MINIMIZATION OF A GINZBURG-LANDAU TYPE FUNCTIONAL WITH BOUNDARY CONDITION WHICH IS NOT S^1-VALUED

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1. Introduction

Let G be a bounded, simply connected smooth domain of $\mathbb{R}^2$ and g a smooth boundary condition. The asymptotic behavior, as ε goes to 0, of the minimizers $\{u_\varepsilon\}$ of the Ginzburg-Landau type energy

$$E_\varepsilon(u) = \frac{1}{2}\int_G |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_G (1-|u|^2)^2$$

over the set

$$H^1_g = \{u \in H^1(G,\mathbb{C});\ u = g \text{ on } \partial G\},$$

was a subject of an extensive study in recent years, see Bethuel-Brezis-Hélein [BBH1, BBH2], Struwe [St], F.H. Lin [L], del Pino-Felmer [DF], Comte-Mironescu [CM] and the references therein. In all these works the boundary condition g was assumed to be S^1-valued. Here we are interested in the case where g is not necessarily S^1-valued, but an arbitrary smooth map from ∂G to $\mathbb{C}$ with the only requirement that it is never zero, so the degree d of g with respect to the origin is well defined. We will consider here mostly the more interesting case $d \neq 0$, so we will assume in the sequel without lost of generality that $d > 0$.

Recall that in the case of an S^1-valued g it was shown in [BBH2] (see also [S], [DF]) that there exist a subsequence $\varepsilon_n \to 0$ and d points $a_1, \dots, a_d \in G$ such that

$u_{\varepsilon_n} \to u_*$ in $C^{1,\alpha}_{\text{loc}}(\overline{G} \setminus \{a_1, \dots, a_d\})$ with $u_* \in C^\infty(\overline{G} \setminus \{a_1, \dots, a_d\}, S^1)$ a singular harmonic map with degree 1 around each a_j (it is called the *canonical harmonic map* associated to the points $a_1, \dots, a_d$, the configuration of degrees $(1, \dots, 1)$ and the boundary condition g (see [BBH2])). Moreover, we have the following estimate for the energy:

$$E_\varepsilon(u_\varepsilon) = \pi d |\log \varepsilon| + O(1) \text{ as } \varepsilon \to 0,$$

(even a more precise estimate is proved in [BBH2]) and the points $a_1, \dots, a_d$ minimize a certain *renormalized energy* over G^d. In our case, we expect an additional term of the order $O(\frac{1}{\varepsilon})$ to appear which is due to boundary interaction. However it is not clear a priori in the case $\min |g| < 1$ whether there is an energy gain for u_ε in having its zeros in a distance $o_\varepsilon(1)$ from the boundary. The answer to this question turns out to be negative, but it requires a detailed analysis.

A first basic step in this analysis consists of trying to seperate the energy contribution due to the modulus $|g|$ from that due to the phase $g/|g|$. Following an observation of Lassoued and Mironescu [LM] we have (see Lemma 2.1 below)

$$E_\varepsilon(u_\varepsilon) = \frac{1}{2}\int_G |\nabla \rho_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_G (1-\rho_\varepsilon^2)^2 + \frac{1}{2}\int_G \rho_\varepsilon^2 |\nabla v_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_G \rho_\varepsilon^4 (1-|v_\varepsilon|^2)^2,$$

where ρ_ε is the solution to the *scalar* minimization problem :

$$\min E_\varepsilon(\rho) \qquad \text{for } \rho \in H^1_{|g|}(G, \mathbb{R}),$$

and $v_\varepsilon \doteq u_\varepsilon/\rho_\varepsilon$. We may thus write $E_\varepsilon(u_\varepsilon) = E_\varepsilon(\rho_\varepsilon) + \widetilde{E}_\varepsilon(v_\varepsilon)$ where

$$\widetilde{E}_\varepsilon(v) = \frac{1}{2}\int_G \rho_\varepsilon^2 |\nabla v|^2 + \frac{1}{4\varepsilon^2}\int_G \rho_\varepsilon^4 (1-|v|^2)^2,$$

and the study of u_ε can then be devided into two parts: first study the minimizer for the scalar problem, namely ρ_ε, then study v_ε, the minimizer of $\widetilde{E}_\varepsilon$ over $H^1_{g/|g|}$. The study of the scalar problem is relatively easy, and is carried out in section 2. In particular we have (see Proposition 2.2) that

$$E_\varepsilon(\rho_\varepsilon) = \frac{1}{\varepsilon\sqrt{2}}\int_{\partial G} (\tfrac{2}{3} - |g| + \tfrac{|g|^3}{3}) + O(1), \qquad \text{as } \varepsilon \text{ goes to } 0.$$

We are left then with the term $\widetilde{E}_\varepsilon(v_\varepsilon)$. It turns out that

$$\widetilde{E}_\varepsilon(v_\varepsilon) = \pi d |\log \varepsilon| + O(1), \qquad \text{as } \varepsilon \text{ goes to } 0.$$

The upper bound is quite easy to establish but the upper bound requires most of section 3 below. In any case, the lower bound combined with the above lead us to the following result:

Theorem A. *Let G and g be as above. Then for every $u \in H^1_g$ we have*

$$E_\varepsilon(u) \geq \frac{1}{\varepsilon\sqrt{2}}\int_{\partial G} (\tfrac{2}{3} - |g| + \tfrac{|g|^3}{3}) + \pi d |\log \varepsilon| - C, \quad \forall \varepsilon > 0,$$

for some constant $C = C(G, g)$.

Theorem A generalizes the result of [BBH2] which treated the case of an S^1-valued g (the original proof worked for a starshaped domain, this assumption was later removed by Struwe [St] and del Pino-Flemer [DF], see also Sandier [Sa] for a different approach). It is the main tool in the proof of the following convergence result (see Section 4):

Theorem B. *For a subsequence* $\varepsilon_n \to 0$ *we have* $v_{\varepsilon_n} \to u_*$ *in* $Cloc(\overline{G}\setminus\{a_1,\ldots,a_d\})$ *and in* $C^k_{loc}(G\setminus\{a_1,\ldots,a_d\})$ *for all* $k\geq 1$, *for some* d *points* $a_1,\ldots,a_d\in G$, *with* u_* *the canonical harmonic map associated to the points* $a_1,\ldots,a_d$, *the configuration of degrees* $(1,\ldots,1)$ *and the boundary condition* $g/|g|$. *The last convergence remains true also for* u_{ε_n}.

Following [BBH2] we can also determine the location of the points $a_1,\ldots,a_d$. Recall first that the renormalized energy $W(\bar b)$ associated to a configuration of d distinct points in G, $\bar b=(b_1,\ldots,b_d)$, when we fix an S^1-valued boundary condition h on ∂G and a configuration of d degrees all equal to 1, is defined by

$$W(\bar b) = -\pi\sum_{i\neq j}\log|b_i-b_j| - \pi\sum_{i,j}R(b_i,b_j)\,.$$

Here $R(x,y)=\Psi(x,y)-\log|x-y|$ where $\Psi(x,y)$ is the solution of

$$\begin{cases} \Delta\Psi(x,y) = 2\pi\delta_y \quad \text{in } G,\\ \dfrac{\partial\Psi}{\partial\nu} = \dfrac{h\times h_\tau}{d} \quad \text{on } \partial G,\\ \displaystyle\int_{\partial G}\Psi(h\times h_\tau) = 0\,. \end{cases}$$

See [BBH2] and Brezis [B] for more details on the renormalized energy. Next we define as in [BBH2]

$$\gamma \doteq \lim_{\varepsilon\to 0}\{E_\varepsilon(w_\varepsilon)-\pi|\log\varepsilon|\},$$

where w_ε is a minimizer for E_ε corresponding to the boundary condition $g(x)=x/|x|$ for $G=B(0,1)$ (the unit disk). Now we are in a position to state our last main result which gives a more precise information on the configuration $\bar a=(a_1,\ldots,a_d)$ and the energy of u_ε.

Theorem C. *We have*

$$\lim_{\varepsilon\to 0}\{E_\varepsilon(u_\varepsilon)-E_\varepsilon(\rho_\varepsilon)-\pi d|\log\varepsilon|\} = W(\bar a)+d\gamma,$$

where $W(\bar b)$ *is the renormalized energy corresponding to the boundary condition* $h=g/|g|$. *Moreover,* $\bar a$ *minimizes* W *over the set of configurations of* d *distinct points in* G.

2. Preliminary results

This section is devoted mainely to the study of the properties of the minimizers for the scalar problem. We start though with a substitution lemma which is due to Lassoued and Mironescu [LM], we give the proof for completeness.

Lemma 2.1. *Let u_ε be a minimizer for E_ε over $H^1_g = H^1_g(G,\mathbb{C})$ and ρ_ε the minimizer for E_ε over $H_|g|^1(G,\mathbb{R})$. Setting $v_\varepsilon = u_\varepsilon/\rho_\varepsilon$ we have*

$$E_\varepsilon(u_\varepsilon) = E_\varepsilon(\rho_\varepsilon) + \frac{1}{4\varepsilon^2}\int_G (1-\rho_\varepsilon^2)^2 + \frac{1}{2}\int_G \rho_\varepsilon^2|\nabla v_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_G \rho_\varepsilon^4(1-|v_\varepsilon|^2)^2\,. \tag{2.1}$$

Proof. We first calculate:

$$\begin{aligned} E_\varepsilon(u_\varepsilon) = E_\varepsilon(\rho_\varepsilon v_\varepsilon) &= \frac{1}{2}\int_G |\nabla(\rho_\varepsilon v_\varepsilon)|^2 + \frac{1}{4\varepsilon^2}\int_G (1-|\rho_\varepsilon v_\varepsilon|^2)^2 \\ &= E_\varepsilon(\rho_\varepsilon) + \frac{1}{2}\int_G \rho_\varepsilon^2|\nabla v_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_G \rho_\varepsilon^4(1-|v_\varepsilon|^2)^2 \\ &+ \frac{1}{4}\int_G (|v_\varepsilon|^2-1)(2|\nabla\rho_\varepsilon|^2 + \frac{2}{\varepsilon^2}\rho_\varepsilon^2(\rho_\varepsilon^2-1)) + \frac{1}{4}\int_G \nabla(\rho_\varepsilon^2)\nabla(|v_\varepsilon|^2-1)\,. \end{aligned}$$

Denoting $\widetilde{E}_\varepsilon(v) \doteq \frac{1}{2}\int_G \rho_\varepsilon^2|\nabla v|^2 + \frac{1}{4\varepsilon^2}\int_G \rho_\varepsilon^4(1-|v|^2)^2$, we get by Green's theorem:

$$E_\varepsilon(u_\varepsilon) = E_\varepsilon(\rho_\varepsilon) + \widetilde{E}_\varepsilon(v_\varepsilon) + \frac{1}{4}\int_G (|v_\varepsilon|^2-1)(-\Delta(\rho_\varepsilon^2) + 2|\nabla\rho_\varepsilon|^2 + \frac{2}{\varepsilon^2}\rho_\varepsilon^2(\rho_\varepsilon^2-1))\,.$$

But the last integral is zero since the Euler-Lagrange equation $-\Delta\rho_\varepsilon = \frac{1}{\varepsilon^2}(1-\rho_\varepsilon^2)\rho_\varepsilon$ implies that $\Delta(\rho_\varepsilon^2) = \frac{2\rho_\varepsilon^2}{\varepsilon^2}(\rho_\varepsilon^2-1) + 2|\nabla\rho_\varepsilon|^2$. □

In the above we refered to ρ_ε as "the minimizer". This is justified by the following lemma.

