On a singular perturbation problem involving a "circular-well" potential

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Abstract

We study the asymptotic behavior, as a small parameter ε goes to 0, of the minimizers for a variational problem which involves a "circular-well" potential, i.e., a potential vanishing on a closed smooth curve in $\mathbb{R}^2$. We thus generalize previous results obtained for the special case of the Ginzburg-Landau potential.

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1 Introduction.

Let Γ be a simple closed C^4-curve in $\mathbb{R}^2$ with length $l(\Gamma)$ which bounds a bounded domain Ω. We define a "circular-well potential" as a function $W : \mathbb{R}^2 \to [0, \infty)$ satisfying

$$W > 0 \text{ on } \mathbb{R}^2 \setminus \Gamma \quad \text{and} \quad W = 0 \text{ on } \Gamma\,. \tag{1.1}$$

We shall assume that W too is of class C^4 and make two additional assumptions. Since W attains its minimal value zero on Γ we have clearly $W_n = 0$ on Γ, where W_n denotes the derivative in the direction of the exterior normal to $\partial\Omega = \Gamma$. We assume then that we are in the generic case, i.e., that

$$W_{nn} > 0 \text{ on } \Gamma\,. \tag{1.2}$$

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Finally, we add a technical assumption on the behavior of W at infinity: there exists $R_0 > 0$ such that

$$\frac{\partial W}{\partial r} \geq 0 \text{ for } |z| = r > R_0 \,. \tag{1.3}$$

Let G be a simply connected, bounded domain in $\mathbb{R}^2$ with boundary of class C^2. For a given smooth boundary condition $g : \partial G \to \mathbb{R}^2$ (more assumptions will be imposed on g later on) set

$$H^1_g(G, \mathbb{R}^2) = \{u \in H^1(G, \mathbb{R}^2) \,:\, u = g \text{ on } \partial G\}.$$

For every $\varepsilon > 0$ let

$$E_\varepsilon(u) = \int_G |\nabla u|^2 + \frac{W(u(x))}{\varepsilon^2} \,,$$

and consider the minimization problem:

$$\min\{E_\varepsilon(u) \,:\, u \in H^1_g(G, \mathbb{R}^2)\} \,. \tag{1.4}$$

We are interested in the asymptotic behavior of the minimizers $\{u_\varepsilon\}_{\varepsilon>0}$ and their energies $\{E_\varepsilon(u_\varepsilon)\}_{\varepsilon>0}$, as $\varepsilon \to 0$. In the case where g is $\mathit{\Gamma}$-valued, this study can be done by using the methods developed by Bethuel, Brezis and Hélein in [4] and [5], and by Struwe in [12], for the case of the Ginzburg-Landau (GL) potential $W(z) = (1 - |z|^2)^2$ (i.e., for S^1-valued g). Therefore, we shall focus on the case where g is not necessarily $\mathit{\Gamma}$-valued.

Special cases of this problem were already studied by us in previous works. In [1] we considered the case of the GL-energy for g which takes its values in $\mathbb{R}^2 \setminus \{0\}$. In [3] we studied the more subtle case of the GL-energy with boundary condition which is allowed to have isolated zeros. In [2] we obtained the first result dealing with more general potentials, namely W which is a function of the Euclidean distance to $\mathit{\Gamma}$ (with the assumption that the image of g is "close enough" to $\mathit{\Gamma}$). The main object of the present article is to extend the results of [2] to the case of a general "circular-well" potential satisfying (1.1)–(1.3).

As we saw in [1, 3], the incompatibility of the boundary condition with the potential is responsible to the appearance of a boundary layer near ∂G, developed by the minimizer u_ε. This results in a contribution of the order $O(\frac{1}{\varepsilon})$ to the energy $E_\varepsilon(u_\varepsilon)$. On the other hand, in the interior of the domain G we find the "topological singularities" which contribute a term of the order $O(|\log \varepsilon|)$ to the energy. The main difficulty in this type of problems is to separate efficiently between these two contributions. In the case of the GL-energy (for simplicity we consider only nonvanishing g), this separation was quite easy to achieve, thanks to the following energy decomposition formula which is due to

Mironescu and Lassoued [9]. Writing each admissible $u \in H^1(G,\mathbb{R}^2)$ as $u = \rho_\varepsilon v$, where ρ_ε is the minimizer of the *scalar* GL-problem

$$\min\{\int_G |\nabla\rho|^2 + \frac{1}{\varepsilon^2}(1-\rho^2)^2 \,:\, \rho \in H^1_{|g|}(G)\}\,, \tag{1.5}$$

we have

$$\int_G |\nabla u|^2 + \frac{1}{\varepsilon^2}(1-|u|^2)^2 = \int_G |\nabla\rho_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-\rho_\varepsilon^2)^2 + \int_G \rho_\varepsilon^2|\nabla v|^2 + \frac{1}{\varepsilon^2}\rho_\varepsilon^4(1-|v|^2)^2. \tag{1.6}$$

The formula (1.6) was used in [1] to prove that

$$u_{\varepsilon_n} \to u_0 = e^{i\phi}\prod_{j=1}^{D}\frac{z-a_j}{|z-a_j|} \text{ in } C^k_{\mathrm{loc}}(G\setminus\{a_1,\dots,a_D\}), \tag{1.7}$$

where D is the absolute value of the degree of $g/|g|$. For a general potential W we cannot expect an analogous formula to (1.6). Still, in the case treated in [3], where W is a function of the distance to Γ, i.e., $W(u) = F(\mathrm{dist}(u,\Gamma))$, we do have a kind of an analogue to ρ_ε, given by the minimizer d_ε to the problem

$$\min\{\int_G |\nabla h|^2 + \frac{F(h)}{\varepsilon^2} \,:\, h \in H^1(G),\ h(x) = \mathrm{dist}(g(x),\Gamma) \text{ on } \partial G\}\,. \tag{1.8}$$

It turns out that the energy of d_ε, i.e., the minimal value for the minimization problem (1.8), gives indeed the leading term in $E_\varepsilon(u_\varepsilon)$, of the order $O(\frac{1}{\varepsilon})$. It is also useful in establishing a convergence result analogous to (1.7), but the analysis is much more involved than in the GL case (see [3] for details).

