EXTREMAL FUNCTIONS FOR HARDY'S INEQUALITY WITH WEIGHT

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1. Introduction

Hardy's inequality for a bounded domain $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary asserts that

$$\int_\Omega |\nabla u|^2 \ge \mu \int_\Omega (u/\delta)^2, \quad \forall u \in H_0^1(\Omega), \tag{1.1}$$

where μ is a positive constant and $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ (see e.g. [7]). The best constant in (1.1), i.e.

$$\mu(\Omega) = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega (u/\delta)^2}, \tag{1.2}$$

depends on Ω. For convex domains $\mu(\Omega) = 1/4$ ([5, 6]), but there are smooth bounded domains with $\mu(\Omega) < 1/4$ ([3, 4, 5]). Brezis and Marcus [2, Theorem I] studied the quantity

$$J_\lambda^\Omega = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 - \lambda \int_\Omega u^2}{\int_\Omega (u/\delta)^2}, \quad \forall \lambda \in \mathbb{R}, \tag{1.3}$$

and showed that, for a C^2 bounded domain Ω, there exists a finite constant $\lambda^* = \lambda^*(\Omega)$ such that

$$\begin{cases} J_\lambda = 1/4, & \forall \lambda \le \lambda^*, \\ J_\lambda < 1/4, & \forall \lambda > \lambda^*. \end{cases} \tag{1.4}$$

Moreover , the infimum in (1.3) is achieved if and only if $\lambda > \lambda^*$. In [2] they also studied the following generalization of (1.3):

$$J_\lambda = J_\lambda(p, q, \eta) = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega p|\nabla u|^2 - \lambda \int_\Omega \eta (u/\delta)^2}{\int_\Omega q(u/\delta)^2}, \quad \forall \lambda \in \mathbb{R}, \tag{1.5}$$

where p, q, η satisfy

$$\begin{aligned} &p, q \in C^1(\overline{\Omega}), \text{ and } p, q > 0 \text{ in } \overline{\Omega}, \\ &\eta \in C^0(\overline{\Omega}), \text{ and } \eta > 0 \text{ in } \Omega, \ \eta = 0 \text{ on } \partial\Omega. \end{aligned} \tag{1.6}$$

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Under the normalization

$$\max_{\partial\Omega} \frac{q}{p} = 1, \tag{1.7}$$

it was proved that (1.4) remains valid in this more general setting, and that the infimum in (1.5) is achieved if $\lambda > \lambda^*$ and it is not achieved if $\lambda < \lambda^*$. The question whether the infimum is achieved in the critical case $\lambda = \lambda^*$ remained open.

Here we give an answer to this question (under slightly stronger assumptions on p, q, η than in (1.6)). Assume that p, q, η satisfy

$$\begin{aligned} &p, q \in C^2(\overline{\Omega}) \ \text{ and } p, q > 0 \text{ in } \overline{\Omega}, \\ &\eta \in \mathrm{Lip}(\overline{\Omega}) \ \text{ and } \eta > 0 \text{ in } \Omega,\ \eta = 0 \text{ on } \partial\Omega. \end{aligned} \tag{1.8}$$

We denote $\Sigma = \partial\Omega$ and define the following quantity (possibly infinite)

$$I(p,q) = \int_\Sigma \frac{d\sigma}{\sqrt{1 - (q(\sigma)/p(\sigma))}}\,. \tag{1.9}$$

Our main result is the following,

Theorem 1. *Assume the weight functions satisfy* (1.8) *and* (1.7). *Then, for $\lambda = \lambda^*$ the infimum in* (1.5) *is achieved if and only if $I(p,q) < \infty$.*

Remark 1.1. Note that in view of (1.7), the assumption $p, q \in C^2(\overline{\Omega})$ implies that for $N = 2$ we always have $I(p,q) = \infty$ and therefore the infimum is never achieved for $\lambda = \lambda^*$. Obviously the same assertion holds for $N = 1$.

The nonexistence part relies on the construction of a subsolution, following the same strategy as in [2]. The proof of existence is new; it uses the construction of a supersolution in H^1, in a neighborhood of the boundary, which serves to control the behavior of a specific minimizing sequence.

As mentioned above, if $\lambda > \lambda^*$ the infimum in (1.5) is achieved by some function $u_\lambda \in H_0^1(\Omega)$. It can be easily seen (see [2]) that u_λ is unique under the normalization:

$$u_\lambda > 0 \text{ in } \Omega \quad \text{and} \quad \int_\Omega u_\lambda^2 = 1\,. \tag{1.10}$$

In view of Theorem 1, this observation remains valid in the critical case $\lambda = \lambda^*$, provided that $I(p,q) < \infty$. Our next result describes the behavior of u_λ as $\lambda \searrow \lambda^*$ in either of the two cases: $I(p,q) < \infty$ and $I(p,q) = \infty$. In fact, the first case is used in the proof of Theorem 1.

Theorem 2.

(i) *If* $I(p,q)<\infty$ *then* $u_\lambda \to u_{\lambda^*}$ *strongly in* $H^1(\Omega)$ *as* $\lambda \searrow \lambda^*$.