Lemma 2.2. *There is a unique solution to the equation*

$$\begin{cases} -\Delta\rho = \dfrac{1}{\varepsilon^2}(1-\rho^2)\rho & \text{in } G, \\ \rho \ge 0 & \text{in } G \\ \rho = |g| & \text{on } \partial G. \end{cases} \tag{2.2}$$

In particular, the unique solution is ρ_ε, the unique minimizer of E_ε over $H^1_{|g|}(G,\mathbb{R})$. Moreover, we have for all $x \in G$:

$$\min(a,1) \le \rho_\varepsilon(x) \le \max(b,1) \tag{2.3}$$

and

$$|\nabla\rho_\varepsilon(x)| \le \frac{C}{\varepsilon} \tag{2.4}$$

where $a \doteq \min_{\partial G}|g|$ and $b \doteq \max_{\partial G}|g|$.

Proof. The uniqueness of the solution follows from the fact that the function $f(t) = (1-t^2)t$ satisfies $f(t)/t \searrow$ on $\mathbb{R}_+$ (see Brezis-Oswald [BO]). Since any minimizer satisfies the Euler-Lagrange equation, and $E_\varepsilon(\rho) = E_\varepsilon(|\rho|)$, the positivity and the uniqueness of the minimizer ρ_ε follows. The estimate (2.3) follows from the uniqueness and the fact that $\min(a,1)$ is a subsolution and $\max(b,1)$ is a supersolution. Finally, the gradient estimate (2.4) follows from standard elliptic estimates, as in [BBH1]. $\square$

More precise pointwise estimates for ρ_ε and its gradient are given in the next proposition.

Notation. For every $x \in G$ we denote $\delta(x) = d(x, \partial G)$.

Proposition 2.1. *For all $x \in G$ we have, with $\delta = \delta(x)$:*

$$|1-\rho_\varepsilon(x)| \le Ce^{-\frac{\delta}{2\varepsilon}} \tag{2.5}$$

and

$$|\nabla\rho_\varepsilon(x)| \le \frac{C}{\delta}\big[(\tfrac{\delta}{\varepsilon})^2+1\big]e^{-\frac{\delta}{2\varepsilon}}\,. \tag{2.6}$$

Proof. Fix any xG. If $a<1$ we define for y in the disk $B(x,\delta)$ (with $\delta=\delta(x)$) the function

$$w_1(r) = \tanh(\tanh^{-1} a + \tfrac{\delta^2-r^2}{3\delta\varepsilon}), \qquad \text{where } r = r(y) = |y-x|.$$

If $a \ge 1$ we define $w_1(r) \equiv 1$. A direct calculation gives

$$-\Delta w_1 = -w_1'' - \frac{w_1'}{r} = \frac{8r^2}{9\delta^2\varepsilon^2}(1-w_1^2)w_1 + \frac{4}{3\delta\varepsilon}(1-w_1^2) \le \frac{8}{9\varepsilon^2}(1-w_1^2)w_1 + \frac{4}{3\delta\varepsilon}(1-w_1^2)\,.$$

For $\varepsilon \le \varepsilon_0$ we have $\frac{4}{3\delta\varepsilon} \le \frac{a}{9\varepsilon^2}$, hence $-\Delta w_1 \le \frac{1}{\varepsilon^2}(1-w_1^2)w_1$. Since $w_1 = a \le \rho_\varepsilon$ on $\partial B(x,\delta)$ it follows that w_1 is a subsolution for (2.2) with $G = B(x,\delta)$ (clearly $w_1 \equiv 1$ is a subsolution in the case $a \ge 1$). Similarly we define

$$w_2(r) = \coth(\coth^{-1} b + \tfrac{\delta^2-r^2}{3\delta\varepsilon})$$

if $b>1$ and $w_2 \equiv 1$ otherwise. The same calculation as above gives

$$-\Delta w_2 = \frac{8r^2}{9\delta^2\varepsilon^2}(1-w_2^2)w_2 + \frac{4}{3\delta\varepsilon}(1-w_2^2) \ge \frac{8}{9\varepsilon^2}(1-w_2^2)w_1 + \frac{4}{3\delta\varepsilon}(1-w_2^2)\,,$$

and for $\varepsilon \le \varepsilon_0$ we have $-\Delta w_2 \le \frac{1}{\varepsilon^2}(1-w_2^2)w_2$. Since $w_2 \ge b$ on ∂G it follows that w_2 is a supersolution (this is trivial in the case $b<1$). By uniqueness of the solution for (2.2) we conclude that $w_1 \le \rho_\varepsilon \le w_2$, hence

$$|1-\rho_\varepsilon| \le Ce^{-\frac{2(\delta^2-\varepsilon^2)}{3\delta\varepsilon}} \qquad \text{on } B(x,\delta).$$

In particular

$$|1-\rho_\varepsilon| \le Ce^{-\frac{\delta}{2\varepsilon}} \qquad \text{on } B(x,\delta/2)$$

which implies (2.5).

In order to prove (2.6), we define the function $\tilde\rho_\varepsilon(y) = \rho_\varepsilon(x+\frac{\delta}{2}y)$ on $B(0,1)$. It satisfies

$$-\Delta(\tilde\rho_\varepsilon - 1) = (\tfrac{\delta}{2\varepsilon})^2(1-\tilde\rho_\varepsilon^2)\tilde\rho_\varepsilon\,.$$

By standard elliptic estimates we have

$$|\nabla\tilde\rho_\varepsilon(0)| \le C\Big[\|\Delta(\tilde\rho_\varepsilon-1)\|_{L^\infty(B(0,1))}\| + \|\tilde\rho_\varepsilon - 1\|_{L^\infty(B(0,1))}\Big] \le C\Big[(\tfrac{\delta}{\varepsilon})^2+1\Big]e^{-\frac{\delta}{2\varepsilon}}.$$

The result follows since $\nabla\tilde\rho_\varepsilon(0) = \frac{\delta}{2}\nabla\rho_\varepsilon(x)$. $\square$

Next we are ready to present an exact estimate for $E_\varepsilon(\rho_\varepsilon)$.

Proposition 2.2. *We have*

$$\frac{1}{\varepsilon\sqrt{2}}\int_{\partial G}(\tfrac{2}{3}-|g|+\tfrac{|g|^3}{3})-C\le E_\varepsilon(\rho_\varepsilon)\le\frac{1}{\varepsilon\sqrt{2}}\int_{\partial G}(\tfrac{2}{3}-|g|+\tfrac{|g|^3}{3})+C\,.$$

Proof. The proof of the upper bound is based on an explicit construction using a technique due to Modica [M] (see also Sternberg [S]). We postpone it to the appendix and we give below the proof of the lower bound which is too motivated by [M]. We fix a vector field $V(x)$ on $\overline{G}$ which satisfies $|V(x)|\le 1$ on G and $V(x)=n(x)$ on ∂G, where $n(x)$ is the unit normal to ∂G at x. Then we have by Cachy-Schwarz inequality and Green's theorem:

$$\begin{aligned}E_\varepsilon(\rho_\varepsilon)\ge\frac{1}{\varepsilon\sqrt{2}}\int_G|(1-\rho_\varepsilon^2)\nabla\rho_\varepsilon|&\ge\frac{1}{\varepsilon\sqrt{2}}\int_G\nabla(\tfrac{2}{3}-\rho_\varepsilon+\tfrac{\rho_\varepsilon^3}{3})\cdot V\\&=\frac{1}{\varepsilon\sqrt{2}}\int_{\partial G}(\tfrac{2}{3}-|g|+\tfrac{|g|^3}{3})-\frac{1}{\varepsilon\sqrt{2}}\int_G(\tfrac{2}{3}-\rho_\varepsilon+\tfrac{\rho_\varepsilon^3}{3})\operatorname{div}V\,.\end{aligned}$$

Note that $\frac{2}{3}-\rho_\varepsilon+\frac{\rho_\varepsilon^3}{3}=\frac{1}{3}(1-\rho_\varepsilon)^2(\rho_\varepsilon+2)$, so by using the upper bound $E_\varepsilon(\rho_\varepsilon)\le C/\varepsilon$ we conclude

$$\Big|\frac{1}{\varepsilon\sqrt{2}}\int_G(\tfrac{2}{3}-\rho_\varepsilon+\tfrac{\rho_\varepsilon^3}{3})\operatorname{div}V\Big|\le\frac{C}{\varepsilon}\int_G(1-\rho_\varepsilon^2)^2\le C.\quad\square$$

We are now in a position to give an upper bound for $E_\varepsilon(u_\varepsilon)$:

Lemma 2.3. *Any minimizer u_ε satisfies*

$$E_\varepsilon(u_\varepsilon)\le E_\varepsilon(\rho_\varepsilon)+\pi d|\log\varepsilon|+C\,. \tag{2.7}$$

Proof. It would be enough to constract $\tilde{u}_\varepsilon\in H^1_g$ for which the estimate (2.7) is valid. We first fix d distinct points $a_1,\dots,a_d$ in G, then $0<r<\min\{|a_i-a_j|;\,1\le i<j<d\}$ and a smooth map $w:G\to S^1$ such that $w=g$ on ∂G and $w(x)=\frac{x-a_i}{|x-a_i|}$ on $\partial B(a_i,r)$, $i=1,\dots,d$. Next we define a map w_ε as follows : $w_\varepsilon=w$ in $G\setminus\bigcup_{i=1}^d B(a_i,r)$ and in each $B(a_i,r)$ w_ε is defined as the minimizer for E_ε with the bouundary condition $\frac{x-a_i}{|x-a_i|}$ on $\partial B(a_i,r)$. By [BBH2] we have

$$E_\varepsilon(w_\varepsilon)=\pi d|\log\varepsilon|+O(1),\qquad\text{as }\varepsilon\text{ goes to }0. \tag{2.8}$$

Finally we set $\tilde{u}_\varepsilon=\rho_\varepsilon w_\varepsilon$. It is clear from Proposition 2.1 and (2.8) that

$$E_\varepsilon(w_\varepsilon)=E_\varepsilon(\rho_\varepsilon)+\pi d|\log\varepsilon|+O(1),$$

and the result follows. $\square$

3. A lower bound for the energy

Recall that we are writing $u_\varepsilon=\rho_\varepsilon v_\varepsilon$ where ρ_ε is the minimzer for E_ε over $H^1_{|g|}(G,\mathbb{R})$ and that $E_\varepsilon(u_\varepsilon)=E_\varepsilon(\rho_\varepsilon)+\widetilde{E}_\varepsilon(v_\varepsilon)$ (Lemma 2.1). We already know (Proposition 2.1) that $E_\varepsilon(\rho_\varepsilon)=\frac{1}{\varepsilon\sqrt{2}}\int_{\partial G}(\frac{2}{3}-|g|+\frac{|g|^3}{3})+O(1)$ and that (see Lemma 2.3) $\widetilde{E}_\varepsilon(v_\varepsilon)\le\pi d|\log\varepsilon|+C$. The main result of this section gives a lower bound for $\widetilde{E}_\varepsilon(v_\varepsilon)$:

Proposition 3.1.

$$\widetilde{E}_\varepsilon(v_\varepsilon) \geq \pi d|\log\varepsilon| - C. \tag{3.1}$$

Remark. Set $a \doteq \min\{|g(x)|;\, x \in \partial G\}$, so that $a > 0$ by our assumption. If $a \geq 1$ the result of Proposition 3.1 is a direct consequence of the results in [BBH2, DF, St]. The case $a < 1$ however requires additional arguments.