In the case of a potential W satisfying only (1.1)–(1.3) we cannot associate to our problem any scalar minimization problem, like in (1.5) or (1.8), so new techniques are needed in order to separate between the two different types of contribution to the energy. A basic tool in the identification of the contribution from the boundary layer is a certain distance function to Γ, with respect to a degenerate Riemannian metric, associated with W. It is defined on $\mathbb{R}^2$ by

$$\Psi(\zeta) = \inf_{\substack{\gamma\in \mathrm{Lip}([0,1],\mathbb{R}^2),\\ \gamma(0)\in\Gamma,\,\gamma(1)=\zeta}} \int_0^1 \big(W(\gamma(t))\big)^{1/2}\,|\gamma'(t)|\,dt\,. \tag{1.9}$$

Since the integral in (1.9) is invariant w.r.t. rescaling, we may replace the interval $[0,1]$ by any other closed interval. It is not difficult to see that $\Psi \in \mathrm{Lip}(\mathbb{R}^2)$ and that it is a solution of the eikonal-type equation

$$|\nabla\Psi(\zeta)|^2 = W(\zeta) \quad \text{a.e. on } \mathbb{R}^2\,, \tag{1.10}$$

with $\Psi = 0$ on Γ. Functions of this type appeared in works on related problems by many authors, c.f. Sternberg [11] and Fonseca–Tartar [7].

For each $\lambda > 0$, the set $\Omega_\lambda = \{x \in \mathbb{R}^2 \,:\, \Psi(x) < \lambda\}$ is a neighborhood of Γ. In Proposition 5.1 in the appendix it is proved that Ψ is of class C^2 in some neighborhood of Γ (certainly it cannot have this regularity everywhere). There exists then a $\lambda_0 > 0$ satisfying

$$\Psi \in C^2(\overline{\Omega}_{\lambda_0}) \,. \tag{1.11}$$

When (1.11) holds, Ω_{λ_0} can be covered by a system of non-intersecting gradient lines of Ψ. In particular, for each $x \in \Omega_{\lambda_0}$ there exists a unique gradient line which passes through it and we shall denote by $\tilde{s}(x)$ its intersection point with Γ. The map $\tilde{s}$ can be viewed as a *projection* from Ω_{λ_0} onto Γ, which is different, in general, from the Euclidean nearest point projection. We shall always assume that the smooth boundary condition $g : \partial G \to \mathbb{R}^2$ satisfies

$$\mathrm{Image}(g) \subset \Omega_{\lambda_0} \,. \tag{1.12}$$

Therefore, by (1.11)–(1.12) the map $\tilde{s}(g) : \partial G \to \Gamma$ is continuous (actually it is even of class C^1) and we can use it to define the *degree* D of g by

$$D = \deg(\tilde{s}(g), \partial G). \tag{1.13}$$

We shall assume in the sequel, without loss of generality, that $D \geq 0$. We denote

$$\lambda_1 := \max\{\Psi(g(x)) \,:\, x \in \partial G\} \,, \tag{1.14}$$

so that $0 \leq \lambda_1 < \lambda_0$.

Our first result gives the asymptotic behavior of the energy of u_ε as ε goes to zero.

Theorem 1. *Let $g : \partial G \to \mathbb{R}^2$ be a smooth map satisfying* (1.12) *of degree $D \geq 0$. Then,*

$$E_\varepsilon(u_\varepsilon) = \frac{2}{\varepsilon}\int_{\partial G} \Psi(g(\sigma))\, d\sigma + D\,\frac{l^2(\Gamma)}{2\pi}\log\frac{1}{\varepsilon} + O(1), \quad \forall \varepsilon > 0. \tag{1.15}$$

The proof of the upper-bound in (1.15) is quite straightforward. On the other hand, the proof of the lower-bound is much more delicate. A key estimate is a lower-bound for the energy of u_ε on $G_{c_0\varepsilon^\alpha} = \{x \in G \,:\, \mathrm{dist}(x, \partial G) \leq c_0\varepsilon^\alpha\}$ for some $\alpha \in (1/2, 1)$ and $c_0 > 0$ (see (3.30) below):

$$\begin{aligned} E_\varepsilon(u_\varepsilon, G_{c_0\varepsilon^\alpha}) &= \int_{G_{c_0\varepsilon^\alpha}} |\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon^2} \\ &\geq \frac{2}{\varepsilon}\int_{\partial G} \Psi(g(\sigma))\, d\sigma + K\int_{\partial G} \frac{|u_\varepsilon(\sigma, c_0\varepsilon^\alpha) - \tilde{s}(g(\sigma))|^2}{\varepsilon^\alpha}\, d\sigma - C \,. \end{aligned}$$

Here we used the (σ,δ)-coordinates where $\sigma(x)$ is the nearest point projection of x on ∂G and $\delta(x)=\mathrm{dist}(x,\partial G)$ (see Section 2 below). The second step consists of showing that

$$E_\varepsilon(u_\varepsilon, G\setminus G_{c_0\varepsilon^\alpha}) + K\int_{\partial G}\frac{|u_\varepsilon(\sigma,c_0\varepsilon^\alpha)-\tilde{s}(g(\sigma))|^2}{\varepsilon^\alpha}\,d\sigma \geq D\frac{l^2(\Gamma)}{2\pi}\log\frac{1}{\varepsilon}-C\,. \tag{1.16}$$

The proof of (1.16) follows by the same arguments used in [2], adapting the methods in [5, 12].

The energy estimate (1.15) leads to our second main result which deals with the convergence of $\{u_\varepsilon\}$. Let $\tau: S^1\to\Gamma$ be an orientation preserving C^2-map satisfying

$$|\tau'(s)| = \frac{l(\Gamma)}{2\pi},\quad \forall s\in S^1\,. \tag{1.17}$$

Theorem 2. *Assume the same hypotheses as in Theorem 1. Then, there exist a subsequence $\varepsilon_n\to 0$ and D points $a_1,\dots,a_D$ in G such that*

$$u_{\varepsilon_n}\to u_* = \tau\big(e^{i\phi_0}\prod_{j=1}^{D}\frac{z-a_j}{|z-a_j|}\big)\ \text{in } C_{loc}(G\setminus\{a_1,\dots,a_D\})\,, \tag{1.18}$$

where ϕ_0 is a smooth harmonic function which is determined by the constraint $u_*=\tilde{s}(g)$ on ∂G.

The following example demonstrates the importance of using the projection $\tilde{s}$ in Theorem 2 (rather than the usual Euclidean projection).

Example 1. Take $\Gamma=S^1$ and fix $a\neq 0$ in the unit disc $B(0,1)$. Set

$$\Psi(z)=\frac{2}{3}-\left|\frac{z-a}{1-\bar{a}z}\right|+\frac{1}{3}\cdot\left|\frac{z-a}{1-\bar{a}z}\right|^3,$$

and define W on $B(0,1)$ by

$$W(z)=|\nabla\Psi(z)|^2=\frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}\left(1-\left|\frac{z-a}{1-\bar{a}z}\right|^2\right)^2. \tag{1.19}$$

Then complete the definition of W outside $B(0,1)$ in such a way that W will be a C^4-function on $\mathbb{R}^2$ satisfying (1.1)–(1.3). Since $\Psi\in C^\infty(B(0,1)\setminus\{a\})$, we know, thanks to Proposition 5.1, that Ψ coincides in $B(0,1)\setminus\{a\}$ with the function defined in (1.9). The level curves of Ψ inside $B(0,1)$ are the circles which are the images, by the Möbius transformation $m_a(z)=\frac{z+a}{1+\bar{a}z}$, of the circles centered at 0, see Figure 1. Now consider

$G = B(0,1)$ and the boundary condition $g(e^{i\theta}) = be^{i\theta}$ for some $b \in (0, |a|)$. Then, applying Theorem 2 to this example we see that the degree $D = \deg(\tilde{g})$ is zero since image$(\tilde{s}(g))$ covers only part of $\Gamma = S^1$. Hence the limit map $u_* = e^{i\phi_0}$ is smooth. Note that $\deg(g/|g|) = 1$, so using the Euclidean nearest point projection instead of $\tilde{s}$ would lead to a wrong result.