(ii) *If* $I(p,q)=\infty$ *then, as* $\lambda \searrow \lambda^*$, u_λ *converges strongly in* $W^{1,p_0}(\Omega)$, $\forall p_0 \in [1,2)$, *to a function* u_* *which is the unique positive solution (up to a multiplicative constant) of*

$$-\operatorname{div}(p\nabla u)=\frac{q}{4\delta^2}u+\frac{\lambda^*\eta}{\delta^2}u \quad \text{in } \Omega. \tag{1.11}$$

Our last result shows how the existence or nonexistence of a minimizer for $\lambda=\lambda^*$ are reflected in the differentiability properties of J_λ at λ^*.

Corollary 1. *The function* J_λ *is differentiable at* λ^* *if and only if* $I(p,q)=\infty$. *More precisely,*

$$(J_{\lambda^*})'_+=\begin{cases}0 & \text{if } I(p,q)=\infty,\\ -(\int_\Omega \frac{\eta u_{\lambda^*}^2}{\delta^2})/(\int_\Omega \frac{q u_{\lambda^*}^2}{\delta^2}) & \text{if } I(p,q)<\infty\,.\end{cases} \tag{1.12}$$

2. Proof of Theorem 1

We first introduce some notations. For $\beta>0$ we denote

$$\Omega_\beta=\{x\in\Omega;\ \delta(x)<\beta\},\quad \Sigma_\beta=\{x\in\Omega;\ \delta(x)=\beta\}.$$

Since Ω is of class C^2, there exists $\beta_0\in(0,1)$ such that for every $x\in\Omega_{\beta_0}$ there exists a unique nearest point projection $\sigma(x)\in\Sigma$. We first assume that $p\equiv 1$ and we will show later how to treat the general case.

For the nonexistence part we will argue by contradiction and rely on the following Proposition which is a variant of Theorem III in [2]. Consider the operator:

$$\mathbf{L}u=-\Delta u-\frac{q}{4\delta^2}u+\frac{\eta}{\delta^2}u\,. \tag{2.1}$$

Proposition 2.1. *Suppose that* q *satisfies* (1.7) *and* (1.8) *(with* $p\equiv 1$*) and that*

$$\int_\Sigma \frac{d\sigma}{\sqrt{1-q(\sigma)}}=\infty\,. \tag{2.2}$$

In addition, suppose that $\eta\in C(\overline{\Omega})$ *and that* $|\eta|\le C\delta$, *where* C *is a constant. If* $0\le u\in H_0^1(\Omega)$ *and satisfies*

$$\mathbf{L}u\ge 0 \quad \text{in } \Omega, \tag{2.3}$$

then $u\equiv 0$.

The proof of Proposition 2.1 is by contradiction. Assuming $u \not\equiv 0$, then $u > 0$ in Ω by the maximum principle. In the next two lemmas we construct a positive subsolution v (i.e. $\mathbf{L}v \le 0$) which is used as a lower bound for u. In these lemmas we assume the assumptions of Proposition 2.1, except for (2.2) which is not needed. We define the operators

$$\mathbf{L}_s u = -\Delta u - \frac{sq}{4\delta^2} u + \frac{\eta}{\delta^2} u, \quad \forall s \in (0,1]. \tag{2.4}$$

Note that in particular $\mathbf{L}_1 = \mathbf{L}$.

Lemma 2.1. *For any $s \in (0,1]$ and $x \in \Omega_{\beta_0}$ set $v_s(x) = \delta(x)^{\alpha_s(x)}$ with*

$$\alpha_s(x) = \left(1 + \sqrt{1 - sq(\sigma(x)) + \delta(x)}\,\right)/2\,, \tag{2.5}$$

which is well defined since $\max_{\Sigma} q = 1$. Then, there exists a constant $C > 0$ such that

$$|\mathbf{L}_s v_s| \le C|\log \delta|\delta^{-1} \quad \text{in } \Omega_{\beta_0},\ \forall s \in (0,1]. \tag{2.6}$$

Proof. For simplicity we drop the indices and write $v = v_s$ and $\alpha = \alpha_s$. All the following computations are performed in Ω_{β_0}. Note first that

$$\nabla \log v = (\log \delta)\nabla\alpha + \alpha\frac{\nabla\delta}{\delta}, \tag{2.7}$$

hence

$$|\nabla \log v|^2 = (\log\delta)^2|\nabla\alpha|^2 + \frac{\alpha^2}{\delta^2} + 2\alpha\frac{\log\delta}{\delta}\nabla\alpha\nabla\delta, \tag{2.8}$$

where we used the identity $|\nabla\delta| = 1$. Next,

$$\Delta \log v = \frac{\Delta v}{v} - |\nabla \log v|^2, \tag{2.9}$$

so that

$$\Delta v = v(\Delta \log v + |\nabla \log v|^2). \tag{2.10}$$

Similarly,

$$\Delta \log \delta = \frac{\Delta\delta}{\delta} - |\nabla\log\delta|^2 = \frac{\Delta\delta}{\delta} - \frac{1}{\delta^2}. \tag{2.11}$$