Since in the sequel our analysis will depend on some monotonicity properties of ρ_ε, it will be convenient to replace it by another function, $\tilde{\rho}_\varepsilon$ which has these properties. We shall use the following lemma.

Lemma 3.1. *Let Ω be a bounded smooth domain of $\mathbb{R}^2$, and g_1, g_2 two nonnegative functions on $\partial\Omega$ which satisfy $g_1 \leq g_2$. For $i = 1,\ 2$ let $\rho_{\varepsilon,i}$ be a nonnegative solution of*

$$\begin{cases} -\Delta\rho_{\varepsilon,i} = \dfrac{1}{\varepsilon^2}(1 - |\rho_{\varepsilon,i}|^2)\rho_{\varepsilon,i} & \text{in } \Omega \\ \rho_{\varepsilon,i} = g_i \quad \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Then $\rho_{\varepsilon,1} \leq \rho_{\varepsilon,2}$ on Ω.

Proof. It is a direct consequence of the uniqueness of the solution (Lemma 2.2) and the method of sub and super solution.

Notation. For $r > 0$ we denote $G_r = \{x \in G; \delta(x) > r\}$.

Since G is smooth, there exists an r_0 such that for all $x \in G \setminus G_{r_0}$ there is a unique $P(x) \in \partial G$ with $d(x, P(x)) = \delta(x)$. We shall use this notation $P(x)$ for the nearest point projection on ∂G throughout this article. For $\varepsilon \leq r_0^2$ and $x \in G$ satisfying $\delta(x) \leq \varepsilon^{1/2}$ we define the function $\tilde{\rho}_{\varepsilon,x}(y)$ on $B(x, \delta(x))$ as the solution of (3.1) with $\Omega = B(x, \delta(x))$ and $g_i \equiv a$. We then define the function $\tilde{\rho}_\varepsilon$ on G by

$$\tilde{\rho}_\varepsilon(y) = \begin{cases} \tilde{\rho}_{\varepsilon,\tilde{y}}(y), & \text{for } d(y, \partial G) \leq \varepsilon^{1/2} \\ h_\varepsilon(y), & \text{for } d(y, \partial G) > \varepsilon^{1/2} \end{cases}$$

where h_ε is the solution for (3.1) with $\Omega = G_{\varepsilon^{1/2}}$ and $g_i(x) = \tilde{\rho}_{\varepsilon,P(x)}(x)$ on $\partial G_{\varepsilon^{1/2}}$. Note that on $G \setminus G_{\varepsilon^{1/2}}$ $\tilde{\rho}_\varepsilon(y)$ is actually a function of $\delta(y)$ which increases with $\delta(y)$.

From Lemma 3.1 it follows that

$$\tilde{\rho}_\varepsilon(x) \leq \rho_\varepsilon(x) \quad \text{on } G\,. \tag{3.3}$$

By (3.3) it is clear that

$$\widetilde{E}_\varepsilon(v_\varepsilon) \geq \frac{1}{2}\int_G \tilde{\rho}_\varepsilon^2|\nabla v_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_G \tilde{\rho}_\varepsilon^4(1-|v_\varepsilon|^2)^2 \geq \frac{1}{2}\int_G \tilde{\rho}_\varepsilon^2|\nabla \tilde{v}_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_G \tilde{\rho}_\varepsilon^4(1-|\tilde{v}_\varepsilon|^2)^2,$$

where $\tilde{v}_\varepsilon$ is defined as a minimizer for the energy

$$\tilde{E}_{\varepsilon,1}(v) \doteq \frac{1}{2}\int_G \tilde{\rho}_\varepsilon^2|\nabla v|^2 + \frac{1}{4\varepsilon^2}\int_G \tilde{\rho}_\varepsilon^4(1 - |v|^2)^2$$

over $H^1_{g/|g|}(G, \mathbb{C})$. We clearly have

$$\tilde{E}_{\varepsilon,1}(\tilde{v}_\varepsilon) \leq \pi d|\log\varepsilon| + C\,. \tag{3.4}$$

The result of Proposition 3.1 will follow once we show that

$$\tilde{E}_{\varepsilon,1}(\tilde{v}_\varepsilon) \geq \pi d|\log\varepsilon| - C\,. \tag{3.5}$$

We shall use a variant of the method of Struwe [St]. We start with the following lemma:

Lemma 3.2. *There exist two constants C_1, C_2 such that*

$$\frac{1}{\varepsilon^2}\int_{G\cap B(x,r)}(1-|\tilde v_\varepsilon|^2)^2 \le C_1 f_\varepsilon(r) + C_2\varepsilon^{1/4}, \qquad \forall x\in G_{2r}\cup\partial G,\ \forall r\in[\varepsilon^{7/8},\varepsilon^{3/4}]$$

where

$$f_\varepsilon(r) = f_\varepsilon(r,x) \doteq r\int_{\partial B(x,r)\cap G}\{\frac{1}{2}\tilde\rho_\varepsilon^2|\nabla\tilde v_\varepsilon|^2 + \frac{\tilde\rho_\varepsilon^4}{4\varepsilon^2}(1-|\tilde v_\varepsilon|^2)^2\}.$$

Proof. The proof relies on Pohozaev's identity. Clearly v_ε satisfies

$$\begin{cases} -div(\tilde\rho_\varepsilon^2\nabla\tilde v_\varepsilon) = \dfrac{\tilde\rho_\varepsilon^4}{\varepsilon^2}(1-|\tilde v_\varepsilon|^2)\tilde v_\varepsilon & \text{in } G \\ \tilde v_\varepsilon = \dfrac{g}{|g|} & \text{on } \partial G. \end{cases} \tag{3.6}$$

Note that from (3.6) it follows, using a scaling argument (see [St]) that

$$|\nabla\tilde v_\varepsilon| \le \frac{C}{\varepsilon}. \tag{3.7}$$

Assume first that $\delta(x)\ge 2r$. We choose x as our new origin and we multiply the Euler equation (3.6) by $y\nabla\tilde v_\varepsilon$ and integrate over $B(x,r)$. We find, denoting by n the exterior unit normal to the boundary:

(3.8)
$$0 = \int_{\partial B(x,r)}\tilde\rho_\varepsilon^2\frac{\partial\tilde v_\varepsilon}{\partial n}(y\nabla\tilde v_\varepsilon) - \frac{1}{2}\int_{\partial B(x,r)}\tilde\rho_\varepsilon^2|\nabla\tilde v_\varepsilon|^2(y\cdot n) - \frac{1}{4\varepsilon^2}\int_{\partial B(x,r)}\tilde\rho_\varepsilon^4(1-|\tilde v_\varepsilon|^2)^2(y\cdot n)$$
$$+\frac{1}{2\varepsilon^2}\int_{B(x,r)}\tilde\rho_\varepsilon^4(1-|\tilde v_\varepsilon|^2)^2 + \int_{B(x,r)}|\nabla\tilde v_\varepsilon|^2\tilde\rho_\varepsilon(y\nabla\tilde\rho_\varepsilon) + \frac{1}{\varepsilon^2}\int_{B(x,r)}(y\nabla\tilde\rho_\varepsilon)\tilde\rho_\varepsilon^3(1-|\tilde v_\varepsilon|^2)^2\,.$$

Note that
$$\frac{\partial\tilde v_\varepsilon}{\partial n}(y\nabla\tilde v_\varepsilon) - \frac{1}{2}(y\cdot n)|\nabla\tilde v_\varepsilon|^2 = \frac{r}{2}(|\frac{\partial\tilde v_\varepsilon}{\partial n}|^2 - |\frac{\partial\tilde v_\varepsilon}{\partial\tau}|^2)\,.$$
So clearly all the boundary integrals in (3.8) are bounded by $Cf_\varepsilon(r)$. To finish the proof in this case we esimate the two righthand side terms in (3.8). By the upper bound (3.4) the and (2.6) (note that $\delta(y)\ge\varepsilon^{7/8}$ for all $y\in B(x,r)$ when ε is small enough) both of them are bounded by $C|\log\varepsilon|\exp(-(\frac{1}{\varepsilon})^{1/10})\le C\varepsilon^{1/4}$. The result then follows in this case.