Figure 1: The solid lines are gradient lines of Ψ in the Example. The dashed lines are level curves of Ψ (both are circles in our case). The image of g is the circle $B(0,b)$ (heavy solid line).

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2 An upper-bound for the energy

This section is devoted to the proof of the easier part of Theorem 1, namely, the upper-bound. We start by introducing some notation. We recall first some properties of the distance function δ to ∂G (see [8, Sec. 14.6]). There exists a $b_0 > 0$ such that the distance function $\delta(x)$ to ∂G is C^2 on $G_{b_0} := \{x \in G : \delta(x) \leq b_0\}$. Moreover, denoting by $\sigma(x)$ the Euclidean nearest point projection of a point $x \in G_{b_0}$ on ∂G, we have that the map $x \mapsto (\sigma(x), \delta(x))$ is a C^1-diffeomorphism of $G_{b_0} = \{x \in G : \delta(x) \le b_0\}$ on $\partial G \times [0, b_0]$. The map $H_t(\sigma) : \partial G \to \Sigma_t := \{x \in G : \delta(x) = t\}$ given by $H_t(\sigma) = \sigma - t\vec{\nu}$ ($\vec{\nu}$ stands for the exterior normal to ∂G) is also a C^1-diffeomorphism and its Jacobian satisfies

$$|\mathrm{Jac}\, H_t(\sigma) - 1| \le \tilde{c}t, \quad \forall (t, \sigma) \in (0, b_0) \times \partial G\,. \tag{2.1}$$

We shall often identify a point $x \in G_{b_0}$ with its (σ, δ)-coordinates: $(\sigma(x), \delta(x))$.

We next establish similar notation in the neighborhood of Γ. We denote by $\tilde{\delta}(x)$ the signed distance function to Γ, with the convention that $\tilde{\delta}$ is negative inside Γ and positive outside. For any $\eta > 0$ set

$$\Gamma_\eta = \{x \in \mathbb{R}^2 : |\tilde{\delta}(x)| < \eta\}\,. \tag{2.2}$$

Again, by [8, Sec. 14.6], there exists $\eta_0 > 0$ such that $\tilde{\delta} \in C^2(\Gamma_{\eta_0})$, each $x \in \Gamma_{\eta_0}$ has a unique nearest point projection $S(x) \in \Gamma$ and the map S is of class C^1 on Γ_{η_0}. Consider an arclength parameterization of Γ:

$$\gamma : [0, l(\Gamma)) \to \Gamma\,. \tag{2.3}$$

For each point $x \in \Gamma_{\eta_0}$ we associate a coordinate $\tilde{\sigma} = \tilde{\sigma}(x)$ given by $\tilde{\sigma}(x) = \gamma^{-1}(S(x))$. With the convention that the variable $\tilde{\sigma}$ is taken modulo $l(\Gamma)$ we obtain that the map $x \mapsto (\tilde{\sigma}(x), \tilde{\delta}(x))$ is a local diffeomorphism of class C^2 of the set Γ_{η_0}.

From our assumptions (1.1)–(1.2) it follows that we may write locally in Γ_{η_0}, using these coordinates,

$$W(\tilde{\sigma}, \tilde{\delta}) = a(\tilde{\sigma}, \tilde{\delta})\tilde{\delta}^2\,, \tag{2.4}$$

for some positive function a of class C^2. We shall denote

$$\alpha(x_0) = 2\sqrt{a(x_0)}, \quad \forall x_0 \in \Gamma. \tag{2.5}$$

Note that by (1.10)–(1.11) we also have in Ω_{λ_0}: $\Psi(\tilde{\sigma}, \tilde{\delta}) = b(\tilde{\sigma}, \tilde{\delta})\tilde{\delta}^2$ for some continuous positive function b. Therefore,

$$\Psi(x) \sim \tilde{\delta}^2(x) \sim W(x) \quad \text{in a neighborhood of } \Gamma\,. \tag{2.6}$$

From Proposition 5.1 in the Appendix it follows that there exists a solution $X(x_0, r)$ to the problem

$$\begin{cases} \frac{\partial X}{\partial r} = \frac{2\nabla\Psi(X)}{\alpha(x_0)r}\,, \\ X(x_0, 0) = x_0, \ \forall x_0 \in \Gamma\,, \end{cases} \tag{2.7}$$

which is a C^1-diffeomorphism of a set $\mathcal{N}$ of the form

$$\mathcal{N} = \{(\tilde{s}, r) \,:\, \tilde{s} \in \mathit{\Gamma}, r \in (-r_1(\tilde{s}), r_2(\tilde{s}))\}\,,$$

with $r_1(\tilde{s}), r_2(\tilde{s}) > 0$, $\forall \tilde{s} \in \mathit{\Gamma}$, onto some neighborhood of $\mathit{\Gamma}$ which contains Ω_{λ_0}, such that $X(x_0, r) \in \Omega$ for $r > 0$. In particular, for some $\beta > 0$ we have

$$\beta \le \left|\frac{\partial X}{\partial \tilde{s}}(\tilde{s}, r)\right| \le \frac{1}{\beta}\,, \quad \forall (\tilde{s}, r) \in X^{-1}(\Omega_{\lambda_0})\,. \tag{2.8}$$

For each $x \in \overline{\Omega}_{\lambda_0}$ we can associate a unique number $r(x)$ satisfying $x = X(\tilde{s}(x), r(x))$ ($r(x) > 0$ if and only if $x \in \Omega$).

