{\eta/2}^{(j)}$, $j = 1, \dots, n$.

(ii) $z_{\varepsilon,\eta} = w_\varepsilon$ on $A_\eta := \Omega \setminus \bigcup_{j=1}^{n} T_\eta^{(j)}$.

(iii) $\int_\Omega |\nabla(z_{\varepsilon,\eta} - w_\varepsilon)|^2 = O_\varepsilon(\eta)$.

Since $A_{\eta/2}$ is simply connected there exists $u_{\varepsilon,\eta} \in C^1(A_{\eta/2}, S^1)$ s.t. $u_{\varepsilon,\eta}^2 = z_{\varepsilon,\eta}$ in $A_{\eta/2}$. Put $v_{\varepsilon,\eta} = \bar{u}_{\varepsilon,\eta} = 1/u_{\varepsilon,\eta}$.

For each $j \in \{1, \dots, n\}$ consider the two "long sides" of $\partial T_{\eta/2}^{(j)}$, γ_j^- and γ_j^+, so that the ordering $\{\gamma_j^-, \gamma_j, \gamma_j^+\}$ corresponds to the orientation that we defined on $\partial\omega_{i_+(j)}$. We have $u_{\varepsilon,\eta} = v_{\varepsilon,\eta} \equiv \sigma_j^-$ on γ_j^- and $u_{\varepsilon,\eta} = v_{\varepsilon,\eta} \equiv \sigma_j^+$ on γ_j^+ with $\sigma_j^\pm \in \{-1, 1\}$. Put $\sigma_j = \sigma_j^+/\sigma_j^-$. Take any smooth function h_j on $\overline{T}_{\eta/2}^{(j)}$ that satisfies

$$h_j \equiv 0 \text{ on } \gamma_j^- \quad \text{and} \quad h_j \equiv \pi\Big(\frac{1 - \sigma_j}{2}\Big) \text{ on } \gamma_j^+ \, .$$

Then, complete the definition of $u_{\varepsilon,\eta}$ and $v_{\varepsilon,\eta}$ to $\bigcup_{j=1}^{n} T_{\eta/2}^{(j)}$ by setting

$$u_{\varepsilon,\eta} = v_{\varepsilon,\eta} = \sigma_j^- e^{ih_j} \text{ on } T_{\eta/2}^{(j)}, \ j = 1, \dots, n \, .$$

It follows that for some $\boldsymbol{e}^{(1)}, \boldsymbol{e}^{(2)} \in \mathcal{V}_0$ we have $u_{\varepsilon,\eta} \in \mathcal{E}_{\boldsymbol{e}^{(1)}}$ and $v_{\varepsilon,\eta} \in \mathcal{E}_{\boldsymbol{e}^{(2)}}$. Since $u_{\varepsilon,\eta}/v_{\varepsilon,\eta} = z_{\varepsilon,\eta}$ in Ω, we still have

$$\boldsymbol{e}^{(2)} - \boldsymbol{e}^{(1)} = \boldsymbol{d} \, . \tag{3.15}$$

For each $j = 1, \dots, n$ define a vector $\boldsymbol{a}^{(j)} \in \mathcal{V}_0$ by:

$$a_i^{(j)} = \begin{cases} 1 & \text{if } i = i_+(j) \, , \\ -1 & \text{if } i = i_-(j) \, , \\ 0 & \text{otherwise} \, . \end{cases}$$

It is easy to see that each vector in $\mathcal{V}_0$ can be expressed as a linear combination, with integer coefficients, of $\boldsymbol{a}^{(1)}, \dots, \boldsymbol{a}^{(n)}$. In particular, for $\boldsymbol{b} := \boldsymbol{d}^{(1)} - \boldsymbol{e}^{(1)} = \boldsymbol{d}^{(2)} - \boldsymbol{e}^{(2)}$ (see (3.15)) there exist $\alpha_1, \dots, \alpha_n \in \mathbb{Z}$ such that

$$\boldsymbol{b} = \sum_{j=1}^{n} \alpha_j \boldsymbol{a}^{(j)} \, .$$

For each $j = 1, \dots, n$, take a smooth function H_j on $\overline{T}_{\eta/2}^{(j)}$ that satisfies

$$H_j \equiv 0 \text{ on } \gamma_j^- \quad \text{and} \quad H_j \equiv 2\pi\alpha_{i_+(j)} \text{ on } \gamma_j^+ \, .$$