By (2.11) we get

$$\begin{aligned} \Delta \log v &= \Delta[\alpha(\log\delta)] = \alpha(\Delta\log\delta) + \frac{2}{\delta}\nabla\alpha\nabla\delta + (\log\delta)\Delta\alpha \\ &= \frac{\alpha\Delta\delta}{\delta} - \frac{\alpha}{\delta^2} + \frac{2}{\delta}\nabla\alpha\nabla\delta + (\log\delta)\Delta\alpha. \end{aligned} \tag{2.12}$$

Finally, plugging (2.8) and (2.12) into (2.10) yields

$$\begin{aligned}\Delta v = {} & \alpha(\alpha-1)\delta^{\alpha-2} + \left[\alpha\Delta\delta + 2(1+\alpha\log\delta)\nabla\alpha\nabla\delta\right]\delta^{\alpha-1} \\ & + \left[(\log\delta)\Delta\alpha + (\log\delta)^2|\nabla\alpha|^2\right]\delta^{\alpha}.\end{aligned} \tag{2.13}$$

Since by (2.5) $\alpha(1-\alpha) = (sq\circ\sigma - \delta)/4$, we infer from (2.13) that

$$\begin{aligned}\mathbf{L}_s v = {} & \frac{1}{4}\left(s\,q\circ\sigma - \delta - sq\right)\delta^{\alpha-2} - \left[\alpha\Delta\delta + 2(1+\alpha\log\delta)\nabla\alpha\nabla\delta\right]\delta^{\alpha-1} \\ & - \left[(\log\delta)\Delta\alpha + (\log\delta)^2|\nabla\alpha|^2\right]\delta^{\alpha} + \eta\delta^{\alpha-2}.\end{aligned} \tag{2.14}$$

Note that

$$\nabla\alpha = \frac{1}{4}(1 - s\,q\circ\sigma + \delta)^{-1/2}(\nabla\delta - s\nabla(q\circ\sigma)),$$

which yields (since $q \le 1$ on Σ)

$$|\nabla\alpha| \le \frac{C}{\delta^{1/2}}. \tag{2.15}$$

In addition

$$\begin{aligned}\Delta\alpha = {} & -\frac{1}{8}(1 - sq\circ\sigma + \delta)^{-3/2}\,|\nabla\delta - s\nabla(q\circ\sigma)|^2 \\ & + \frac{1}{4}(1 - sq\circ\sigma + \delta)^{-1/2}\,(\Delta\delta - s\Delta(q\circ\sigma))\end{aligned}$$

gives

$$|\Delta\alpha| \le \frac{C}{\delta^{3/2}}. \tag{2.16}$$

Combining (2.14),(2.15),(2.16) and using the fact that $|q(\sigma(x)) - q(x)| \le C\delta(x)$ we obtain

$$|\mathbf{L}_s v| \le C(\delta^{\alpha-1} + |\log\delta|\delta^{\alpha-3/2} + |\log\delta|^2\delta^{\alpha-1}). \tag{2.17}$$

Finally, since $\alpha \ge 1/2$ it follows that

$$|\mathbf{L}_s v| \le C|\log\delta|\delta^{-1},$$

where all the constants C are independent of s. □

Lemma 2.2. *Set*

$$m \equiv \min\{q(\sigma);\, \sigma\in\Sigma\} \in (0,1] \tag{2.18}$$

and let α_0 be the unique root of $\alpha_0(1-\alpha_0) = m/8$ in $(1/2,1)$. For any $s \in (1/2,1)$ let $U_s = v_s + \delta^{\alpha_0}$. Then, there exists $\beta \in (0,\beta_0)$ such that

$$\mathbf{L}\,U_s < 0 \ \textit{in}\ \Omega_\beta,\ \forall s \in (1/2,1). \tag{2.19}$$

Proof. For $\beta < \beta_0$ small enough we have

$$\text{(2.20)}\qquad \begin{aligned} \mathbf{L}\,\delta^{\alpha_0} &= \alpha_0(1-\alpha_0)\delta^{\alpha_0-2} - \alpha_0\delta^{\alpha_0-1}\Delta\delta - \frac{q}{4}\delta^{\alpha_0-2} + \eta\delta^{\alpha_0-2} \\ &= (\frac{m}{8} - \frac{q}{4})\delta^{\alpha_0-2} + O(\delta^{\alpha_0-1}) \le -\frac{m}{16}\delta^{\alpha_0-2} \quad \text{ in } \Omega_\beta\,. \end{aligned}$$

So by (2.6) we infer that, if β is chosen small enough, then

$$\mathbf{L}\,U_s = \mathbf{L}\,v_s + \mathbf{L}\,\delta^{\alpha_0} \le \mathbf{L}_s v_s + \mathbf{L}\,\delta^{\alpha_0} \le C|\log\delta|\delta^{-1} - \frac{m}{16}\delta^{\alpha_0-2} < 0 \quad \text{on } \Omega_\beta,\ \forall s \in (1/2, 1)\,.$$