For $x_0 \in \mathit{\Gamma}$ let

$$\gamma^+_{x_0} : (-\infty, t^+(x_0)] \to \overline{\Omega}_{\lambda_0} \cap \Omega$$

be the path given by

$$\gamma^+_{x_0}(t) = X(x_0, e^{\alpha t/2}) \quad (\text{with } \alpha = \alpha(x_0),\ \text{ see (2.5)}). \tag{2.9}$$

with $\Psi(\gamma^+_{x_0}(t^+(x_0))) = \lambda_0$. Let $x \in \Omega_{\lambda_0} \cap \Omega$ satisfy $\tilde{s}(x) = x_0$. Then, $\gamma^+_{x_0}$ satisfies:

$$\begin{cases} \dot{\gamma}^+_{x_0} = \nabla\Psi(\gamma^+_{x_0}), \\ \gamma^+_{x_0}(-\infty) = \tilde{s}(x) \quad \text{and} \quad \gamma^+_{x_0}(\frac{2\log r(x)}{\alpha}) = x\,. \end{cases} \tag{2.10}$$

In a similar manner we define for each $x_0 \in \mathit{\Gamma}$ a path $\gamma^-_{x_0} : (-\infty, t^-(x_0)] \to \overline{\Omega}_{\lambda_0} \setminus \Omega$ by $\gamma^-_{x_0}(t) = X(x_0, -e^{\alpha t/2})$ which satisfies

$$\begin{cases} \dot{\gamma}^-_{x_0} = \nabla\Psi(\gamma^-_{x_0}), \\ \gamma^-_{x_0}(-\infty) = \tilde{s}(x) \quad \text{and} \quad \gamma^-_{x_0}(\frac{2\log(-r(x))}{\alpha}) = x\,, \end{cases} \tag{2.11}$$

for every $x \in \overline{\Omega}_{\lambda_0} \setminus \Omega$ such that $\tilde{s}(x) = x_0$. It will be convenient to set also $\gamma^0_{x_0}(t) \equiv x_0, \forall x_0 \in \mathit{\Gamma}, \forall t$.

We are now ready to state and prove the upper-bound part of Theorem 1.

Proposition 2.1. *Under the hypotheses of Theorem 1 we have*

$$E_\varepsilon(u_\varepsilon) \le \frac{2}{\varepsilon}\int_{\partial G} \Psi(g(\sigma))\,d\sigma + D\,\frac{l^2(\mathit{\Gamma})}{2\pi}\log\frac{1}{\varepsilon} + C,\ \forall \varepsilon > 0. \tag{2.12}$$

Proof. Clearly it is enough to construct $\{v_\varepsilon\} \subset H^1_g(G,\mathbb{R}^2)$ such that,

$$E_\varepsilon(v_\varepsilon) \le \frac{2}{\varepsilon}\int_{\partial G}\Psi(g(s))\,ds + D\,\frac{l^2(\Gamma)}{2\pi}\log\frac{1}{\varepsilon} + C\,.$$

Denoting $S(\sigma)=\operatorname{sgn}(r(g(\sigma)))$ for $\sigma\in\partial G$, we define v_ε for ε small by:

$$v_\varepsilon(\sigma,\delta) = \begin{cases} \gamma^{S(\sigma)}_{\tilde{s}(g(\sigma))}\big(\frac{2\log|r(g(\sigma))|}{\alpha} - \frac{\delta}{\varepsilon}\big) & \delta \le \varepsilon^{1/2}\,, \\ \frac{2\varepsilon^{1/2}-\delta}{\varepsilon^{1/2}} v_\varepsilon(\sigma,\varepsilon^{1/2}) + \frac{\delta-\varepsilon^{1/2}}{\varepsilon^{1/2}}\tilde{s}(g(\sigma)) & \varepsilon^{1/2} < \delta \le 2\varepsilon^{1/2}\,, \\ \tilde{s}(g(\sigma)) & 2\varepsilon^{1/2} < \delta \le b_0\,. \end{cases}$$

It remains to define v_ε on $G\setminus G_{b_0}$. We choose D points $a_1,\dots,a_D\in G\setminus G_{b_0}$ and then r_0 such that

$$0 < r_0 \le \frac{1}{2}\min\{\min_{i\neq j}|a_i-a_j|, \min_i \delta(a_i)-b_0\}\,.$$

On $(G\setminus G_{b_0})\setminus \bigcup_{j=1}^{D} B(a_j,r_0)$ we set $v_\varepsilon = F_0$ where F_0 is any Γ-valued C^1-map satisfying:

$$F_0(x) = \begin{cases} \tilde{s}(g(\sigma(x))) & \text{on } \Sigma_{b_0} = \partial(G\setminus G_{b_0}) \\ \tau(\frac{x-a_j}{|x-a_j|}) & \text{on } \partial B(a_j,r_0),\ j=1,\dots,D \text{ (see (1.17))}. \end{cases}$$

Finally, on each $B(a_j,r_0)$ we define $v_\varepsilon(x) = f_\varepsilon(|x-a_j|)\cdot\tau(\frac{x-a_j}{|x-a_j|})$ where the scalar function f_ε is defined by:

$$f_\varepsilon(r) = \begin{cases} 1 & \text{for } \varepsilon < r \le r_0, \\ \frac{r}{\varepsilon} & \text{for } 0 \le r \le \varepsilon. \end{cases}$$

It is easy to verify that

$$E_\varepsilon(v_\varepsilon, B(a_j,r_0)) = \frac{l^2(\Gamma)}{2\pi}\log\frac{1}{\varepsilon} + C, \quad j=1,..,D,$$

which implies that

$$E_\varepsilon(v_\varepsilon, G\setminus G_{b_0}) \le D\,\frac{l^2(\Gamma)}{2\pi}\log\frac{1}{\varepsilon} + C. \tag{2.13}$$

It remains to estimate $E_\varepsilon(v_\varepsilon, G_{b_0})$. Since X is a C^1-solution of (2.7), it follows that $\gamma^\pm_{x_0}$ satisfy

$$|\gamma^\pm_{x_0}(t) - x_0|, |\dot{\gamma}^\pm_{x_0}(t)| \le Ce^{ct} \;\text{ and }\; \Psi(\gamma^\pm_{x_0}(t)) \le Ce^{ct}, \forall x_0\in\Gamma,$$

for some positive constants c, C. Using these estimates we conclude easily that

$$E_\varepsilon(v_\varepsilon, G_{b_0} \setminus G_{\varepsilon^{1/2}}) \le C\,. \tag{2.14}$$

Finally, on $G_{\varepsilon^{1/2}}$ we have

$$\frac{\partial v_\varepsilon}{\partial \delta}(\sigma,\delta) = -\frac{1}{\varepsilon}\cdot \dot{\gamma}_{\tilde{s}(g(\sigma))}\Big(\frac{2\log r(g(\sigma))}{\alpha} - \frac{\delta}{\varepsilon}\Big) = -\frac{1}{\varepsilon}\cdot \nabla\Psi(v_\varepsilon(\sigma,\delta))\,.$$