Then define a function H on Ω by

$$H(x) = \begin{cases} H_j(x) & \text{if } x \in T^{(j)}_{\eta/2} \text{ for some j}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Finally, set

$$\tilde{u}_{\varepsilon,\eta} = e^{iH} u_{\varepsilon,\eta} \quad \text{and} \quad \tilde{v}_{\varepsilon,\eta} = e^{iH} v_{\varepsilon,\eta}\,.$$

We have that $\tilde{u}_{\varepsilon,\eta} \in \mathcal{E}_{\boldsymbol{d}^{(1)}}, \tilde{v}_{\varepsilon,\eta} \in \mathcal{E}_{\boldsymbol{d}^{(2)}}$ and $\tilde{u}_{\varepsilon,\eta}/\tilde{v}_{\varepsilon,\eta} = z_{\varepsilon,\eta}$. Therefore,

$$|\tilde{u}_{\varepsilon,\eta} - \tilde{v}_{\varepsilon,\eta}| = |z_{\varepsilon,\eta} - 1| \text{ in } \Omega.$$

Combining it with property (iii) of $z_{\varepsilon,\eta}$ and (3.14) we get that

$$\begin{aligned} \lim_{\eta\to 0} \int_\Omega |\nabla(\tilde{u}_{\varepsilon,\eta} - \tilde{v}_{\varepsilon,\eta})|^2 &= \lim_{\eta\to 0} \int_\Omega \big|\nabla|\tilde{u}_{\varepsilon,\eta} - \tilde{v}_{\varepsilon,\eta}|\big|^2 = \lim_{\eta\to 0} \int_\Omega \big|\nabla|z_{\varepsilon,\eta} - 1|\big|^2 \\ &= \int_\Omega \big|\nabla|w_\varepsilon - 1|\big|^2 + o_\varepsilon(1) = \Big(\frac{2}{\pi}\Big)^2 I(\boldsymbol{d}) + o_\varepsilon(1)\,. \end{aligned}$$

Letting $\varepsilon \to 0$ we infer that $\delta^2(\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)}) \le (\frac{2}{\pi})^2 I(\boldsymbol{d})$ which together with Theorem 2 gives the result (3.13). $\square$

Remark 3.1. Property (C) is satisfied, for example, when $d_1, d_2, \ldots, d_n > 0$. More generally, it is satisfied if there is no nontrivial subset of indices $J \subsetneq \{0, 1, \ldots, n\}$ for which $\sum_{i\in J} d_i = 0$. On the other hand, we shall give an example in the next section that shows that both property (C) and the conclusion of Theorem 4 do not always hold.

We close this section with two open problems:

Open problem 1: Is it true that $\delta(\mathcal{E}_{\boldsymbol{d}^{(1)}}, \mathcal{E}_{\boldsymbol{d}^{(2)}})$ is never attained for $\boldsymbol{d}^{(1)} \neq \boldsymbol{d}^{(2)}$, i.e., is the assumption made in Corollary 3.1, that equality holds in (3.12), unnecessary?
Open problem 2: Given $\boldsymbol{d} \neq \boldsymbol{0}$, is property (C) a necessary condition for the equality in (3.13) to hold for all $\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)}$ satisfying $\boldsymbol{d}^{(2)} - \boldsymbol{d}^{(1)} = \boldsymbol{d}$?

4 An example

In the beginning of this section we shall study the distance between homotopy classes of S^1-valued maps defined on a certain *graph*. We shall later use it to produce an example for which the conclusion of Theorem 4 does not hold. Consider the graph $G = F \cup E_1 \cup E_2$, where $F = \partial B_1(0)$, $E_1 = \partial B_1(2e^{is})$ and $E_2 = \partial B_1(2e^{-is})$, for some $s \in (\pi/6, \pi/2)$, so that E_1 and E_2 do not intersect each other and they touch F at the points e^{is} and e^{-is}, respectively. Here $B_1(p)$ is the unit disc centered at p. We define:

$$H^1(G, S^1) := C(G, S^1) \cap H^1(F, S^1) \cap H^1(E_1, S^1) \cap H^1(E_2, S^1)\,.$$

To each $u \in H^1(G, S^1)$ we associate the vector $\boldsymbol{d} = (d_1, d_2, d_3)$ of the degrees of u on F, E_1 and E_2, respectively. This induces a decomposition $H^1(G, S^1) = \bigcup_{\boldsymbol{d} \in \mathbb{Z}^3} \mathcal{E}_{\boldsymbol{d}}$ as in the previous sections. The distance between two homotopy classes is given by, analogously to (1.2) and (2.1),

$$\delta_G^2(\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)}) = \inf \Big\{ \int_G |(u_1 - u_2)'|^2 : u_1 \in \mathcal{E}_{\boldsymbol{d}^{(1)}}, u_2 \in \mathcal{E}_{\boldsymbol{d}^{(2)}} \Big\}. \tag{4.1}$$

We shall need the following simple lemma whose proof is postponed after the description of our example.