In the remaining case, i.e. when $x\in\partial G$, we denote $D = G\cap B(x,r)$ and we find using Pohozaev's identity as above (again we choose x as the origin)
(3.9)
$$0 = \int_{\partial D}\tilde\rho_\varepsilon^2\frac{\partial\tilde v_\varepsilon}{\partial n}(y\nabla\tilde v_\varepsilon) - \frac{1}{2}\int_{\partial D}\tilde\rho_\varepsilon^2|\nabla\tilde v_\varepsilon|^2(y\cdot n) - \frac{1}{4\varepsilon^2}\int_{\partial D}\tilde\rho_\varepsilon^4(1-|\tilde v_\varepsilon|^2)^2(y\cdot n)$$
$$+\frac{1}{2\varepsilon^2}\int_D\tilde\rho_\varepsilon^4(1-|\tilde v_\varepsilon|^2)^2 + \int_D|\nabla\tilde v_\varepsilon|^2\tilde\rho_\varepsilon(y\nabla\tilde\rho_\varepsilon) + \frac{1}{\varepsilon^2}\int_D(y\nabla\tilde\rho_\varepsilon)\tilde\rho_\varepsilon^3(1-|\tilde v_\varepsilon|^2)^2$$
$$\doteq I_1+\cdots+I_6\,.$$

We have clearly

$$I_4 \geq \frac{a^4}{2\varepsilon^2}\int_D (1-|\tilde v_\varepsilon|^2)^2. \tag{3.10}$$

Next note that for every $y\in D$ we have either $y\cdot\nabla\tilde\rho_\varepsilon \geq 0$, or, in case $y\cdot\nabla\tilde\rho_\varepsilon < 0$, the angle $\alpha(y)$ between y and $\nabla\tilde\rho_\varepsilon(y)$ (which in turn has the same direction as the vector $P(y)y$) is bounded by $\frac{\pi}{2}+k|y|$ for some k. In the latter case we have

$$|y\cdot\nabla\tilde\rho_\varepsilon| \leq |y||\nabla\tilde\rho_\varepsilon||\cos\alpha(y)| \leq k|y|^2|\nabla\tilde\rho_\varepsilon| \leq C\varepsilon^{3/4+3/4-1} = C\varepsilon^{1/2}.$$

Here we used the fact that (2.4) continues to hold for $\tilde\rho_\varepsilon$ as it is easy to see by the same proof. Since clearly $\tilde E_{\varepsilon,1}(\tilde v_\varepsilon) \leq C|\log\varepsilon|$, we get that

$$I_5+I_6 \geq -C\varepsilon^{1/2}|\log\varepsilon| \geq -C\varepsilon^{1/4}\,. \tag{3.11}$$

We are left with the boundary integrals I_1, I_2, I_3. Each one of them can be written as

$$I_j = I_{j,1}+I_{j,2} \quad (j=1,2,3)$$

where $I_{j,1}$ is the integral on $\partial D\cap\partial G$ while $I_{j,2}$ is the integral over $\partial D\cap G$. Clearly

$$|I_{1,2}|+|I_{2,2}|+|I_{3,2}| \leq Cf_\varepsilon(r) \quad \text{and} \quad I_{3,1}=0\,. \tag{3.12}$$

Next note that although we do not have necessarily $y\cdot n\geq 0$ on $\partial D\cap\partial G$, the angle $\alpha(y)$ between y and n is bounded by $\frac{\pi}{2}+k|y|$ whenever $y\cdot n<0$. Hence

$$\begin{aligned} |I_{2,1}+I_{2,2}| &= \left|\int_{\partial D\cap\partial G}\tilde\rho_\varepsilon^2\tfrac12(y\cdot n)\Big|\frac{\partial\tilde v_\varepsilon}{\partial n}\Big|^2 + \int_{\partial D\cap\partial G}\tilde\rho_\varepsilon^2\Big[(y\cdot\tau)\frac{\partial\tilde v_\varepsilon}{\partial n}\frac{\partial\tilde v_\varepsilon}{\partial\tau} - \tfrac12(y\cdot n)\tilde\rho_\varepsilon^2\Big|\frac{\partial\tilde v_\varepsilon}{\partial\tau}\Big|^2\Big]\right| \\ &\leq C\int_{\partial D\cap\partial G}|y|^2\Big|\frac{\partial\tilde v_\varepsilon}{\partial n}\Big|^2 + C\varepsilon^{3/4}\int_{\partial D\cap\partial G}\Big|\frac{\partial\tilde v_\varepsilon}{\partial n}\Big| + C\varepsilon^{3/2}\,. \end{aligned}$$

Using (3.7) we finally conclude that

$$|I_{2,1}+I_{2,2}| \leq C\big(\varepsilon^{3\cdot3/4-2}+\varepsilon^{3/4+3/4-1}\big) \leq C\varepsilon^{1/4}\,. \tag{3.13}$$

Combining (3.10) – (3.13) we are led to the desierd conclusion. □

Lemma 3.3. *For any $\alpha\in(3/4,1)$ there exists a constant $C(\alpha)$ such that*

$$\frac{1}{\varepsilon^2}\int_{B(x,\varepsilon^\alpha)\cap G}(1-|\tilde v_\varepsilon|^2)^2 \leq C(\alpha), \quad \forall x\in G\,.$$

Proof. We may assume that $\alpha<7/8$. Assume first that $\delta(x)\geq 2\varepsilon^\alpha$. Then by Fubini's theorem and (3.4) we may find some $r\in(\varepsilon^\alpha,\varepsilon^{3/4})$ such that

$$r\int_{\partial B(x,r)\cap G}\Big\{\frac12\tilde\rho_\varepsilon^2|\nabla\tilde v_\varepsilon|^2 + \frac{\tilde\rho_\varepsilon^4}{4\varepsilon^2}(1-|\tilde v_\varepsilon|^2)^2\Big\} \leq C'(\alpha)\,. \tag{3.14}$$

The result now follows from Lemma 3.2. When $\delta(x)<2\varepsilon^\alpha$, we have

$$G\cap B(x,\varepsilon^\alpha)\subset G\cap B(P(x),4\varepsilon^\alpha)$$

and we may apply Lemma 3.2 again, to find $r\in(4\varepsilon^\alpha,\alpha^{3/4})$ such that (3.14) holds with x replaced by $P(x)$, to conclude. □

Lemma 3.4. *For every α, β such that $3/4 \le \alpha < \beta \le 7/8$ there exists a constant $C(\alpha,\beta)$ such that*

$$\widetilde{E}_\varepsilon(\tilde{v}_\varepsilon; B(x,\varepsilon^\alpha) \cap G) \ge C(\alpha,\beta)|\log\varepsilon|$$

whenever $|\tilde{v}_\varepsilon(z)| < 9/10$ for some $z \in B(x,\varepsilon^\beta)$ and x satisfies : either $\delta(x) \ge 2\varepsilon^\alpha$ or $x \in \partial G$.

Proof. Note first that the assumption $|\tilde{v}_\varepsilon(z)| < 9/10$ implies that

$$\frac{1}{\varepsilon^2}\int_{B(z,\varepsilon)\cap G}(1-|\tilde{v}_\varepsilon|^2)^2 \ge c_0 \tag{3.15}$$

for some constant $c_0 > 0$. This follows directly from (3.7) as in [BBH2, Theorem III.3]. Lemma 3.2 then implies the existence of $c_1 > 0$ such that $f_\varepsilon(r,x) \ge c_1$ for all $r \in (\varepsilon^\beta, \varepsilon^\alpha)$. Finally

$$\tilde{E}_{\varepsilon,1}(\tilde{v}_\varepsilon; B(x,\varepsilon^\alpha)\cap G) \ge \int_{\varepsilon^\beta}^{\varepsilon^\alpha}\frac{c_1}{r}\,dr = c_1(\beta-\alpha)|\log\varepsilon|,$$

as claimed. □

Lemma 3.5. *There exist a constant λ and an integer N (both independent of ε) such that the set*

$$S^i_\varepsilon \doteq \{x \in G_{\varepsilon^{4/5}}; |\tilde{v}_\varepsilon(x)| < 9/10\}$$

can be covered by a disjoint union of no more than N discs of radius $\lambda\varepsilon$ with centers in S^i_ε.

Proof. We can first find $N(\varepsilon)$ mutually disjoint discs $\{B(x^\varepsilon_j, \varepsilon^{5/6})\}_{j=1}^{N(\varepsilon)}$ with $x^\varepsilon_j \in S^i_\varepsilon$ for all j, such that

$$\bigcup_{x\in S^i_\varepsilon} B(x,\varepsilon^{5/6}) \subset \bigcup_{j=1}^{N(\varepsilon)} B(x^\varepsilon_j, 5\varepsilon^{5/6})\,. \tag{3.16}$$

We claim that $N(\varepsilon) \le N_1$ for some N_1, uniformly in ε. Indeed this follows from Lemma 3.4 which gives $\tilde{E}_{\varepsilon,1}(\tilde{v}_\varepsilon; B(x,\varepsilon^{5/6})) \ge C(5/6,7/8)|\log\varepsilon|$, combined with (3.4). Next we can find $M(\varepsilon)$ mutually disjoint discs $\{B(y^\varepsilon_j,\varepsilon)\}_{j=1}^{M(\varepsilon)}$ with $y^\varepsilon_j \in S^i_\varepsilon$ such that

$$S^i_\varepsilon \subset \bigcup_{j=1}^{M(\varepsilon)} B(y^\varepsilon_j, 5\varepsilon)\,.$$

By (3.16) we get in particular that

$$\bigcup_{j=1}^{M(\varepsilon)} B(y^\varepsilon_j,\varepsilon) \subset \bigcup_{j=1}^{N(\varepsilon)} B(x^\varepsilon_j, 5\varepsilon^{5/6})\,. \tag{3.17}$$

By Lemma 3.3

$$\tfrac{1}{\varepsilon^2}\int_{B(x^\varepsilon_j, 5\varepsilon^{5/6})}(1-|\tilde{v}_\varepsilon|^2)^2 \le C$$

so combining it with (3.15) (applied to $B(y^\varepsilon_j,\varepsilon)$) we conclude that $M(\varepsilon) \le M$ for some M, uniformly in ε. If the discs $\{B(y^\varepsilon_j,\varepsilon)\}_{j=1}^{M(\varepsilon)}$ are mutually disjoint, these discs satisfy the requirements with $N = M$ and $\lambda = 5$. Otherwise, we subsequently multiply the radii of all discs by 2 and delete some discs, and after a finite number of steps ($\le M$) we shall be left with a collection of $N \le M$ disjoint discs of radius $\lambda = 5\cdot 2^k\varepsilon$. □

Lemma 3.6. *There exist a constant λ' and an integer N' (both independent of ε) such that the set*

$$S^b_\varepsilon \doteq \{x \in G \setminus G_{\varepsilon^{4/5}};\, |\tilde v_\varepsilon(x)| < 9/10\}$$

can be covered by a disjoint union of no more than N' discs of radius $\lambda'\varepsilon$ with centers in S^b_ε.