Since by the construction of v_ε, $-\frac{\partial \Psi(v_\varepsilon)}{\partial\delta} \ge 0$, it follows from (2.10)–(2.11),(1.10) and (2.1) that

$$\begin{aligned} I_1 := \int_{G_{\varepsilon^{1/2}}} \Big|\frac{\partial v_\varepsilon}{\partial \delta}\Big|^2 + \frac{W(v_\varepsilon)}{\varepsilon^2} &= -\frac{2}{\varepsilon}\int_{G_{\varepsilon^{1/2}}} \nabla\Psi(v_\varepsilon)\,\frac{\partial v_\varepsilon}{\partial\delta} \\ = \frac{2}{\varepsilon}\int_{G_{\varepsilon^{1/2}}} -\frac{\partial(\Psi(v_\varepsilon))}{\partial\delta} &\le \frac{2}{\varepsilon}\int_{\partial G}\int_0^{\varepsilon^{1/2}} -\frac{\partial(\Psi(v_\varepsilon))}{\partial\delta}(1+\tilde{c}\delta)\,d\delta\,d\sigma\,. \end{aligned} \tag{2.15}$$

Next, for each $\sigma \in \partial G$ we have

$$\begin{aligned} &\int_0^{\varepsilon^{1/2}} -\frac{\partial(\Psi(v_\varepsilon(\sigma,\delta)))}{\partial\delta}(1+\tilde{c}\delta)\,d\delta = \\ &\qquad \Psi(v_\varepsilon(\sigma,0)) - \Psi(v_\varepsilon(\sigma,\varepsilon^{1/2}))(1+\tilde{c}\varepsilon^{1/2}) + \int_0^{\varepsilon^{1/2}} \tilde{c}\,\Psi(v_\varepsilon(\sigma,\delta))\,d\delta \le C\,, \end{aligned} \tag{2.16}$$

where C is independent of σ and ε. An immediate consequence of (2.16) is that $I_1 \le \frac{C}{\varepsilon}$. In particular, $\int_{G_{\varepsilon^{1/2}}} W(v_\varepsilon) \le C\varepsilon$ and using (2.6) we obtain that

$$\int_{G_{\varepsilon^{1/2}}} \Psi(v_\varepsilon) \le C\varepsilon\,. \tag{2.17}$$

Integrating (2.16) over $\sigma \in \partial G$ and using (2.17) yields

$$I_1 \le \frac{2}{\varepsilon}\int_{\partial G} \Psi(g(\sigma))\,d\sigma + O(1). \tag{2.18}$$

It is easy to verify that $|\frac{\partial v_\varepsilon}{\partial\sigma}| \le C$ and therefore

$$I_2 := \int_{G_{\varepsilon^{1/2}}} \Big|\frac{\partial v_\varepsilon}{\partial\sigma}\Big|^2 \le C\,. \tag{2.19}$$

The result follows by combining (2.18)–(2.19) with (2.13)–(2.14). □

3 A lower-bound of the energy

In this section we shall prove the lower-bound part of Theorem 1. We begin with a simple lemma which provides two basic estimates satisfied by the minimizer u_ε. The proofs are very similar to those of the analogous results for the Ginzburg-Landau functional (see [4]).

Lemma 3.1. *There exist two positive constants $\widetilde{C}_1$ and $\widetilde{C}_2$ such that for every ε we have*

$$|u_\varepsilon| \le \widetilde{C}_1 \quad \text{in } G, \tag{3.1}$$

and

$$\|\nabla u_\varepsilon\|_{L^\infty(G)} \le \frac{\widetilde{C}_2}{\varepsilon}\,. \tag{3.2}$$

Proof. We claim that (3.1) holds with

$$\widetilde{C}_1 = \max(\max_{x\in\partial G}|g(x)|, R_0) \quad \text{(see (1.3))}.$$

Indeed, assuming by negation that (3.1) does not hold, we get easily from (1.3) that the function

$$\tilde{u}_\varepsilon(x) = \begin{cases} u_\varepsilon(x) & \text{if } |u_\varepsilon(x)| \le \widetilde{C}_1, \\ \widetilde{C}_1 \frac{u_\varepsilon(x)}{|u_\varepsilon(x)|} & \text{if } |u_\varepsilon(x)| > \widetilde{C}_1, \end{cases}$$

belongs to $H^1_g(G,\mathbb{R}^2)$ and satisfies $E_\varepsilon(\tilde{u}_\varepsilon) < E_\varepsilon(u_\varepsilon)$, contradicting the minimizing property of u_ε.

The Euler-Lagrange equation satisfied by u_ε is

$$\Delta u_\varepsilon = \big(\frac{1}{2\varepsilon^2}\big)\nabla W(u_\varepsilon)\,. \tag{3.3}$$

Applying standard elliptic estimates for the rescaled function $v_\varepsilon(y) = u_\varepsilon(\varepsilon y)$ (like in [4, 12]) yields the L^∞-bound for the gradient. □

The main result of this section is the following

Proposition 3.1. *There exists a constant $C > 0$ such that,*

$$E_\varepsilon(u_\varepsilon) \ge \frac{2}{\varepsilon}\int_{\partial G}\Psi(g(\sigma))\,d\sigma + D\,\frac{l^2(\Gamma)}{2\pi}\log\frac{1}{\varepsilon} - C\,,\ \forall\varepsilon\in(0,1). \tag{3.4}$$

The proof relies on several lemmas.

Lemma 3.2. *Let I be a compact subinterval of $(1/2,1)$. Then, there exist constants $a=a(I)\in(0,1)$ and $\mu=\mu(I), C=C(I)>0$ and, for any $\alpha\in I$ and $\varepsilon\in(0,1)$, a set $J=J_{\alpha,\varepsilon}\subset(0,C(I))$ satisfying meas$(J)\geq\mu$, such that*

$$\int_{G_{c_0\varepsilon^\alpha}} |\nabla u_\varepsilon|^2+\frac{W(u_\varepsilon)}{\varepsilon^2}\geq\frac{2}{\varepsilon}\int_{\partial G}\Psi(g(\sigma))\,d\sigma-C \tag{3.5}$$

and

$$\int_{\Sigma_{c_0\varepsilon^\alpha}}\Psi(u_\varepsilon(\sigma))\,d\sigma\leq C\varepsilon^{1+a}\,, \tag{3.6}$$

for all $c_0\in J$.