□

Proof of Proposition 2.1. Without loss of generality we may assume that $\eta \ge 0$, because (2.3) remains valid if η is replaced by $|\eta|$. We argue by contradiction and assume that $u \not\equiv 0$. Then by the maximum principle $u > 0$ in Ω. We fix $\beta > 0$ as in Lemma 2.2. Note that for $s \in (1/2, 1)$ the function U_s defined in Lemma 2.2 belongs to $H^1(\Omega_\beta)$. Clearly there exists $\varepsilon > 0$ such that $\varepsilon U_s \le u$ on Σ_β, $\forall s \in (1/2, 1)$. Since $w_s \doteq \varepsilon U_s - u \le 0$ on Σ_β we have $w_s^+ \in H_0^1(\Omega_\beta)$. By (2.3) and (2.19) we have

$$\text{(2.21)}\qquad \mathbf{L} w_s \le 0 \quad \text{ in } \Omega_\beta.$$

Testing (2.21) against w_s^+ yields

$$\text{(2.22)}\qquad \int_{\Omega_\beta} |\nabla w_s^+|^2 - \frac{q}{4\delta^2}(w_s^+)^2 + \frac{\eta}{\delta^2}(w_s^+)^2 \le 0\,.$$

But by a result of Brezis-Marcus [2, (4.11)] we have also

$$\text{(2.23)}\qquad \int_{\Omega_\beta} |\nabla w_s^+|^2 \ge \int_{\Omega_\beta} \frac{q}{4\delta^2}(w_s^+)^2\,.$$

Combining (2.22) and (2.23) gives $w_s^+ \equiv 0$ in Ω_β, $\forall s \in (1/2, 1)$. Passing to the limit as $s \to 1$ we find

$$\text{(2.24)}\qquad u \ge \varepsilon v_1 \text{ on } \Omega_\beta,$$

with

$$\text{(2.25)}\qquad v_1 = \delta^{(1+\sqrt{1-q\circ\sigma+\delta})/2}\,.$$

On the other hand we claim that

$$\text{(2.26)}\qquad \frac{v_1}{\delta} \notin L^2(\Omega_\beta)\,.$$

By (2.24) this implies that $u/\delta \notin L^2(\Omega_\beta)$ which, in view of the assumption that $u \in H_0^1(\Omega)$ contradicts Hardy's inequality (1.1).

In order to establish (2.26) note first that for some $c > 0$ we have (see (1.4) in [2]):

$$\int_{\Omega_\beta} \frac{v_1^2}{\delta^2} \geq c \int_\Sigma \int_0^\beta t^{\sqrt{1-q(\sigma)+t}-1}\, dt d\sigma \,.$$

Since

$$\sqrt{1-q(\sigma)+t} - \sqrt{1-q(\sigma)} = \frac{t}{\sqrt{1-q(\sigma)+t} + \sqrt{1-q(\sigma)}} \leq t^{1/2}\,,$$

it follows that

$$\begin{aligned} t^{\sqrt{1-q(\sigma)+t}-1} &= t^{\sqrt{1-q(\sigma)+t}-\sqrt{1-q(\sigma)}}\, t^{\sqrt{1-q(\sigma)}-1} \\ &\geq t^{\sqrt{t}}\, t^{\sqrt{1-q(\sigma)}-1} \geq c_0 t^{\sqrt{1-q(\sigma)}-1} \quad (\text{with } c_0 = (1/e)^{2/e}). \end{aligned}$$

Hence:

$$\int_{\Omega_\beta} \frac{v_1^2}{\delta^2} \geq cc_0 \int_\Sigma \int_0^\beta t^{\sqrt{1-q(\sigma)}-1}\, dt d\sigma = cc_0 \int_\Sigma \frac{\beta^{\sqrt{1-q(\sigma)}}}{\sqrt{1-q(\sigma)}}\, d\sigma \geq cc_0\beta \int_\Sigma \frac{d\sigma}{\sqrt{1-q(\sigma)}}\,.$$

Therefore (2.26) follows from (2.2).

□

Proof of Theorem 1, nonexistence part. Suppose $I(p,q) = \infty$ and assume by contradiction that a minimizer u for (1.5) does exist. Then we may assume $u > 0$ in Ω and u solves

$$-\operatorname{div}(p\nabla u) - \frac{q}{4\delta^2}u - \frac{\lambda^*\eta}{\delta^2}u = 0 \quad \text{in } \Omega.$$

The function $\widetilde{u} = \sqrt{p}\, u$ satisfies the equation

$$-\Delta\widetilde{u} - \frac{q}{4p\delta^2}\widetilde{u} - \frac{\lambda^*\eta}{p\delta^2}\widetilde{u} = \big(-\frac{\Delta p}{2p} + \frac{|\nabla p|^2}{4p^2}\big)\widetilde{u}\,.$$

Therefore, by Proposition 2.1, $u \equiv 0$. Contradiction. □

For the existence part of Theorem 1 we need the following lemma.