Proof. We can first find $N(\varepsilon)$ mutually disjoint discs $\{B(x^\varepsilon_j, \varepsilon^{3/4}/5)\}_{j=1}^{N(\varepsilon)}$ such that

$$\bigcup_{x\in S^i_\varepsilon} B(x,\varepsilon^{3/4}/5) \subset \bigcup_{j=1}^{N(\varepsilon)} B(x^\varepsilon_j,\varepsilon^{3/4})\,,$$

where each x^ε_j is a point on ∂G which is of the form $P(z)$ with $z \in S^b_\varepsilon$. Then we get as in Lemma 3.5 that $N(\varepsilon) \le N_1$ uniformly in ε. The rest of the proof is almost identical to the proof of Lemma 3.5 and is thus omitted. $\square$

Combining the results of the above two lemmas we conclude that the set

$$S_\varepsilon \doteq \{x \in G;\, |\tilde v_\varepsilon(x)| < 9/10\}$$

can be covered by no more then $N(\varepsilon) \le N$ discs $\{B(y^\varepsilon_j, \lambda\varepsilon)\}$ with $y^\varepsilon_j \in S$ for all j, where N and λ are independent of ε. Passing to a subsequence $\varepsilon_n \to 0$, we may assume that $N(\varepsilon_n)$ equals N for all n, and that $y^{\varepsilon_n}_j \to l_j \in \overline{G}$ for all j. If the distinct limit points are $a_1, \dots, a_{N_1}$ with $N_1 \le N$, then we set for all $k = 1, \dots, N_1$

$$\Lambda_k = \left\{i \in \{1,2,\dots,N\};\, y^{\varepsilon_n}_i \to a_k\right\}.$$

THe argument of [BBH2, Lemma V.1] shows that

$$\left|\deg\bigl(\tilde v_{\varepsilon_n}, \partial(G \cap B(y^{\varepsilon_n}_i, \lambda\varepsilon_n))\bigr)\right| \le C \qquad \text{uniformly in } n.$$

Passing to a further subsequence if necessary we may assume that for all i the degree $d_i = \deg\bigl(\tilde v_{\varepsilon_n}, \partial(G \cap B(y^{\varepsilon_n}_i, \lambda\varepsilon_n))\bigr)$ is independent of n and then set

$$\kappa_j = \sum_{i\in\Lambda_j} d_i \quad \forall j = 1,\dots,N_1\,.$$

A priori some of the a_j's may belong to ∂G. We shall see later that this cannot happen. We shall need the following lemma.

Lemma 3.7. *Let $x \in \partial G$ and r_1, r_2 be such that $\varepsilon_n^{7/8} < r_1 \le r_2/2$ and such that the domain*

$$D(n) \doteq G \cap B(x,r_2) \setminus \overline{B(x,r_1)}$$

does not intersect any of the discs $\{B(y^{\varepsilon_n}_j, \lambda\varepsilon_n)\}$, $j = 1,\dots,N$. We assume also that $r_2 \le c(G)$ with $c(G)$ having the property that for all $r \le c(G)$ $\partial B(x,r)$ intersects ∂G in exactly two points and the length of $\partial B(x,r)\cap G$ is bounded by $10\pi r/9$. Then

$$\frac{1}{2}\int_{D(n)} \tilde\rho^2_{\varepsilon_n}|\nabla \tilde v_{\varepsilon_n}|^2 \ge \frac{3\pi}{2} d_0^2 \log(\frac{r_2}{r_1}) - C\,,$$

where $d_0 = deg(\tilde v_{\varepsilon_n}, \partial D(n))$

Proof. We may assume that $d_0 \neq 0$, otherwise the result is clear. We may write on $D(n) : \tilde v_{\varepsilon_n} = |\tilde v_{\varepsilon_n}|\tilde w_{\varepsilon_n}$ with $|\tilde v_{\varepsilon_n}| \geq 9/10$ by assumption. We shall denote

$$Q(n) = G \setminus \overline{G_{\varepsilon_n |\log \varepsilon_n|^2}}\,.$$

For any $r \in [r_1, r_2]$ we have by assumption

$$\int_{\partial(B(x,r)\cap G)} |\tilde w_{\varepsilon_n}| \geq 2\pi d_0\,. \tag{3.18}$$

By Cauchy-Schwarz:

$$\left(\int_{\partial B(x,r)\cap Q(n)} |\nabla \tilde w_{\varepsilon_n}|\right)^2 \leq C\varepsilon_n |\log \varepsilon_n|^2 \int_{\partial B(x,r)\cap Q(n)} |\nabla \tilde v_{\varepsilon_n}|^2 ,$$

so

$$C\varepsilon_n |\log\varepsilon_n|^3 \geq C\varepsilon_n |\log\varepsilon_n|^2 \int_{D(n)} |\nabla \tilde v_{\varepsilon_n}|^2 \geq \int_{r_1}^{r_2} \left(\int_{\partial B(x,r)\cap Q(n)} |\nabla \tilde w_{\varepsilon_n}|\right)^2$$

Hence
(3.19)

$$\text{meas}\{r \in [r_1, r_2]; \int_{\partial B(x,r)\cap Q(n)} |\nabla \tilde w_{\varepsilon_n}| > 1/10\} \leq 100C\varepsilon_n|\log\varepsilon_n|^3 \leq \varepsilon_n^{7/8} \leq r_1\,,$$

provided ε_n is small enough. By $(3.18) - (3.19)$ and Cauchy-Schwarz we get

$$\begin{aligned}\int_{D(n)} |\nabla \tilde w_{\varepsilon_n}|^2 \geq \int_{r_1}^{r_2} \int_{\partial B(x,r)\setminus Q(n)} |\nabla \tilde w_{\varepsilon_n}|^2 &\geq \int_{2r_1}^{r_2} (2\pi|d_0| - \tfrac{1}{10})^2 (\tfrac{9}{10\pi r})\, dr \\ &\geq 3\pi d_0^2 \log(\frac{r_2}{r_1}) - C\,.\end{aligned} \tag{3.20}$$

By Proposition 2.1 we clearly have

$$\tilde\rho_{\varepsilon_n}(x) \geq 1 - \frac{C}{|\log\varepsilon_n|^2}, \quad \text{on } G \setminus Q(n)\,, \tag{3.21}$$

hence

$$\frac12 \int_{D(n)} \tilde\rho^2_{\varepsilon_n} |\nabla \tilde v_{\varepsilon_n}|^2 \geq \frac12 \big(1 - \frac{C}{|\log\varepsilon_n|^2}\big)(3\pi d_0^2 \log(\frac{r_2}{r_1}) - C) \geq \frac32 \pi d_0^2 \log(\frac{r_2}{r_1}) - C\,,$$

as claimed. □

Proof of Proposition 3.1. Assume that for some j_0 $a_{j_0} \in \partial G$. By relabeling we may assume that the sequences $y_1^{\varepsilon_n}, \dots, y_m^{\varepsilon_n}$ converge to a_{j_0}. We denote as usual $d_i = \deg(\tilde v_{\varepsilon_n}, \partial(G \cap B(y_i^{\varepsilon_n}, \lambda\varepsilon_n)))$ so that $\sum_{i=1}^m d_i = \kappa_{j_0}$. In the sequel we shall omit the superscript ε_n in order to simplify the notations. For $\alpha \in [7/8, 1)$ that will be determined later, we consider the discs $\{B(y_i, \varepsilon_n^\alpha)\}_{i=1}^m$. After l_1 iterations ($l_1 \leq m$), each consisting of multiplying all radii by 9, deleting some discs and replacing if necessary y_i by $P(y_i)$, we may find a new collection of discs $\{B(z_{j,1}, r_1)\}_{j=1}^{k_1}$ with $r_1 = 9^{l_1}\varepsilon_n^\alpha$ such that

(i) Each $B(y_i, \varepsilon_n^\alpha)$ is contained in some $B(z_{j,1}, r_1)$;
(ii) Each $z_{j,1}$ is of the form y_i or $P(y_i)$ for some i;
(iii) $|z_{i,1} - z_{j,1}| \geq 4r_1 \quad \forall i \neq j$;
(iv) For each j either $z_{j,1} \in \partial G$ or $\delta(z_{j,1}) \geq 2r_1$.

We denote $d_{j,1} = \deg(\tilde{v}_{\varepsilon_n}, \partial(G \cap B(z_{j,1}, r_1)))$ for all j, so as above we have $\sum_{j=1}^{k_1} d_{j,1} = \kappa_{j_0}$. Next we define

$$R_1 = \min\big\{\{\delta(z_{i,1});\ z_{i,1} \in G\}\,,\ \{|z_{i,1} - z_{j,1}|;\ i \neq j\}\big\}\,.$$

After l_2 iterations as above ($l_2 \leq m$), we can find a new collection of discs $\big\{B(z_{j,2}, r_2)\big\}_{j=1}^{k_2}$ with $r_2 = 9^{l_2} R_1$ such that

(i) Each $B(z_{i,1}, r_1)$ is contained in some $B(z_{j,2}, r_2)$;
(ii) Each $z_{j,2}$ is of the form $z_{i,1}$ or $P(z_{i,1})$ for some i;
(iii) $|z_{i,2} - z_{j,2}| \geq 4r_2 \qquad \forall i \neq j$;
(iv) For each j either $z_{j,1} \in \partial G$ or $\delta(z_{j,1}) \geq 2r_2$.

We denote anlogously to the above $d_{j,2} = \deg(\tilde{v}_{\varepsilon_n}, \partial(G \cap B(z_{j,2}, r_2)))$ so $\sum_{j=1}^{k_2} d_{j,2} = \kappa_{j_0}$.

We next claim that

(3.22)
$$\frac{1}{2}\int_{G\cap B(z_{j,1},R_1)\setminus B(z_{j,1},r_1)} \tilde{\rho}_{\varepsilon_n}^2 |\nabla \tilde{v}_{\varepsilon_n}|^2 \geq \begin{cases} \pi d_{j,1}^2 \log(R_1/r_1) - C & \text{if } z_{j,1} \in G, \\ \frac{3}{2}\pi d_{j,1}^2 \log(R_1/r_1) - C & \text{if } z_{j,1} \in \partial G\,. \end{cases}$$

Indeed, looking first at the case $z_{j,1} \in G$, we have by [BMR, Th.]

$$\frac{1}{2}\int_{B(z_{j,1},R_1/2)\setminus B(z_{j,1},r_1)} |\nabla \tilde{v}_{\varepsilon_n}|^2 \geq \pi d_{j,1}^2 \log(R - 1/r_1) - C$$

since by (3.4)

$$\frac{1}{r_1^2}\int_G (1 - |\tilde{v}_{\varepsilon_n}|^2)^2 \leq \frac{1}{\varepsilon_n^{2\alpha}}\int_G (1 - |\tilde{v}_{\varepsilon_n}|^2)^2 \leq C\varepsilon_n^{2(1-\alpha)}|\log \varepsilon_n| \leq C\,.$$

Using (3.21) we get (3.22) in this case. In the remaining case when $z_{j,1} \in \partial G$, (3.22) follows directly from Lemma 3.7.

Continuing in this way we get a sequence

$$r_1 < R_1 < r_2 = 9^{l_2} R_1 < R_2 < r_3 = 9^{l_3} R_2 < \cdots < r_M = 9^{l_M} R_{M_1}$$

with the corresponding discs $\{B(z_{i,j}, r_j),\ j = 1, \ldots, M, i = 1, \ldots, k_j\}$, such that the disc $B(z_{1,M}, r_M)$ contains all the previous discs with $z_{1,M} \in \partial G$ or $z_{1,M} \in G$ such that $\delta(z_{1,M}) \geq 2r_M$. In the first case we set

$$R_M = r_M\,, \quad z_{1,M+1} = z_{1,M}\,, \quad r_{M+1} = R_M\,.$$

In the second case we set

$$R_M = \delta(z_{1,M}), \quad z_{1,M+1} = P(z_{1,M}), \quad r_{M+1} = 2R_M\,.$$

We denote as usual $d_{i,j} = \deg(\tilde{v}_{\varepsilon_n}, \partial(G \cap B(z_{i,j}, r_j)))$ and in particular $d_{1,M+1} = \kappa_{j_0}$.