Lemma 4.1. *Let $u, v \in H^1(S^1, S^1)$ satisfy: $\deg u = \deg v = k \neq 0$ and $|(u - v)(1)| = \eta > 0$. Then,*

$$\int_{S^1} |(u - v)'|^2 \geq \frac{2\eta^2}{\pi}\,, \tag{4.2}$$

and this bound is optimal.

The next result gives an example of non-optimality of the bound of Theorem 2 for maps in $H^1(G, S^1)$.

Proposition 4.1. *Let $\boldsymbol{d}^{(1)} = (1, 1, 1)$ and $\boldsymbol{d}^{(2)} = (2, 1, 1)$. Then,*

$$\delta_G^2(\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)}) > \frac{8}{\pi}\,. \tag{4.3}$$

Proof. Take any $u_1 \in \mathcal{E}_{\boldsymbol{d}^{(1)}}$ and $u_2 \in \mathcal{E}_{\boldsymbol{d}^{(2)}}$ and let $w = u_2 - u_1$. Put $\eta = \max(|w(e^{is})|, |w(e^{-is})|)$ which we may assume w.l.o.g. to be achieved at $e^{is} = E_1 \cap F$. By Lemma 4.1

$$\int_{E_1} |w'|^2 \geq \frac{2\eta^2}{\pi}\,. \tag{4.4}$$

Since at some point on F we must have $|w| = 2$, we deduce from Theorem 1(i) and the argument used there, that for some $0 \leq s_1, s_2$ satisfying $s_1 + s_2 = 2s$ or $s_1 + s_2 = 2\pi - 2s$ we have

$$\int_F |w'|^2 \geq \max\Big(\frac{8}{\pi}, \frac{(2-\eta)^2}{s_1} + \frac{(2-\eta)^2}{s_2}\Big)\,. \tag{4.5}$$

From (4.4)–(4.5) we infer

$$\int_G |w'|^2 \geq g(\eta) := \frac{2\eta^2}{\pi} + \max\Big(\frac{8}{\pi}, \frac{4(2-\eta)^2}{2\pi - 2s}\Big)\,. \tag{4.6}$$

Evidently,

$$\gamma_0 := \min_{\eta \in [0,2]} g(\eta) > \frac{8}{\pi}\,, \tag{4.7}$$

and the result follows. $\square$

Proof of Lemma 4.1. Put $w = u - v$. We may assume w.l.o.g. that $w(1) = (u-v)(1) = \eta i$. There are two possibilities:

(i) $w(e^{i\theta_0}) = 0$ for some $\theta_0 \in (0, 2\pi)$. Then, by the Cauchy-Schwarz inequality

$$\int_{S^1} |w'|^2 \geq \int_{S^1} \big||w|'\big|^2 \geq \frac{\eta^2}{\theta_0} + \frac{\eta^2}{2\pi - \theta_0} \geq \frac{2\eta^2}{\pi}\,.$$

(ii) $\gamma := \min\{|w(e^{i\theta})| \,:\, \theta \in [0, 2\pi)\} > 0\,.$

In this case the winding number of w w.r.t. 0 is also k. In particular, there is some $\theta_1 \in (0, 2\pi)$ for which $w(e^{i\theta_1}) = -ti$, with $t \geq \gamma$. Writing $w = w_1 + iw_2$ we have

$$\int_{S^1} |w'|^2 \geq \int_{S^1} |w_2'|^2 \geq \frac{(\eta+t)^2}{\theta_1} + \frac{(\eta+t)^2}{2\pi - \theta_1} \geq 2\frac{(\eta+t)^2}{\pi} > \frac{2\eta^2}{\pi}\,,$$

and (4.2) follows in this case too.