Lemma 2.3. *Assume that q, η satisfy the assumptions of Proposition 2.1, except for* (2.2). *Set $\bar{v} = v_1 - \delta^{\alpha_0}$ with v_1 given in* (2.25) *and α_0 as defined in Lemma 2.2. Then there exists $\beta \in (0, \beta_0)$ such that $\bar{v} > 0$ in $\Omega_\beta \cup \Sigma_\beta$ and*

$$-\Delta\bar{v} - \frac{q}{4\delta^2}\bar{v} - \frac{\lambda\eta}{\delta^2}\bar{v} \geq 0 \qquad \text{in } \Omega_\beta\,,\ \forall\lambda \leq \lambda^* + 1\,. \tag{2.27}$$

If, in addition,

$$\int_\Sigma \frac{d\sigma}{\sqrt{1-q(\sigma)}} < \infty\,, \tag{2.28}$$

then $\bar v \in H^1(\Omega_\beta)$.

Proof. By (2.20) and (2.6) we obtain

$$-\Delta\bar v - \frac{q}{4\delta^2}\bar v - \frac{\lambda\eta}{\delta^2}\bar v \ge \frac{m}{16}\delta^{\alpha_0-2} + O(|\log\delta|\delta^{-1}) \ge 0, \quad \forall\lambda\le\lambda^*+1,$$

for δ sufficiently small. This proves (2.27).

Next we can choose $\beta<\beta_0$ such that

$$\alpha_1(x) = (1+\sqrt{1-q(\sigma(x))+\delta(x)})/2 < \alpha_0 \quad \text{in } \Omega_\beta\cup\Sigma_\beta$$

(implying $\bar v>0$ in $\Omega_\beta\cup\Sigma_\beta$).

Finally we show that under the assumption (2.28) we have $\bar v\in H^1(\Omega_\beta)$. Clearly $\delta^{\alpha_0}\in H^1$ and thus it suffices to prove that $v_1\in H^1$. Using (2.7) we find

$$\nabla v_1 = v_1\nabla\log v_1 = \delta^{\alpha_1}\big[(\log\delta)\nabla\alpha_1 + \alpha_1\frac{\nabla\delta}{\delta}\big]\,.$$

By (2.15) we get

$$|\nabla v_1|^2 \le C\big[\delta^{2\alpha_1-1}(\log\delta)^2 + \delta^{2\alpha_1-2}\big] \le C\delta^{2\alpha_1-2}\,. \tag{2.29}$$

From [2, (1.4)] we have for some $c>0$:

$$\int_{\Omega_\beta}\delta^{2\alpha_1-2} \le \frac{1}{c}\int_\Sigma\int_0^\beta t^{\sqrt{1-q(\sigma)}-1}\,dt\,d\sigma = \frac{1}{c}\int_\Sigma\frac{\beta^{\sqrt{1-q(\sigma)}}}{\sqrt{1-q(\sigma)}}\,d\sigma<\infty\,, \quad \text{(using (2.28))}. \tag{2.30}$$

Combining (2.29)–(2.30) yields that $v_1\in H^1(\Omega_\beta)$. □

Proof of Theorem 1 when $p\equiv 1$, existence part. Recall that we assume that (2.28) is satisfied. We fix a sequence $\{\lambda_n\}$ such that $\lambda_n<\lambda^*+1$ for all n, and $\lambda_n\searrow\lambda^*$. By [2, Theorem I] we know that for every n, the infimum $\mu_n\equiv J_{\lambda_n}<1/4$ in (1.3) is achieved by a function $v_n\in H_0^1(\Omega)$ which satisfies

$$\begin{cases} -\Delta v_n = \dfrac{\mu_n q}{\delta^2}v_n + \dfrac{\lambda_n\eta}{\delta^2}v_n & \text{in } \Omega \\ \quad v_n>0 \quad \text{in } \Omega. \end{cases} \tag{2.31}$$

We choose the normalization

$$\int_\Omega|\nabla v_n|^2 = 1\,. \tag{2.32}$$

Passing to a subsequence, we may assume that $v_n\rightharpoonup u$ weakly in $H^1(\Omega)$, $v_n\to u$ a.e. in Ω, and $v_n\to u$ strongly in $L^2(\Omega)$ for some function $u\in H_0^1(\Omega)$. We are going to prove that $v_n\to u$ strongly in $H^1(\Omega)$. This implies that $u\not\equiv 0$ and thus u is a minimizer for J_{λ^*}.

Note that for each $\beta > 0$ the function v_n satisfies

$$-\Delta v_n = c_n(x) v_n \quad \text{in } \Omega \setminus \Omega_\beta, \text{ with } |c_n(x)| \le \frac{C}{\beta^2}.$$

Hence by standard elliptic estimates we also have

$$\{v_n\} \text{ is bounded in } L^\infty_{\rm loc}(\Omega). \tag{2.33}$$

Next we fix $\beta_1 > 0$ satisfying the conclusion of Lemma 2.3. By (2.33) we have, in particular, for some $\gamma > 0$

$$v_n \le \gamma \bar{v} \quad \text{on } \Sigma_{\beta_1},\ \forall n, \tag{2.34}$$

with $\bar{v}$ as in Lemma 2.3. We next claim that

$$v_n \le \gamma \bar{v} \quad \text{on } \Omega_{\beta_1},\ \forall n. \tag{2.35}$$