Next we claim that

$$\text{(3.23)} \qquad \int_{G\cap B(z_{1,M+1},r_{M+1})} \tilde{\rho}_{\varepsilon_n}^2 |\nabla \tilde{v}_{\varepsilon_n}|^2 \geq \pi|\kappa_{j_0}|\log(r_{M+1}/\varepsilon_n) - C\,.$$

Clearly we may suppose that $\kappa_{j_0} \neq 0$. First note that the analogue to (3.22) continues to hold for $z_{i,j}$, $j > 1$, i.e.:
(3.24)

$$\frac{1}{2}\int_{G\cap B(z_{i,j},R_j)\setminus B(z_{i,j},r_j)} \tilde\rho_{\varepsilon_n}^2 |\nabla \tilde v_{\varepsilon_n}|^2 \geq \begin{cases} \pi d_{i,j}^2 \log(R_j/r_j) - C & \text{if } z_{i,j} \in G, \\ \frac{3}{2}\pi d_{i,j}^2 \log(R_j/r_j) - C & \text{if } z_{i,j} \in \partial G\,. \end{cases}$$

Assume first that none of the points $\{z_{j,1}\}_{j=1}^{k_1}$ lies on ∂G. Summing on the inequalities (3.24) we get
(3.25)

$$\begin{aligned}\frac{1}{2}\int_{G\cap B(z_{1,M+1},r_{M+1})\setminus\bigcup_{k=1}^{k_1} B(z_{k,1},r_1)} \tilde\rho_{\varepsilon_n}^2 |\nabla \tilde v_{\varepsilon_n}|^2 &\geq \sum_{j=1}^{M+1}\sum_{i=1}^{k_j} d_{i,j}^2 \log(R_j/r_j) - C \\ &\geq \pi|\kappa_{j_0}|\log(R_{M+1}/r_1) - C\,.\end{aligned}$$

Also
(3.26)

$$\frac{1}{2}\int_{B(z_{j,1},r_1)\setminus\bigcup\{B(y_i,\lambda\varepsilon_n);y_i\in B(z_{j,1},r_1)\}} \tilde\rho_{\varepsilon_n}^2 |\nabla \tilde v_{\varepsilon_n}|^2 \geq \pi d_{j,1}^2 \log(\tfrac{r_1}{\lambda\varepsilon_n}) - C, \quad j = 1,\dots,k_1$$

by [BMR, Th] since

$$\frac{1}{\varepsilon_n^2}\int_{B(z_{j,1},r_1)} (1-|\tilde v_{\varepsilon_n}|^2)^2 \leq C$$

by Lemma 3.3 (we alsoused (3.4)). Combining (3.25) with (3.26) we are led to the result in this case.

In the second case, we distinguish two possibilities. The first is:

(3.27) $\qquad$ there exist $z_{1,t_1},\dots,z_{1,t_L} \in G$ such that $\displaystyle\sum_{i=1}^{L} d_{1,t_i} = \kappa_{j_0}\,.$

In this case we may use (3.26) only for $j = t_i$, $i = 1,\dots,L$. Adding the resulting inequalities to (3.25) we obtain (3.23) as above. Finally if (3.27) is not satisfied then necessarily there esist $s_1, s_2, \dots, s_{M+1}$ such that

$$\begin{cases} z_{s_1,1}, z_{s_2,2},\dots, z_{s_{M+1},M+1} \in \partial G, \\ B(z_{s_1,1},r_1) \subset B(z_{s_2,2},r_2)\cdots \subset B(z_{s_{M+1},M+1},r_{M+1}), \\ d_{s_i,i} \neq 0, \qquad i = 1,\dots,M+1\,. \end{cases}$$

We then get by (3.24) that
(3.28)

$$\frac{1}{2}\int_{G\cap B(z_{s_j,j},R_j)\setminus B(z_{s_j,j},r_j)} \tilde\rho_{\varepsilon_n}^2 |\nabla \tilde v_{\varepsilon_n}|^2 \geq \tfrac{3}{2}\pi d_{s_j,j}^2 \log(R_j/r_j) - C, \quad j = 1,\dots,M+1.$$

Summing the inequalities (3.24), taking into account (3.28), leads to

$$\begin{aligned}\frac{1}{2}\int_{G\cap B(z_{1,M+1},r_{M+1})\setminus\bigcup_{k=1}^{k_1} B(z_{k,1},r_1)} \tilde\rho_{\varepsilon_n}^2 |\nabla \tilde v_{\varepsilon_n}|^2 &\geq \pi(|\kappa_{j_0}|+1)\log(\tfrac{R_{M+1}}{\varepsilon_n^{\alpha}}) - C \\ &\geq \pi|\kappa_{j_0}|\log(\tfrac{R_{M+1}}{\varepsilon_n}) - C,\end{aligned}$$

if $\alpha \geq |\kappa_{j_0}|/(|k_{j_0}|+1)$. We thus *fix* an $\alpha[7/8, 1)$ satisfying this inequality. Such a choice is possible since $|\kappa_{j_0}|$ is bounded by some constant (recall also that we are assuming $\kappa_{j_0} \neq 0$). We have thus established (3.23).

Next, let us fix $\nu > 0$ such that

$$\nu < \min\{\{|a_i - a_j|/2;\, i \neq j\}\,,\, \{\delta(a_i)/2;\, a_i \in G\}\}\,.$$

By (3.23) and Lemma 3.7 we have

$$\begin{aligned}\frac{1}{2}\int_{B(z_{1,M+1},\nu/2)} \tilde{\rho}^2_{\varepsilon_n}|\nabla \tilde{v}_{\varepsilon_n}|^2 &\geq \pi|\kappa_{j_0}|\log(\nu/\varepsilon_n) + \frac{\pi}{2}|\kappa_{j_0}|\log(\nu/r_{M+1})| - C\\ &= \pi|\kappa_{j_0}||\log(\nu/\varepsilon_n)| + |\kappa_{j_0}|b_{j_0,n}\,,\end{aligned}$$

with $\lim_{n\to\infty} b_{j_0,n} = +\infty$. We then conclude that

$$\frac{1}{2}\int_{B(a_j,\nu)} \tilde{\rho}^2_{\varepsilon_n}|\nabla \tilde{v}_{\varepsilon_n}|^2 \geq \pi|\kappa_j||\log(\nu/\varepsilon_n)| + |\kappa_j|b_{j,n}, \quad \text{for } a_j \in \partial G \tag{3.29}$$

with $\lim_{n\to\infty} b_{j,n} = +\infty$. On the other hand we have by [BMR, Th]:

$$\frac{1}{2}\int_{B(a_j,\nu)} \tilde{\rho}^2_{\varepsilon_n}|\nabla \tilde{v}_{\varepsilon_n}|^2 \geq \pi|\kappa_j||\log(\nu/\varepsilon_n)| - C, \quad \text{for } a_j \in G \tag{3.30}$$

(here again we used Lemma 3.3 and (3.4)). Summing (3.29) – (3.30) over all j yields

$$\frac{1}{2}\int_G \tilde{\rho}^2_{\varepsilon_n}|\nabla \tilde{v}_{\varepsilon_n}|^2 \geq \pi\sum_{j=1}^{N_1}|\kappa_j||\log\varepsilon_n| + \sum_{a_j\in\partial G}|\kappa_j|b_{j,n} - C\,. \tag{3.31}$$

Proposition 3.1 clearly follows from (3.31). In fact, (3.31) gives the adidtional information: $\kappa_j = 0$ whenever $a_j \in \partial G$. □

4. The convergence result

This section is devoted to the proof of Theorems B and C in the introduction. A basic estimate for the convergence result is given in the following lemma. Recall that we write $u_\varepsilon = \rho_\varepsilon v_\varepsilon$ where ρ_ε is the minimizer for E_ε over $H^1_{|g|}(G,\mathbb{R})$ and that we denote $a = \min_{\partial G}|g|$.

Lemma 4.1. *We have*

$$\frac{1}{\varepsilon^2}\int_G (1-|v_\varepsilon|^2)^2 \leq C\,. \tag{4.1}$$

Proof. We use a variant of an argument of del Pino-Felmer [DF]. For each $\varepsilon > 0$ we denote by $\bar{\rho}_\varepsilon$ the unique solution (see Lemma 2.2) of

$$\begin{cases} -\Delta\bar{\rho}_\varepsilon = \dfrac{1}{\varepsilon^2}(1-\bar{\rho}^2_\varepsilon)\bar{\rho}_\varepsilon & \text{in } G,\\ \bar{\rho}_{\varepsilon_n} \geq 0 & \text{in } G\\ \bar{\rho}_{\varepsilon_n} = \min(a,1) & \text{on } \partial G,\end{cases} \tag{4.2}$$

By the uniqueness of the solution to (4.2) and the method of sub and super solution it is easy to see that

$$\bar\rho_{2\varepsilon}(x) \le \bar\rho_\varepsilon(x) \le \rho_\varepsilon(x), \qquad \forall x \in G. \tag{4.3}$$

We shall denote by $\bar u_\varepsilon$ a minimizer of E_ε for the boundary condition $\min(a,1)g/|g|$ and write $\bar u_\varepsilon = \bar\rho_\varepsilon \bar v_\varepsilon$. By Proposition 3.1 we have

$$\begin{aligned}\pi d|\log\varepsilon| - C &\le \frac{1}{2}\int_G \bar\rho_{2\varepsilon}^2 |\nabla \bar v_{2\varepsilon}|^2 + \frac{1}{16\varepsilon^2}\int_G \bar\rho_{2\varepsilon}^4 (1-|\bar v_{2\varepsilon}|^2)^2 \\ &\le \frac{1}{2}\int_G \bar\rho_{2\varepsilon}^2 |\nabla v_\varepsilon|^2 + \frac{1}{16\varepsilon^2}\int_G \bar\rho_{2\varepsilon}^4 (1-|v_\varepsilon|^2)^2 \\ &\le \frac{1}{2}\int_G \bar\rho_{\varepsilon}^2 |\nabla v_\varepsilon|^2 + \frac{1}{16\varepsilon^2}\int_G \bar\rho_{\varepsilon}^4 (1-|v_\varepsilon|^2)^2 \\ &\le \frac{1}{2}\int_G \rho_{\varepsilon}^2 |\nabla v_\varepsilon|^2 + \frac{1}{16\varepsilon^2}\int_G \rho_{\varepsilon}^4 (1-|v_\varepsilon|^2)^2 .\end{aligned} \tag{4.4}$$