Finally, we show that (4.2) is optimal. We use a construction similar to the one used in the proof of Theorem 1(i). Assume first that $\eta < 2$. Fix any small $\varepsilon > 0$. First, set

$$u^{(\varepsilon)}(\theta) = v^{(\varepsilon)}(\theta) = \exp\Big(2\pi k \frac{\theta + \pi}{\varepsilon} i\Big),\ \theta \in [-\pi, \varepsilon - \pi]\,.$$

On $[\varepsilon - \pi, 0]$ let $\rho(\theta) = \rho^{(\varepsilon)}(\theta)$ be the linear function satisfying $\rho(\varepsilon - \pi) = 0$ and $\rho(0) = \eta$. Let $\alpha(\theta) = \sin^{-1}\rho(\theta)/2$ and then set

$$u^{(\varepsilon)}(\theta) = e^{i\alpha(\theta)} \text{ and } v^{(\varepsilon)}(\theta) = e^{-i\alpha(\theta)},\ \theta \in [\varepsilon - \pi, 0]\,.$$

Finally, on $[0, \pi]$ use a similar construction to the one used on $[\varepsilon - \pi, 0]$, to get $u^{(\varepsilon)}$ and $v^{(\varepsilon)}$ conjugate to each other, and $|u^{(\varepsilon)} - v^{(\varepsilon)}|$ equals to a linear function changing from η back to 0. It is elementary to check that the above construction yields

$$\int_{S^1} \big|(u^{(\varepsilon)} - v^{(\varepsilon)})'\big|^2 = \eta^2\Big(\frac{1}{\pi - \varepsilon} + \frac{1}{\pi}\Big)\,.$$

The case $\eta = 2$ follows from the case $\eta < 2$ by a standard approximation argument. □

Next we use the example of Proposition 4.1 to construct a perforated domain in $\mathbb{R}^2$ for which a case of strict inequality in (3.12) occurs. This domain is obtained by a certain thickening of the graph G. For any small ε consider an annulus $F^{(\varepsilon)} = B_{1+\varepsilon}(0)\backslash B_{1-\varepsilon}(0)$ and two other annuli $E_1^{(\varepsilon)} = B_{1+\varepsilon}(z_1)\backslash B_{1-\varepsilon}(z_1)$, $E_2^{(\varepsilon)} = B_{1+\varepsilon}(z_2) \setminus B_{1-\varepsilon}(z_2)$ with $z_1 = (2+3\varepsilon)e^{is}$ and $z_2 = (2+3\varepsilon)e^{-is}$. Finally, connect $F^{(\varepsilon)}$ to $E_1^{(\varepsilon)}$ and $E_2^{(\varepsilon)}$ by adding two small "tubes", $Q_1^{(\varepsilon)}$ and $Q_2^{(\varepsilon)}$ of length and width $\sim \varepsilon$, around the lines of phase $\theta = \pm s$. The resulting domain will be denoted by $G^{(\varepsilon)}$ (see Figure 2). We would like to extend the arguments of Proposition 4.1 to the new domain. Note that strictly speaking the degree vectors

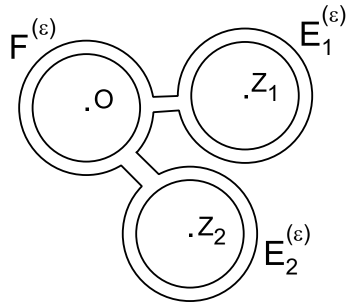

Figure 2: The domain $G^{(\varepsilon)}$

now are four-dimensional (see (3.1)), with d_0 corresponding to the degree along the outer boundary. However, with a slight abuse of notation we shall suppress d_0 and stick to the notation of the one-dimensional case.

We have the following two-dimensional analogue to Proposition 4.1.

Proposition 4.2. *Let $\boldsymbol{d}^{(1)}$ and $\boldsymbol{d}^{(2)}$ be as in Proposition 4.1. Then, for small enough ε we have*

$$\delta^2(\boldsymbol{d}^{(1)},\boldsymbol{d}^{(2)}) > H(\boldsymbol{d}^{(2)}-\boldsymbol{d}^{(1)})\,. \tag{4.8}$$

Proof. Let $\boldsymbol{d}=\boldsymbol{d}^{(2)}-\boldsymbol{d}^{(1)}=(1,0,0)$. The inequality (4.8) is a direct consequence of (4.3) and the following two estimates:

$$H(\boldsymbol{d})=\frac{16\varepsilon}{\pi}+O(\varepsilon^2)\,, \tag{4.9}$$

and

$$\delta^2(\boldsymbol{d}^{(1)},\boldsymbol{d}^{(2)}) \geq 2\gamma_0\varepsilon+O(\varepsilon^{3/2})\,, \tag{4.10}$$

where γ_0 is defined in (4.7).