Proof. For $\alpha\in I$ and any $c>0$ we have by the Cauchy-Schwarz inequality

$$\begin{aligned}\int_{G_{c\varepsilon^\alpha}} |\nabla u_\varepsilon|^2+\frac{W(u_\varepsilon)}{\varepsilon^2}&\geq\frac{2}{\varepsilon}\int_{G_{c\varepsilon^\alpha}}|\nabla u_\varepsilon|\left(W(u_\varepsilon)\right)^{1/2}\\&\geq\frac{2}{\varepsilon}\int_{G_{c\varepsilon^\alpha}}|\nabla(\Psi(u_\varepsilon))|\geq\frac{2}{\varepsilon}\int_{G_{c\varepsilon^\alpha}}\nabla(\Psi(u_\varepsilon))\cdot V,\end{aligned}$$

for every C^1 vector field V such that $|V|\leq 1$ on $G_{c\varepsilon^\alpha}$. Choosing $V=-\nabla\delta$ yields,

$$\begin{aligned}\int_{G_{c\varepsilon^\alpha}} |\nabla u_\varepsilon|^2+\frac{W(u_\varepsilon)}{\varepsilon^2}&\geq\frac{2}{\varepsilon}\int_{\partial G}\Psi(g(\sigma))\,d\sigma-\frac{2}{\varepsilon}\int_{\Sigma_{c\varepsilon^\alpha}}\Psi(u_\varepsilon(\sigma))\,d\sigma-\frac{2}{\varepsilon}\int_{G_{c\varepsilon^\alpha}}\Psi(u_\varepsilon)\,\mathrm{div}\,V\\&:=I_1+I_2+I_3.\end{aligned} \tag{3.7}$$

By the upper-bound (2.12) we know that $\int_G W(u_\varepsilon)\leq C\varepsilon$, so that by (3.1) and (2.6) we deduce that $W(u_\varepsilon)\sim\Psi(u_\varepsilon)$ and therefore I_3 is bounded (uniformly in ε).

Using $\int_G W(u_\varepsilon)\leq C\varepsilon$ again we obtain that $\int_G\Psi(u_\varepsilon)\leq C\varepsilon$ and we deduce the existence of $c_1\in(0,1)$ such that

$$\int_{\Sigma_{c_1\varepsilon^\alpha}}\Psi(u_\varepsilon(\sigma))\,d\sigma\leq C\varepsilon^{1-\alpha}\,.$$

For $c=c_1$ we get $I_2\leq C\varepsilon^{-\alpha}$ and (3.7) now reads:

$$\int_{G_{c_1\varepsilon^\alpha}} |\nabla u_\varepsilon|^2+\frac{W(u_\varepsilon)}{\varepsilon^2}\geq\frac{2}{\varepsilon}\int_{\partial G}\Psi(g(\sigma))\,d\sigma-C\varepsilon^{-\alpha}\,.$$

Using the upper-bound again we obtain that

$$\int_{G\setminus G_{c_1\varepsilon^\alpha}} |\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon^2} \le C\varepsilon^{-\alpha}.$$

In particular, $\int_{G\setminus G_{c_1\varepsilon^\alpha}} W(u_\varepsilon) \le C\varepsilon^{2-\alpha}$ and there exists then $c_2 \in (1,2)$ such that $\int_{\Sigma_{c_2\varepsilon^\alpha}} W(u_\varepsilon(\sigma))\, d\sigma \le C\varepsilon^{2-2\alpha}$, and therefore also

$$\int_{\Sigma_{c_2\varepsilon^\alpha}} \Psi(u_\varepsilon(\sigma))\, d\sigma \le C\varepsilon^{2-2\alpha}.$$

This last estimate is then plugged back in (3.7) and the argument is repeated.

Let $n = n(\alpha)$ be such that

$$\frac{n-1}{n} \le \alpha < \frac{n}{n+1}\,. \tag{3.8}$$

Clearly, $\sup\{n(\alpha) \text{ satisfying (3.8)} : \alpha \in I\} < \infty$. Applying the above argument n times we obtain the existence of some $c_n \in (n-1, n)$ such that

$$\int_{G_{c_n\varepsilon^\alpha}} |\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon^2} \ge \frac{2}{\varepsilon}\int_{\partial G} \Psi(g(\sigma))\, d\sigma - C\varepsilon^{n-1-n\alpha}\,.$$

Using the upper-bound once more we get that

$$\int_{G\setminus G_{c_n\varepsilon^\alpha}} |\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon^2} \le C\big(\varepsilon^{n-1-n\alpha} + |\log\varepsilon|\big)\,,$$

which leads to

$$\int_{G\setminus G_{c_n\varepsilon^\alpha}} \Psi(u_\varepsilon) \le C\varepsilon^2(\varepsilon^{n-1-n\alpha} + |\log\varepsilon|)\,,$$

and to the existence of a $c_0 \in (n, n+1)$ such that

$$\int_{\Sigma_{c_0\varepsilon^\alpha}} \Psi(u_\varepsilon(\sigma))\, d\sigma \le C\varepsilon^{(n+1)(1-\alpha)} + C\varepsilon^{2-\alpha}|\log\varepsilon| \le C\varepsilon^{1+a}\,, \tag{3.9}$$

for any a satisfying

$$0 < a < \min(1-\alpha, (n+1)(1-\alpha) - 1)\,. \tag{3.10}$$

Clearly, an $a(I)$ satisfying (3.10) for all $\alpha \in I$ can be chosen. Furthermore, the measure of the set of c_0's in $(n, n+1)$ satisfying (3.9) is bounded from below, uniformly in $\alpha \in I \subset\subset (1/2, 1)$ and $\varepsilon \in (0,1)$, by some positive μ. We therefore proved (3.6), and using (3.9) in (3.7) with $c = c_0$ yields that $I_2 \le C$, and (3.5) follows as well. □

The next lemma provides a simple pointwise lower-bound for $|\nabla u_\varepsilon|$. We denote

$$G_0^\varepsilon := \{x \in G \,:\, u_\varepsilon(x) \in \Omega_{\lambda_0}\}\,. \tag{3.11}$$

Lemma 3.3. *We have*

$$|\nabla u_\varepsilon|^2 \geq \frac{|\nabla(\Psi(u_\varepsilon))|^2}{W(u_\varepsilon)} \quad \textit{a.e. in } G \tag{3.12}$$

and

$$|\nabla u_\varepsilon|^2 \geq \beta|\nabla(\tilde{s}(u_\varepsilon))|^2 + \frac{|\nabla(\Psi(u_\varepsilon))|^2}{W(u_\varepsilon)} \quad \textit{in } G_0^\varepsilon\,, \tag{3.13}$$

for some $\beta > 0$, independent of ε.