Note first that (2.27) gives

$$-\Delta(\gamma\bar{v}) - \frac{\mu_n q}{\delta^2}(\gamma\bar{v}) - \frac{\lambda_n \eta}{\delta^2}(\gamma\bar{v}) \ge (\frac{1}{4} - \mu_n)\frac{q}{\delta^2}(\gamma\bar{v}) \quad \text{in } \Omega_{\beta_1}. \tag{2.36}$$

Subtracting (2.36) from (2.31) yields

$$-\Delta(v_n - \gamma\bar{v}) - \frac{\mu_n q}{\delta^2}(v_n - \gamma\bar{v}) - \frac{\lambda_n \eta}{\delta^2}(v_n - \gamma\bar{v}) \le -(\frac{1}{4} - \mu_n)\frac{q}{\delta^2}(\gamma\bar{v}) \quad \text{in } \Omega_{\beta_1}. \tag{2.37}$$

Set

$$w_n = \begin{cases} (v_n - \gamma\bar{v})^+ & \text{on } \Omega_{\beta_1}, \\ 0 & \text{on } \Omega \setminus \Omega_{\beta_1}. \end{cases}$$

Note that by (2.34) $w_n \in H^1_0(\Omega)$. Testing (2.37) against w_n gives

$$\int_\Omega |\nabla w_n|^2 - \frac{\mu_n q}{\delta^2} w_n^2 - \frac{\lambda_n \eta}{\delta^2} w_n^2 \le -(\frac{1}{4} - \mu_n)\int_\Omega \frac{q}{\delta^2}(\gamma\bar{v}) w_n\,. \tag{2.38}$$

Since $\mu_n = J_{\lambda_n}$, the left hand side of (2.38) is nonnegative. Therefore $w_n \equiv 0$ and (2.35) is proved.

Since $v_n \to u$ strongly in $L^2(\Omega)$, (2.34) and the dominated convergence theorem imply that

$$\lim_{n\to\infty} \int_\Omega \frac{q v_n^2}{\delta^2} = \int_\Omega \frac{q u^2}{\delta^2}\,.$$

Testing (2.31) against v_n gives

$$\int_\Omega |\nabla v_n|^2 = \int_\Omega \frac{\mu_n q}{\delta^2} v_n^2 + \frac{\lambda_n \eta}{\delta^2} v_n^2\,. \tag{2.39}$$

The right hand side of (2.39) converges to $\int_\Omega \frac{qu^2}{\delta^2} + \int_\Omega \frac{\lambda^*\eta}{\delta^2}u^2 = \int_\Omega |\nabla u|^2$, i.e.

$$\lim_{n\to\infty}\int_\Omega |\nabla v_n|^2 = \int_\Omega |\nabla u|^2,$$

and the strong convergence $v_n \to u$ in $H^1(\Omega)$ follows. Finally note that we actually proved the strong H^1-convergence $u_\lambda \to u_{\lambda^*}$ as $\lambda \searrow \lambda^*$ (and not only of a subsequence). This follows from the simplicity of the eigenvalue λ^* (as in [2, Remark 3.2]).

Remark 2.1. In the general case when $p \not\equiv 1$ we argue as follows. Let $\lambda > \lambda^*$ and let u_λ be a minimizer for $J_\lambda(p,q,\eta)$. Then u_λ satisfies

$$-\operatorname{div}(p\nabla u_\lambda) - \frac{J_\lambda q}{\delta^2}u_\lambda - \frac{\lambda\eta}{\delta^2}u_\lambda = 0 \quad \text{in } \Omega.$$

and hence $\widetilde{u}_\lambda = \sqrt{p}\, u_\lambda$ satisfies

$$-\Delta\widetilde{u}_\lambda - \frac{J_\lambda q}{p\delta^2}\widetilde{u}_\lambda - \frac{\lambda\eta}{p\delta^2}\widetilde{u}_\lambda - \Big(-\frac{\Delta p}{2p} + \frac{|\nabla p|^2}{4p^2}\Big)\widetilde{u}_\lambda = 0\,. \tag{2.40}$$

This $\widetilde{u}_\lambda$ satisfies a similar equation to the one satisfied by u_λ in the case $p \equiv 1$, except for the last term on the left hand side of (2.40). The argument used in the existence proof of Theorem 1 can be easily adapted to cover this case as well.

□

3. The behavior of u_λ and J_λ near λ^*

Proof of Theorem 2. Case (i) of Theorem 2 was actually proved in the previous section, in the course of the proof of the existence part of Theorem 1. We thus assume that $I(p,q) = \infty$. We shall also assume that $p \equiv 1$, the general case follows from this case by the argument of Remark 2.1. We shall need the following lemma which can be proved by the same argument as in Theorem 2.7 of [1] and Lemma 8 of [5].