On the other hand, by Lemma 2.7

$$\frac{1}{2}\int_G \rho_\varepsilon^2 |\nabla v_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_G \rho_\varepsilon^4 (1-|v_\varepsilon|^2)^2 \le \pi d|\log\varepsilon| + C . \tag{4.5}$$

Substracting (4.4) from (4.5) yields the result. □

By Lemma 4.1 and the gradient estimate

$$|\nabla v_\varepsilon| \le \frac{C}{\varepsilon}, \tag{4.6}$$

which follows as (3.7), we may apply the method of [BBH2, ch IV] in order to locate the "bad discs". More precisely, we cover the set

$$S_\varepsilon \doteq \{x \in G;\, |\tilde v_\varepsilon(x)| < 9/10\}$$

by no more then $N(\varepsilon) \le N$ discs $\{B(y_j^\varepsilon, \lambda\varepsilon)\}$ with $y_j^\varepsilon \in S$ for all j, where N and λ are independent of ε. As in section 3, passing to a subsequence $\varepsilon_n \to 0$, we may assume that $N(\varepsilon_n)$ equals N for all n, and that $y_j^{\varepsilon_n} \to l_j \in \overline{G}$ for all j. If the distinct limit points are $a_1, \dots, a_{N_1}$ with $N_1 \le N$, then we set for all $k = 1, \dots, N_1$

$$\Lambda_k = \left\{ i \in \{1,2,\dots,N\};\, y_i^{\varepsilon_n} \to a_k \right\}.$$

THe argument of [BBH2, Lemma V.1] shows that

$$\left|\deg\bigl(v_{\varepsilon_n}, \partial(G \cap B(y_i^{\varepsilon_n}, \lambda\varepsilon_n))\bigr)\right| \le C \qquad \text{uniformly in } n.$$

Passing to a further subsequence if necessary we may assume that for all i the degree $d_i = \deg\bigl(v_{\varepsilon_n}, \partial(G \cap B(y_i^{\varepsilon_n}, \lambda\varepsilon_n))\bigr)$ is independent of n and then set

$$\kappa_j = \sum_{i \in \Lambda_j} d_i \quad \forall j = 1, \dots, N_1 .$$

As in the proof of Proposition 3.1 we fix $\nu > 0$ such that

$$\nu < \min\{\{|a_i - a_j|/2;\, i \neq j\}\,,\, \{\delta(a_i)/2;\, a_i \in G\}\}\,.$$

The argument that led to (3.29) – (3.30) gives the analogous results for v_{ε_n}, i.e.

$$\frac{1}{2}\int_{B(a_j,\nu)} \rho^2_{\varepsilon_n} |\nabla v_{\varepsilon_n}|^2 \geq \pi|\kappa_j||\log(\nu/\varepsilon_n)| + |\kappa_j| b_{j,n}, \quad \text{for } a_j \in \partial G \tag{4.7}$$

with $\lim_{n\to\infty} b_{j,n} = +\infty$, and

$$\frac{1}{2}\int_{B(a_j,\nu)} \rho^2_{\varepsilon_n} |\nabla v_{\varepsilon_n}|^2 \geq \pi|\kappa_j||\log(\nu/\varepsilon_n)| - C, \quad \text{for } a_j \in G. \tag{4.8}$$

In fact, the proof of (4.8) is simpler since it follows directly from [BMR, Th.] via (4.1). Combining (4.7) – (4.8) we are led to the following analogue of (3.31), namely:

$$\frac{1}{2}\int_{G} \tilde{\rho}^2_{\varepsilon_n} |\nabla v_{\varepsilon_n}|^2 \geq \pi \sum_{j=1}^{N_1} |\kappa_j||\log \varepsilon_n| + \sum_{a_j \in \partial G} |\kappa_j| b_{j,n} - C\,. \tag{4.9}$$

Since $\sum_{j=1}^{N_1} \kappa_j = d$, we may use (4.9) and the upper bound (2.7) to infer that

$$\kappa_j = 0, \qquad \text{for } a_j \in \partial G$$

and

$$\kappa_j \geq 0, \qquad \text{for } a_j \in G.$$

Moreover if we denote for each $\mu \leq \nu$ by $\Omega_\nu \doteq G_\nu \setminus \bigcup_{a_j \in G} \overline{B(a_j,\nu)}$ we get that

$$\int_{\Omega_\mu} |\nabla v_{\varepsilon_n}|^2 \leq C(\mu)\,. \tag{4.10}$$

By (4.10) and (4.1) we deduce the *weak* convergence of a subsequence, still denoted by v_{ε_n}, in $H^1_{\text{loc}}(G\setminus\{a_j \in G\})$ to a limit $v_* \in H^1_{\text{loc}}(G\setminus\{a_j;\, a_j \in G\}, S^1)$. Moreover, by the method of [BBH2, Ch. VI] we get

$$v_{\varepsilon_n} \to v_* \quad \text{in } C^k(K), \forall k,$$

for every compact subset $K \subset G\setminus\bigcup_j\{a_j\}$. In particular it follows as in [BBH2] that v_* is an harmonic map, i.e. it satisfies

$$-\Delta v_* = |\nabla v_*|^2 v_* \quad \text{in } G\setminus\bigcup_j\{a_j\}\,.$$

This implies in fact that also

$$u_{\varepsilon_n} \to v_* \quad \text{in } C^k(K), \forall k,$$

since $\rho_{\varepsilon_n} \to 1$ in $C^k(K)$. Using the arguments of [BBH2] we get also that

$$\kappa_j = \deg(v_*, a_j) = 1, \qquad \text{for all } a_j \in G.$$

Next we are going to prove the uniform convergence of $\{v_{\varepsilon_n}\}$ to v_* on $\overline{G} \bigcup_j \{a_j\}$. First we fix any $x \in \partial G$ and $\mu > 0$ such that $\overline{G} \cap B(x, 2\mu)$ does not contain any of the a_j's. By Fubini's theorem we choose an $R \in [\mu, 2\mu]$ with

$$\int_{G \cap \partial B(x,R)} \{\frac{1}{2}|\nabla v_{\varepsilon_n}|^2 + \frac{1}{4\varepsilon^2}(1 - |v_{\varepsilon_n}|^2)^2\} \leq C\,. \tag{4.11}$$

It follows, in particular, from (4.11) that $v_{\varepsilon_n} \to v_*$ uniformly on $\partial B(x, R) \cap G$. Note that by our assumption $\deg(v_*, \partial(G \cap B(x, R)) = 0$. For each n let us denote by w_{ε_n} a minimizer for E_{ε_n} on $\Omega \doteq G \cap B(x, R)$ with the boundary condition v_* on $\partial\Omega$. By [BBH2, Th. A.3] we have

$$w_{\varepsilon_n} \to v_* \quad \text{in } C^{1,\alpha}\big(B(x, R/2) \cap \overline{G}\big)\,. \tag{4.12}$$

The proof of (4.12) shows also (see [BBH1]) that

$$\frac{1}{\varepsilon_n^2}\int_\Omega (1 - |w_{\varepsilon_n}|^2)^2 \to 0\,. \tag{4.13}$$

Next for each $0 < \eta < R$ we set $\Omega_\eta \doteq \Omega \cap G_\eta$. We have

$$\widetilde{E}_{\varepsilon_n}(v_{\varepsilon_n}; \Omega) \leq \widetilde{E}_{\varepsilon_n}(w_{\varepsilon_n}; \Omega) = \widetilde{E}_{\varepsilon_n}(w_{\varepsilon_n}; \Omega_\eta) + \widetilde{E}_{\varepsilon_n}(w_{\varepsilon_n}; \Omega \setminus \Omega_\eta)\,. \tag{4.14}$$

Passing to the limit in (4.14), using (4.12) – (4.13) and the fact that $\rho_{\varepsilon_n} \to 1$ uniformelly on copact subsets of G, we find that for all small η

$$\overline{\lim}\, \widetilde{E}_{\varepsilon_n}(v_{\varepsilon_n}; \Omega) \leq \frac{1}{2}\int_{\Omega_\eta} |\nabla v_*|^2\,. \tag{4.15}$$

Sending η to 0 in (4.15) we get

$$\overline{\lim}\, \widetilde{E}_{\varepsilon_n}(v_{\varepsilon_n}; \Omega) \leq \frac{1}{2}\int_\Omega |\nabla v_*|^2\,. \tag{4.16}$$

In particular $\{v_{\varepsilon_n}\}$ is bounded in $H^1(\Omega)$ and passing to a subsequence we may assume that $v_{\varepsilon_n} \rightharpoonup v_0 \in H^1(\Omega, S^1)$. From (4.16) it follows that for all small η

$$\overline{\lim}\, \frac{1}{2}\int_{\Omega_\eta} |\nabla v_{\varepsilon_n}|^2 \leq \frac{1}{2}\int_\Omega |\nabla v_*|^2,$$

which implies

$$\frac{1}{2}\int_\Omega |\nabla v_0|^2 \leq \frac{1}{2}\int_\Omega |\nabla v_*|^2.$$

Since $v_0 = v_*$ on $\partial\Omega$ we must have $v_0 \equiv v_*$.

Appendix

We prove below the upper bound of proposition 2.2, i.e. we show that

$$E_\varepsilon(\rho_\varepsilon) \le \frac{1}{\varepsilon\sqrt{2}} \int_{\partial G} (\tfrac{2}{3} - |g| + \tfrac{|g|^3}{3}) + C\,.$$

Clearly it would be enough to construct a function w_ε such that $w_\varepsilon = |g|$ on ∂G and

$$E_\varepsilon(w_\varepsilon) \le \frac{1}{\varepsilon\sqrt{2}} \int_{\partial G} (\tfrac{2}{3} - g + \tfrac{g^3}{3}) + C\,.$$

As Modica [M] and Sternberg [S] we shall use the solution y of

$$\text{(A.1)} \qquad \begin{cases} y'(t) = \frac{1}{\sqrt{2}} \cdot (1 - y(t)^2) \\ y(0) = 0\,. \end{cases}$$

In fact $y(t)$ is given explicitely by $y(t) = \tanh(\frac{t}{\sqrt{2}})$. In the case $\beta \ge 1$ we shall also use $z = \frac{1}{y}$, which is too a solution on $\mathbb{R}_+$ of

$$z'(t) = \tfrac{1}{\sqrt{2}} \cdot (1 - z(t)^2)$$

For some constants $c_1, c_2 > 0$ we have

$$\text{(A.2)} \qquad c_1 e^{-\sqrt{2}t} \le |1 - y(t)| \le c_2 e^{-\sqrt{2}t} \qquad \text{on } \mathbb{R}_+$$

$$\text{(A.3)} \qquad c_1 e^{-\sqrt{2}t} \le |1 - z(t)| \le c_2 e^{-\sqrt{2}t} \qquad \text{on } z^{-1}(\beta, +\infty)\,.$$