Proof of (4.9): For any $w\in\mathcal{E}_{\boldsymbol{d}}$ we have clearly

$$\int_{G^{(\varepsilon)}}|\nabla w|^2 \geq \int_{F^{(\varepsilon)}}|\nabla w|^2 \geq \int_{1-\varepsilon}^{1+\varepsilon}\int_0^{2\pi}\Big|\frac{\partial w}{\partial\theta}\Big|^2\,\frac{d\theta}{r}\,dr \geq 2\pi\log\Big(\frac{1+\varepsilon}{1-\varepsilon}\Big)\,. \tag{4.11}$$

On the other hand, consider the map $w^{(\varepsilon)}$ defined by: $w^{(\varepsilon)}(z)=z/|z|$ on F_ε, $w^{(\varepsilon)}(z)\equiv e^{is}$ on $E_1^{(\varepsilon)}$, and $w^{(\varepsilon)}(z)\equiv e^{-is}$ on $E_2^{(\varepsilon)}$. By a direct construction we can complete the definition of $w^{(\varepsilon)}$ to $Q_1^{(\varepsilon)}\cup Q_2^{(\varepsilon)}$ with an additional cost of energy $O(\varepsilon^2)$. Thus we get

$$\int_{G^{(\varepsilon)}}|\nabla w^{(\varepsilon)}|^2 = 2\pi\log\Big(\frac{1+\varepsilon}{1-\varepsilon}\Big)+O(\varepsilon^2)\,. \tag{4.12}$$

From (4.11)–(4.12) it follows that $I(\boldsymbol{d})=4\pi\varepsilon+O(\varepsilon^2)$ which together with (3.5) implies (4.9).

Proof of (4.10): An explicit simple construction shows that

$$\delta^2(\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)}) \le C\varepsilon\,, \tag{4.13}$$

(here and in the sequel we denote by C different constants which do not depend on ε). Thanks to (4.13), it suffices to consider only pairs of maps $u_j \in \mathcal{E}_{\boldsymbol{d}^{(j)}}$, $j = 1, 2$, satisfying

$$\int_{G^{(\varepsilon)}} |\nabla(u_2 - u_1)|^2 \le C\varepsilon\,.$$

Consider such a pair and set $w = u_2 - u_1$. For each $\alpha \in (s-\varepsilon/2, s+\varepsilon/2)$ consider the two segments:

$$l_1(\alpha) = \{z_1 + re^{i(\pi+\alpha)} : r > 0\} \cap (E_1^{(\varepsilon)} \cup F^{(\varepsilon)})\,,$$
$$l_2(\alpha) = \{z_2 + re^{i(\pi-\alpha)} : r > 0\} \cap (E_2^{(\varepsilon)} \cup F^{(\varepsilon)})\,.$$

By Fubini's theorem there exists $\alpha_0 = \alpha_0(\varepsilon) \in (s - \varepsilon/2, s + \varepsilon/2)$ such that

$$\int_{l_1(\alpha_0)\cup l_2(\alpha_0)} |\nabla w|^2 \le C\,. \tag{4.14}$$

Note that (4.14) implies

$$|w(x_2) - w(x_1)| \le C|x_2 - x_1|^{1/2} \le C\varepsilon^{1/2},\ \forall x_1, x_2 \in l_j(\alpha_0),\ j = 1, 2\,. \tag{4.15}$$

For each $r \in (1 - \varepsilon, 1 + \varepsilon)$ define the following four points:

$$p_j(r) = l_j(\alpha_0) \cap \partial B_r(z_j),\ q_j(r) = l_j(\alpha_0) \cap \partial B_r(0),\ \ j = 1, 2\,.$$

Put

$$\eta(r) = \max(|w(p_1(r))|, |w(p_2(r))|) \text{ and } \zeta(r) = \max(|w(q_1(r))|, |w(q_2(r))|)\,.$$

By the argument of Proposition 4.1 we have for each $r \in (1 - \varepsilon, 1 + \varepsilon)$:

$$\int_{\partial B_r(0)\cup\partial B_r(z_1)\cup\partial B_r(z_2)} |w'|^2 \ge \frac{2\eta^2(r)}{\pi r} + \frac{1}{r}\max\Big(\frac{8}{\pi}, \frac{4(2-\zeta(r))^2}{2\pi - 2s}\Big)\,. \tag{4.16}$$