Proof. The estimate (3.12) follows immediately from the inequality

$$|\nabla(\Psi(u_\varepsilon))|^2 = |\nabla\Psi(u_\varepsilon)\cdot\nabla u_\varepsilon|^2 \leq |\nabla\Psi(u_\varepsilon)|^2\,|\nabla u_\varepsilon|^2 = W(u_\varepsilon)|\nabla u_\varepsilon|^2\,, \text{ a.e. in } G\,. \tag{3.14}$$

Next, at each point $y \in \Omega_{\lambda_0}$ we denote by $\vec{\nu} = \vec{\nu}(y)$ a unit vector in the direction of $\nabla\Psi(y)$ and by $\vec{\tau} = \vec{\tau}(y)$ an orthogonal unit vector, in the direction of $\nabla\tilde{s}(y)$. For $x \in G_0^\varepsilon$ we may write then

$$|\nabla u_\varepsilon|^2 = |\nabla_\nu u_\varepsilon|^2 + |\nabla_\tau u_\varepsilon|^2\,, \tag{3.15}$$

and get a more precise form of (3.14) in this case:

$$|\nabla_\nu u_\varepsilon|^2 = \frac{|\nabla(\Psi(u_\varepsilon))|^2}{W(u_\varepsilon)}\,. \tag{3.16}$$

Finally, since

$$\nabla_\tau u_\varepsilon = \nabla u_\varepsilon \cdot \frac{\nabla\tilde{s}(u_\varepsilon)}{|\nabla\tilde{s}(u_\varepsilon)|} = \frac{\nabla(\tilde{s}(u_\varepsilon))}{|\nabla\tilde{s}(u_\varepsilon)|}$$

and $|\nabla\tilde{s}(y)| \leq \frac{1}{\sqrt{\beta}}, \forall y \in \Omega_{\lambda_0}$ for some $\beta > 0$, we conclude that

$$|\nabla_\tau u_\varepsilon|^2 \geq \beta|\nabla(\tilde{s}(u_\varepsilon))|^2\,,$$

as required. □

In the sequel we shall fix an interval $I \subset\subset (1/2, 1)$ and consider $\alpha \in I$. An important role in the proof of Proposition 3.1 is played by the scalar function $d_{0\varepsilon}$ which is defined as the minimizer for the problem

$$\min\{\int_{G_{c_0\varepsilon^\alpha}} \frac{|\nabla d|^2}{W(u_\varepsilon)+\varepsilon^2} \,:\, d \in H^1(G_{c_0\varepsilon^\alpha}),\, d = \Psi(u_\varepsilon) \text{ on } \partial G_{c_0\varepsilon^\alpha}\}\,. \tag{3.17}$$

The existence and uniqueness of $d_{0\varepsilon}$ is standard. The Euler-Lagrange equation satisfied by $d_{0\varepsilon}$ is

$$\begin{cases} \operatorname{div}\left(\frac{\nabla d_{0\varepsilon}}{W(u_\varepsilon)+\varepsilon^2}\right) = 0 \ \text{ in } \ G_{c_0\varepsilon^\alpha}\,, \\ d_{0\varepsilon} = \Psi(u_\varepsilon) \text{ on } \partial G_{c_0\varepsilon^\alpha}\,, \end{cases} \tag{3.18}$$

for any $c_0 \in J$ (see Lemma 3.2).

Lemma 3.4. *There exists $\varepsilon_0 > 0$ such that for every $\varepsilon \le \varepsilon_0$ and $\alpha \in I$ we have:*

$$0 \le d_{0\varepsilon} \le \lambda_1 \ \ \textit{in } G_{c_0\varepsilon^\alpha} \quad \textit{(see (1.14))}. \tag{3.19}$$

Proof. It is enough to show that

$$d_{0\varepsilon} = \Psi(u_\varepsilon) \le \lambda_1 \ \text{ on } \partial G_{c_0\varepsilon^\alpha} \text{ (for } \varepsilon \text{ small)}, \tag{3.20}$$

and then apply the maximum principle to (3.18). The inequality on ∂G is clear from (1.14). As for the bound on $\Sigma_{c_0\varepsilon^\alpha} = \{x \in G : \delta(x) = c_0\varepsilon^\alpha\}$, note first that by (3.6) and (2.6) we have

$$\int_{\Sigma_{c_0\varepsilon^\alpha}} W(u_\varepsilon) \le C\varepsilon^{1+a}\,.$$

Let $x_0 \in \Sigma_{c_0\varepsilon^\alpha}$ satisfy

$$|\tilde{\delta}(u_\varepsilon(x_0))| = m := \max_{\Sigma_{c_0\varepsilon^\alpha}} |\tilde{\delta}(u_\varepsilon(x))|\,.$$

By (3.2) we have:

$$|\tilde{\delta}(u_\varepsilon(x))| \ge m/2 \text{ for every } x \in \Sigma_{c_0\varepsilon^\alpha} \text{ s.t. } |x_0 - x| \le \frac{m\varepsilon}{2c}\,,$$

for some $c > 0$. For such x we obtain, for some $a_0 > 0$ (see (2.4)):

$$W(u_\varepsilon(x)) \ge a_0\tilde{\delta}^2(u_\varepsilon(x)) \ge \frac{a_0 m^2}{4},$$

so that, for ε small,

$$a_0\frac{m^3\varepsilon}{8c} = \frac{m\varepsilon}{2c}\cdot\frac{a_0 m^2}{4} \le \int_{\Sigma_{c_0\varepsilon^\alpha}} W(u_\varepsilon) \le C\varepsilon^{1+a},$$

which leads to $m \le \varepsilon^b$ for some $b > 0$. Therefore, also $\Psi(u_\varepsilon) \le CW(u_\varepsilon) \le C\varepsilon^{2b} < \lambda_1$ on $\Sigma_{c_0\varepsilon^\alpha}$, for ε small enough, and (3.20) follows. □

By the definition of $d_{0\varepsilon}$, Lemma 3.3 and the upper-bound (2.12) we deduce that

$$\int_{G_{c_0\varepsilon^\alpha}} \frac{|\nabla d_{0\varepsilon}|^2}{W(u_\varepsilon)+\varepsilon^2} \le \int_{G_{c_0\varepsilon^\alpha}} \frac{|\nabla(\Psi(u_\varepsilon))|^2}{W(u_\varepsilon)+\varepsilon^2} \le \frac{C}{\varepsilon}\,. \tag{3.21}$$

Put $d_{1\varepsilon} = \Psi(u_\varepsilon) - d_{0\varepsilon}$. Then,

$$\frac{|\nabla(\Psi(u_\varepsilon))|^2}{W(u_\varepsilon)+\varepsilon^2} = \frac{1}{W(u_\varepsilon)+\varepsilon^2}(|\nabla d_{0\varepsilon}|^2 + 2\nabla d_{0\varepsilon}.\nabla d_{1\varepsilon} + |\nabla d_{1\varepsilon}|^2)\,. \tag{3.22}$$

The motivation for introducing $d_{1\varepsilon}$ is the following simple consequence of (3.22), (3.18) and Green's formula:

$$\int_{G_{c_0\varepsilon^\alpha}} \frac{|\nabla(\Psi(u_\varepsilon))|^2}{W(u_\varepsilon)+\varepsilon^2} = \int_{G_{c_0\varepsilon^\alpha}} \frac{|\nabla d_{0\varepsilon}|^2}{W(u_\varepsilon)+\varepsilon^2} + \frac{|\nabla d_{1\varepsilon}|^2}{W(u_\varepsilon)+\varepsilon^2}\,. \tag{3.23}$$

In fact, from (3.16) and (3.23) we conclude that

$$\int_{G_{c_0\varepsilon^\alpha}} |\nabla_\nu u_\varepsilon|^2 \ge \int_{G_{c_0\varepsilon^\alpha}} \frac{|\nabla d_{0\varepsilon}|^2}{W(u_\varepsilon)+\varepsilon^2} + \frac{|\nabla d_{1\varepsilon}|^2}{W(u_\varepsilon)+\varepsilon^2}\,. \tag{3.24}$$

The next lemma provides a crucial lower-bound for the first term on the r.h.s. of (3.24).