Since ∂G is smooth, there exists some $\mu > 0$ such that $\delta(x) = d(x, \partial G)$ is a smooth function on $\{x \in G;\ \delta(x) < \mu\}$. Moreover, for these x's $|\nabla\delta| = 1$ and there exists a unique point $P(x) \in \partial G$ satisfying $d(x, P(x)) = \delta(x)$. For $\varepsilon > 0$ small enough, we shall represent each $x \in \{y \in G;\ \delta(y) \le 2\sqrt{\varepsilon}\}$ by $x = (\theta, \delta)$ with $\delta = \delta(x)$ and with $\theta = \theta(x) \in [0, 2\pi[$ which is defined below. Let us *fix* a point $A \in \partial G$. We set $\theta(A) = 0$ and $\theta(x) = 0$ whenever $P(x) = A$. Now for every small enough $\delta > 0$, we denote by $s(x)$ the arclength variable on the closed curve $\partial G_\delta = \{y \in G;\ \delta(y) = \delta\}$, starting from the point $A_\delta \in \partial G_\delta$ which satisfies $P(A_\delta) = A$. Let l_δ denote the length of ∂G_δ. We then define

$$\theta(x) = (\tfrac{2\pi}{l_\delta}) s(x) \qquad \text{for } x \in \partial G_\delta\,.$$

Finally, we denote

$$t(\theta) = \begin{cases} y^{-1}(g(\theta, 0)), & \text{for } g(\theta, 0) < 1 \\ z^{-1}(g(\theta, 0)), & \text{for } g(\theta, 0) > 1. \end{cases}$$

In the sequel we will work with the coordinates θ and δ, so in particular ∂G is represented by $\{(\theta, 0);\ \theta \in [0, 2\pi)\}$.

We now define $w_\varepsilon(x)$ for $x = (\theta, \delta) \in G$ by

$$w_\varepsilon(x) = \begin{cases} 1, & \text{if } \delta \ge 2\sqrt{\varepsilon} \text{ or } g(\theta, 0) = 1 \\ y(t(\theta) + \frac{\delta}{\varepsilon}), & \text{if } \delta \in [0, \sqrt{\varepsilon}] \text{ and } g(\theta, 0) < 1 \\ z(t(\theta) + \frac{\delta}{\varepsilon}), & \text{if } \delta \in [0, \sqrt{\varepsilon}] \text{ and } g(\theta, 0) > 1 \\ 1 - (1 - w_\varepsilon(\theta, \sqrt{\varepsilon}))(2 - \frac{\delta}{\sqrt{\varepsilon}}), & \text{if } \delta \in [\sqrt{\varepsilon}, 2\sqrt{\varepsilon}]. \end{cases}$$

Note that $|\nabla s| = |\nabla \delta| = 1$ and $\nabla s \cdot \nabla \delta = 0$, hence we have

$$E_\varepsilon(w_\varepsilon) = \int_0^{2\sqrt{\varepsilon}} \int_0^{l_\delta} \frac{1}{4\varepsilon^2}(1-w_\varepsilon^2)^2 + \frac{1}{2}|\nabla w_\varepsilon|^2 \, ds \, d\delta \ ,$$

with

$$|\nabla w_\varepsilon|^2 = |\frac{\partial w_\varepsilon}{\partial \delta}|^2 + |\frac{\partial w_\varepsilon}{\partial s}|^2 .$$

Now

$$\begin{aligned} \frac{\partial w_\varepsilon}{\partial \delta}(\theta, \delta) &= \frac{\partial \theta}{\partial \delta} \cdot \partial_1 w_\varepsilon + \partial_2 w_\varepsilon , \\ \frac{\partial w_\varepsilon}{\partial s}(\theta, \delta) &= \frac{\partial \theta}{\partial s} \cdot \partial_1 w_\varepsilon = \frac{2\pi}{l_\delta} \cdot \partial_1 w_\varepsilon \end{aligned}$$

and

$$\frac{\partial(\theta, \delta)}{\partial(s, \delta)} = \frac{l_\delta}{2\pi} .$$

We then have

$$\begin{aligned} E_\varepsilon(w_\varepsilon) = & \int_0^{2\sqrt{\varepsilon}} \int_0^{l_\delta} \left\{ \frac{1}{4\varepsilon^2}(1-w_\varepsilon^2)^2 + \frac{1}{2}(\partial_2 w_\varepsilon)^2 \right\} \frac{l_\delta}{2\pi} \, d\theta \, d\delta \\ & + \int_0^{2\sqrt{\varepsilon}} \int_0^{l_\delta} \left\{ \frac{1}{2}(\partial_1 w_\varepsilon)^2 \big[(\frac{2\pi}{l_\delta})^2 + (\frac{\partial \theta}{\partial \delta})^2 \big] + \partial_1 w_\varepsilon \partial_2 w_\varepsilon (\frac{\partial \theta}{\partial \delta}) \right\} \frac{l_\delta}{2\pi} \, d\theta \, d\delta . \end{aligned}$$

So

$$E_\varepsilon(w_\varepsilon) \le I_1 + I_2 + I_3 + I_4 \tag{A.4}$$

where

$$I_1 = \frac{1}{\sqrt{2}\varepsilon} \int_0^{\sqrt{\varepsilon}} \int_0^{2\pi} (1-w_\varepsilon^2)(\partial_2 w_\varepsilon) \frac{l_\delta}{2\pi} \, d\theta \, d\delta \ ,$$

$$I_2 = \int_0^{\sqrt{\varepsilon}} \int_0^{2\pi} \Big\{ |\partial_1 w_\varepsilon||\partial_2 w_\varepsilon||\frac{\partial \theta}{\partial \delta}| + \frac{1}{2}(\partial_1 w_\varepsilon)^2 \big[(\frac{2\pi}{l_\delta})^2 + (\frac{\partial \theta}{\partial \delta})^2 \big] \Big\} \frac{l_\delta}{2\pi} \, d\theta d\delta \ ,$$

$$I_3 = \int_{\sqrt{\varepsilon}}^{2\sqrt{\varepsilon}} \int_0^{2\pi} \frac{1}{4\varepsilon^2}(1-w_\varepsilon^2)^2 \frac{l_\delta}{2\pi} \, d\theta \, d\delta$$

and

$$I_4 = \int_{\sqrt{\varepsilon}}^{2\sqrt{\varepsilon}} \int_0^{2\pi} \Big\{ |\partial_1 w_\varepsilon||\partial_2 w_\varepsilon||\frac{\partial \theta}{\partial \delta}| + \frac{1}{2}(\partial_1 w_\varepsilon)^2 \big[(\frac{2\pi}{l_\delta})^2 + (\frac{\partial \theta}{\partial \delta})^2 \big] + \frac{1}{2}(\partial_2 w_\varepsilon)^2 \Big\} \frac{l_\delta}{2\pi} \, d\theta d\delta .$$

We next estimate each of $I_1 - I_4$. By the smoothness of G it follows that $|\frac{\partial \theta}{\partial \delta}|$, $|\frac{\partial l_\delta}{\partial \delta}|$, $\frac{l_\delta}{2\pi}$ and $\frac{2\pi}{l_\delta}$ are all bounded by a certain constant K depending only on G.

Noticing that $-(1-w_\varepsilon^2)\partial_2 w_\varepsilon = \partial_2(\frac{2}{3} - w_\varepsilon + \frac{w_\varepsilon^3}{3})$ and integrating by parts on δ we get (l_0 is just the length of ∂G) :

$$I_1 = \frac{l_0}{2\sqrt{2}\pi\varepsilon}\int_0^{2\pi}\left(\frac{2}{3} - w_\varepsilon + \frac{w_\varepsilon^3}{3}\right)(\theta,0)d\theta - \frac{l_{\sqrt{\varepsilon}}}{2\sqrt{2}\pi\varepsilon}\int_0^{2\pi}\left(\frac{2}{3} - w_\varepsilon + \frac{w_\varepsilon^3}{3}\right)(\theta,\sqrt{\varepsilon})d\theta$$
$$+ \frac{1}{2\sqrt{2}\pi\varepsilon}\int_0^{\sqrt{\varepsilon}}\int_0^{2\pi}\frac{\partial l_\delta}{\partial\delta}\left(\frac{2}{3} - w_\varepsilon + \frac{w_\varepsilon^3}{3}\right)(\theta,\delta)\,d\theta d\delta \doteq i_1 + i_2 + i_3\,.$$

Clearly

$$i_1 = \frac{1}{\varepsilon\sqrt{2}}\int_{\partial G}(\tfrac{2}{3} - g + \tfrac{g^3}{3})ds.$$

We then claim that i_2 goes to 0 and i_3 stays bounded as ε goes to 0. This follows from the inequality

$$(\tfrac{2}{3} - w_\varepsilon + \tfrac{w_\varepsilon^3}{3})(\theta,\delta) = (1-w_\varepsilon)^2(\tfrac{w_\varepsilon+2}{3})(\theta,\delta) \le ce^{-\frac{\sqrt{2}\delta}{\varepsilon}},$$

for some constant c, where we have used (A.2) and (A.3).

Next we estimate I_2. Note that for some constant c

$$|\partial_2 w_\varepsilon| = \frac{1}{\varepsilon\sqrt{2}}|1-w_\varepsilon^2| \le \frac{1}{\varepsilon\sqrt{2}}ce^{-\sqrt{2}\frac{\delta}{\varepsilon}+t(\theta)}$$

and

$$|\partial_1 w_\varepsilon| = \frac{1}{\sqrt{2}}|1-w_\varepsilon^2|\left|\frac{\partial t}{\partial\theta}\right|.$$

By the definition of t, we obtain:

$$\text{either } \frac{\partial t}{\partial\theta} = \frac{\partial g}{\partial\theta}(y^{-1})'(g(\theta,0))$$
$$\text{or } \frac{\partial t}{\partial\theta} = \frac{\partial g}{\partial\theta}(z^{-1})'(g(\theta,0))\,.$$

Using (A.2) we get:

$$(y^{-1})'(g(\theta,0)) = \left|\frac{1}{y'(y^{-1}(g(\theta,0)))}\right| = \left|\frac{1}{1-(y(t(\theta))^2}\right| \le \frac{1}{ce^{-\sqrt{2}t(\theta)}}\,,$$

and the same estimate is valid when we replace y by z. Moreover, by (A.2)–(A.3)

$$|(1-w_\varepsilon^2)(\theta,\delta)| \le ce^{-(\sqrt{2}t(\theta)+\frac{\delta}{\varepsilon})}$$

so that

$$|\partial_1 w_\varepsilon| \le ce^{-\frac{\sqrt{2}\delta}{\varepsilon}}.$$

Now, a easy calculation shows that the first term in the definition of I_2 stays bounded while the second goes to 0 as ε goes to 0. Sabstituting the above estimates in (A.4) we get the desired conclusion. □

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