By (4.15) there exists a constant η_0 such that:

$$|\eta(r) - \eta_0|, |\zeta(r) - \eta_0| \le C\varepsilon^{1/2}, \forall r \in (1 - \varepsilon, 1 + \varepsilon)\,.$$

Therefore, from (4.16) and (4.6)–(4.7) we infer that

$$\int_{\partial B_r(0)\cup\partial B_r(z_1)\cup\partial B_r(z_2)} |w'|^2 \ge \frac{\gamma_0}{r} - C\varepsilon^{1/2}\,,\ \forall r \in (1 - \varepsilon, 1 + \varepsilon)\,. \tag{4.17}$$

Integrating (4.17) over $r \in (1 - \varepsilon, 1 + \varepsilon)$ yields (4.10). □

Remark 4.1. Let $\boldsymbol{e}^{(1)} = (1,0,0)$ and $\boldsymbol{e}^{(2)} = (2,0,0)$, so that $\boldsymbol{e}^{(2)} - \boldsymbol{e}^{(1)} = \boldsymbol{d}^{(2)} - \boldsymbol{d}^{(1)} = (1,0,0)$. The argument of Theorem 1 shows that $\delta^2_G(\boldsymbol{e}^{(1)}, \boldsymbol{e}^{(2)}) = 8/\pi$. Therefore, Proposition 4.1 shows not only that the bound of Theorem 2 is not optimal in $H^1(G, S^1)$, but also that $\delta_G(\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)})$ is not always a function of $\boldsymbol{d}^{(2)} - \boldsymbol{d}^{(1)}$ only. The same conclusion holds also for $H^1(G^{(\varepsilon)}, S^1)$. Indeed, we claim that $\delta^2(\boldsymbol{e}^{(1)}, \boldsymbol{e}^{(2)}) = H(\boldsymbol{e}^{(2)} - \boldsymbol{e}^{(1)})$. In this special case, we do not need property (C) in its full strength, and we can use (part of) the argument of Theorem 4 once we prove that there is a regular value $e^{i\alpha}$ ($\alpha \in [0, 2\pi)$) of W, for which $W^{-1}(e^{i\alpha})$ contains a curve joining $\partial B_{1-\varepsilon}(0)$ to $\partial B_{1+\varepsilon}(0) \cap \partial G^{(\varepsilon)}$ (W denotes as usual a minimizer in (3.2)). A quick way to see that, is via the co-area formula. Denoting by $\mathcal{H}^1$ the one dimensional Hausdorff measure, we have

$$\int_0^{2\pi} \mathcal{H}^1(\{W^{-1}(e^{i\alpha}) \cap E_1^{(\varepsilon)}\})\, d\alpha = \int_{E_1^{(\varepsilon)}} |\nabla W| \le \Big(\int_{E_1^{(\varepsilon)}} |\nabla W|^2 \Big)^{1/2} \cdot |E_1^{(\varepsilon)}|^{1/2} \le C\varepsilon\,.$$

Therefore, for a.e. $\alpha \in (0, 2\pi)$, $e^{i\alpha}$ is a regular value of W such that $W^{-1}(e^{i\alpha}) \cap E_1^{(\varepsilon)}$ is a curve of length $O(\varepsilon)$. The existence of an α with the desired properties follows.

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References

[1] F. Bethuel, H. Brezis and F. Hélein, Ginzburg-Landau Vortices, Birkhäuser, 1994.

[2] F. Bethuel and X. Zheng, *Density of smooth functions between two manifolds in Sobolev spaces*, J. Func. Anal. **80** (1988), 60–75.

[3] H. Brezis and L. Nirenberg, *Degree theory and BMO. I. Compact manifolds without boundaries*, Selecta Math. (N.S.) **1** (1995), 197–263.

[4] P. Duren, Harmonic mappings in the plane, Cambridge Tracts in Mathematics **156**, Cambridge University Press, Cambridge, 2004.

[5] Y. Kim, J. Rubinstein and P. Sternberg, *Topologically driven patterns in nematic liquid crystals*, J. Math. Phys. **46**, Art. 095110, 2005.

[6] J. Rubinstein and P. Sternberg, *Homotopy classification of minimizers for Ginzburg Landau functionals in multiply connected domains*, Comm. Math. Phys. **179** (1996), 257–264.

[7] B. White, *Homotopy classes in Sobolev spaces and the existence of energy minimizing maps*, Acta Math. **160** (1988), 1–17.