Lemma 3.1. *Assume $\bar{u} \in H^1_{loc}(\Omega_\beta) \cap C(\Omega_\beta)$ and $\underline{u} \in H^1_0(\Omega) \cap C(\Omega_\beta)$ satisfy $\bar{u} > 0$ in Ω_β and*

$$-\Delta\bar{u} + a(x)\bar{u} \geq 0 \quad \textit{in } \Omega_\beta\,,$$

$$-\Delta\underline{u} + a(x)\underline{u} \leq 0 \quad \textit{in } \Omega_\beta\,,$$

for some $\beta > 0$ and $a(x) \in L^\infty_{loc}(\Omega_\beta)$. If $\bar{u} \geq \underline{u}$ on $\Sigma_{\beta/2}$, then $\bar{u} \geq \underline{u}$ on $\Omega_{\beta/2}$.

For a sequence $\lambda_n \searrow \lambda^*$ consider the corresponding minimizers $\{u_{\lambda_n}\}$ with the same normalization as in (1.10), i.e.

$$u_{\lambda_n} > 0 \text{ in } \Omega \quad \text{and} \quad \int_\Omega u_{\lambda_n}^2 = 1\,. \tag{3.1}$$

Since on $\Omega \setminus \Omega_\beta$ the function u_{λ_n} satisfies an equation of the form $-\Delta u_{\lambda_n} = c_n(x) u_{\lambda_n}$ with $|c_n(x)| \le C/\beta^2$, we deduce from (3.1) and standard elliptic estimates that $\{u_{\lambda_n}\}$ is bounded in $L^\infty_{\text{loc}}(\Omega)$. In particular, for some $\gamma > 0$ we have $u_{\lambda_n} \le \gamma \bar{v}$ on $\Sigma_{\beta/2}$, where $\bar{v}$ and β are as in Lemma 2.3. Applying Lemma 3.1 gives

$$u_{\lambda_n} \le \gamma \bar{v} \quad \text{in } \Omega_{\beta/2},\ \forall n\,, \tag{3.2}$$

which implies

$$u_{\lambda_n}(x) \le C\delta(x)^{1/2}\,, \quad \forall x \in \Omega,\ \forall n. \tag{3.3}$$

Next, fix $x \in \Omega$, set $r = \delta(x)/2$ and consider on $B_1 = B_1(0)$ (the unit ball centered at the origin) the function $\tilde{u}_{\lambda_n}(y) = u_{\lambda_n}(x + ry)$ which satisfies

$$-\Delta \tilde{u}_{\lambda_n} = \tilde{c}_n(y)\tilde{u}_{\lambda_n} \text{ in } B_1, \ \text{ with } |\tilde{c}_n(y)| \le C.$$

Using (3.3) and elliptic estimates we infer that

$$|\nabla \tilde{u}_{\lambda_n}(0)| \le C(\|\tilde{u}_{\lambda_n}\|_{\mathrm{L}^\infty(B_1)} + \|\Delta \tilde{u}_{\lambda_n}\|_{\mathrm{L}^\infty(B_1)}) \le Cr^{1/2}\,,$$

which yields by rescaling

$$|\nabla u_{\lambda_n}(x)| \le \frac{C}{\delta(x)^{1/2}}\,, \quad \forall x \in \Omega,\ \forall n. \tag{3.4}$$

By (3.3) and (3.4) we get that

$$\{u_{\lambda_n}\} \text{ is bounded in } W^{1,p}(\Omega),\ \forall p < 2. \tag{3.5}$$

Consequently there exists a subsequence (still denoted by $\{u_{\lambda_n}\}$) such that

$$u_{\lambda_n} \rightharpoonup u_* \text{ weakly in } W^{1,p}_0(\Omega),\ \forall p < 2. \tag{3.6}$$

Furthermore, from the Euler-Lagrange equation (2.31) for u_{λ_n} and standard elliptic estimates we conclude that $\{u_{\lambda_n}\}$ is bounded in $W^{2,r}_{\text{loc}}(\Omega)$ for all $r < \infty$. Therefore there exists a subsequence (which we still denote by $\{u_{\lambda_n}\}$) such that

$$u_{\lambda_n} \to u_* \text{ in } C^1_{\text{loc}}(\Omega). \tag{3.7}$$

In addition, by (3.5) and Hölder's inequality,

$$\sup_n \int_{\Omega_\beta} (u_{\lambda_n}^q + |\nabla u_{\lambda_n}|^q)\, dx \to 0 \quad \text{as } \beta \to 0,\ \forall q < 2. \tag{3.8}$$

Combining (3.7) and (3.8) we conclude that

$$u_{\lambda_n} \to u_* \text{ strongly in } W_0^{1,p}(\Omega),\ \forall p < 2. \tag{3.9}$$

In particular $u_{\lambda_n} \to u_*$ in $L^2(\Omega)$ and consequently $u_* \geq 0$ a.e. in Ω and $u_* \not\equiv 0$ (see (1.10)). In addition, u_* satisfies the equation obtained by passing to the limit in the Euler-Lagrange equation (2.31) for u_{λ_n}, i.e.,

$$-\Delta u_* - \frac{q}{4\delta^2} u_* - \frac{\lambda^* \eta}{\delta^2} u_* = 0 \quad \text{in } \Omega. \tag{3.10}$$

Therefore, by the maximum principle $u_* > 0$ in Ω.

So far we established the convergence of a subsequence to the limit u_*. Next we show that there exists a unique positive solution (up to a multiplicative constant) of (3.10). Clearly this implies the full convergence $u_\lambda \to u_*$ in $W^{1,p_0}(\Omega)$ as $\lambda \searrow \lambda^*$, thus completing the proof of Theorem 2.

Let w be a positive solution of (3.10). Choose $\beta > 0$ which satisfies both the conclusions of Lemma 2.2 and Lemma 2.3. Clearly there exists $\gamma_0 > 0$ such that

$$w \geq \gamma_0 U_s \quad \text{on } \Sigma_{\beta/2},\ \forall s \in (1/2, 1), \tag{3.11}$$

with the family of subsolutions $\{U_s\}$ given by Lemma 2.2. Applying Lemma 2.2 and Lemma 3.1 we conclude that

$$w \geq \gamma_0 U_s \quad \text{on } \Omega_{\beta/2},\ \forall s \in (1/2, 1)\,.$$

Sending s to 1 we infer that

$$w \geq \gamma_0 \bar{v} \quad \text{on } \Omega_{\beta/2}, \tag{3.12}$$

with $\bar{v}$ given in Lemma 2.3. On the other hand, passing to the limit in (3.2) gives

$$u_* \leq \gamma \bar{v} \quad \text{in } \Omega_{\beta/2}\,. \tag{3.13}$$

By (3.12), applied to $w = u_*$, combined with (3.13), we obtain that for some $c_0 > 0$

$$c_0 \bar{v} \leq u_* \leq c_0^{-1} \bar{v} \quad \text{in } \Omega_{\beta/2}. \tag{3.14}$$

By (3.12) and (3.14) there exists $c > 0$ such that $w \geq cu_*$ on Ω. Set

$$c_1 = \inf_{x\in\Omega} \frac{w}{u_*}\,.$$

We claim that $w = c_1 u_*$. Indeed, if this is not true, then $\widetilde{w} = w - c_1 u_*$ is a nontrivial nonnegative solution of (3.10). By the maximum principle $\widetilde{w} > 0$ in Ω, hence by (3.12) applied to $w = \widetilde{w}$, and (3.14) we get that there exists $c_2 > 0$ such that $\widetilde{w} \geq c_2 u_*$ in Ω, which contradicts the definition of c_1. □

Proof of Corollary 1. Fix any two values $\lambda, \nu > \lambda^*$. Then u_λ and u_ν satisfy

$$-\operatorname{div}(p\nabla u_\lambda) = J_\lambda \frac{qu_\lambda}{\delta^2} + \lambda \frac{\eta u_\lambda}{\delta^2}, \tag{3.15}$$

$$-\operatorname{div}(p\nabla u_\nu) = J_\nu \frac{qu_\nu}{\delta^2} + \nu \frac{\eta u_\nu}{\delta^2}. \tag{3.16}$$

Subtracting (3.15) from (3.16) yields that $v \doteq u_\nu - u_\lambda$ satisfies

$$-\operatorname{div}(p\nabla v) - J_\nu \frac{qv}{\delta^2} - \nu \frac{\eta v}{\delta^2} = (J_\nu - J_\lambda)\frac{qu_\lambda}{\delta^2} + (\nu - \lambda)\frac{\eta u_\lambda}{\delta^2}\,. \tag{3.17}$$

Testing (3.17) against u_ν, using integration by parts and (3.16), we obtain

$$\frac{J_\nu - J_\lambda}{\nu - \lambda} = -\frac{\int_\Omega \frac{\eta u_\lambda u_\nu}{\delta^2}}{\int_\Omega \frac{qu_\lambda u_\nu}{\delta^2}}\,. \tag{3.18}$$

Letting ν tend to λ in (3.18) we infer that J_λ is differentiable at λ and that

$$J'_\lambda = -\frac{\int_\Omega \frac{\eta u_\lambda^2}{\delta^2}}{\int_\Omega \frac{qu_\lambda^2}{\delta^2}}\,. \tag{3.19}$$

Assume first that $I(p,q) = \infty$. Then we must have $\lim_{\lambda\searrow\lambda^*} \int_\Omega \frac{qu_\lambda^2}{\delta^2} = \infty$. Indeed, if not, then for a subsequence $\lambda_n \searrow \lambda^*$, $\{u_{\lambda_n}\}$ is bounded in $H^1(\Omega)$, and a further subsequence converges weakly to a minimizer of J_{λ^*}, contradicting Theorem 1. On the other hand, by (1.8) and (3.3) the numerator is bounded. Thus passing to the limit in (3.19) yields $J'_{\lambda^*} = 0$ as claimed. If $I(p,q) < \infty$, then by (i) of Theorem 2 we have $u_\lambda \to u_{\lambda^*}$ in $H^1(\Omega)$ as $\lambda \searrow \lambda^*$. This implies by (1.1) that also

$$\lim_{\lambda\searrow\lambda^*} \int_\Omega \frac{qu_\lambda^2}{\delta^2} = \int_\Omega \frac{qu_{\lambda^*}^2}{\delta^2}\,,$$

so passing to the limit in (3.19) gives (1.12). □

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