Lemma 3.5. *There exists a constant $C_1 = C_1(I)$ such that*

$$\int_{G_{c_0\varepsilon^\alpha}} \frac{|\nabla d_{0\varepsilon}|^2}{W(u_\varepsilon)+\varepsilon^2} + \frac{W(u_\varepsilon)+\varepsilon^2}{\varepsilon^2} \ge \frac{2}{\varepsilon}\int_{\partial G} \Psi(g(\sigma))\,d\sigma - C_1,\ \ \forall\varepsilon\in(0,1),\ \forall\alpha\in I\,.$$

Proof. As in the proof of Lemma 3.2 we have for $V = -\nabla\delta$:

$$\begin{aligned}
\int_{G_{c_0\varepsilon^\alpha}} \frac{|\nabla d_{0\varepsilon}|^2}{W(u_\varepsilon)+\varepsilon^2} + \frac{W(u_\varepsilon)+\varepsilon^2}{\varepsilon^2} &\ge \frac{2}{\varepsilon}\int_{G_{c_0\varepsilon^\alpha}} |\nabla d_{0\varepsilon}| \ge \frac{2}{\varepsilon}\int_{G_{c_0\varepsilon^\alpha}} \nabla d_{0\varepsilon}\cdot V \\
&= \frac{2}{\varepsilon}\int_{\partial G_{c_0\varepsilon^\alpha}} \Psi(u_\varepsilon(\sigma))\,d\sigma - \frac{2}{\varepsilon}\int_{G_{c_0\varepsilon^\alpha}} d_{0\varepsilon}\operatorname{div} V \\
&\ge \frac{2}{\varepsilon}\int_{\partial G} \Psi(g(\sigma))\,d\sigma - \frac{2}{\varepsilon}\int_{G_{c_0\varepsilon^\alpha}} d_{0\varepsilon}\operatorname{div} V - C\,,
\end{aligned} \tag{3.25}$$

where in the last inequality we used (3.6). Therefore, in order to conclude we only need to prove that

$$\int_{G_{c_0\varepsilon^\alpha}} d_{0\varepsilon} \le C\varepsilon\,. \tag{3.26}$$

Fix any $\delta \in (0, c_0\varepsilon^\alpha)$ and let $\delta_0 = \delta/5$. By the upper-bound (2.12) there exists $\delta_1 \in (\delta_0, 2\delta_0)$ such that

$$\int_{\Sigma_{\delta_1}} W(u_\varepsilon) \le \frac{C\varepsilon}{\delta}\,.$$

By the same argument as in the proof of Lemma 3.2 we get that

$$\int_{G\setminus G_{\delta_1}} W(u_\varepsilon) \le \frac{C\varepsilon^2}{\delta} + C\varepsilon^2|\log\varepsilon|\,.$$

Repeating the argument we find $\delta_2 \in (2\delta_0, 3\delta_0)$ such that

$$\int_{G\setminus G_{\delta_2}} W(u_\varepsilon) \le \frac{C\varepsilon^3}{\delta^2} + C\varepsilon^2|\log\varepsilon|\,.$$

Repeating the argument one last time we deduce that there exists $\delta_3 \in (3\delta_0, 4\delta_0)$ such that

$$\int_{G\setminus G_{\delta_3}} W(u_\varepsilon) \le \frac{C\varepsilon^4}{\delta^3} + C\varepsilon^2|\log\varepsilon|\,.$$

Hence, for any $\delta \in (\varepsilon, c_0\varepsilon^\alpha)$ we have,

$$\int_{G\setminus G_{\delta}} W(u_\varepsilon) \le \frac{C\varepsilon^4}{\delta^3} + C\varepsilon^2|\log\varepsilon|. \tag{3.27}$$

Next, using (2.1) and (3.6) we obtain

$$\begin{aligned}\int_{G_{c_0\varepsilon^\alpha}\setminus G_\varepsilon} d_{0\varepsilon} &\le C\int_\varepsilon^{c_0\varepsilon^\alpha}\Big(\int_{\partial G} d_{0\varepsilon}(\sigma,\delta)\,d\sigma\Big)\,d\delta\\ &\le C\int_\varepsilon^{c_0\varepsilon^\alpha}\Big(\int_{\partial G}\big(d_{0\varepsilon}(\sigma,c_0\varepsilon^\alpha)+\int_\delta^{c_0\varepsilon^\alpha}|\nabla d_{0\varepsilon}(\sigma,t)|\,dt\big)\,d\sigma\Big)\,d\delta \qquad (3.28)\\ &\le C\int_\varepsilon^{c_0\varepsilon^\alpha}\Big(\int_{G\setminus G_\delta}|\nabla d_{0\varepsilon}|\Big)\,d\delta + C\varepsilon^{1+a+\alpha}\,.\end{aligned}$$

By the Cauchy-Schwarz inequality, (3.21) and (3.27) we get:

$$\begin{aligned}\int_\varepsilon^{c_0\varepsilon^\alpha}\Big(\int_{G\setminus G_\delta}|\nabla d_{0\varepsilon}|\Big)\,d\delta &\le \int_\varepsilon^{c_0\varepsilon^\alpha}\Big(\int_{G\setminus G_\delta}\frac{|\nabla d_{0\varepsilon}|^2}{W(u_\varepsilon)+\varepsilon^2}\Big)^{1/2}\Big(\int_{G\setminus G_\delta}W(u_\varepsilon)+\varepsilon^2\Big)^{1/2}\,d\delta\\ &\le \frac{C}{\varepsilon^{1/2}}\int_\varepsilon^{c_0\varepsilon^\alpha}\Big(\int_{G\setminus G_\delta}W(u_\varepsilon)+\varepsilon^2\Big)^{1/2}\,d\delta\\ &\le \frac{C}{\varepsilon^{1/2}}\int_\varepsilon^{c_0\varepsilon^\alpha}\big(\frac{\varepsilon^2}{\delta^{3/2}}+\varepsilon|\log\varepsilon|^{1/2}+\varepsilon\big)\,d\delta\\ &\le -C\varepsilon^{3/2}\cdot s^{-1/2}\big]_\varepsilon^{c_0\varepsilon^\alpha} + c\varepsilon^{\alpha+1/2}|\log\varepsilon|^{1/2}+C\varepsilon^{1/2+\alpha}\le C\varepsilon\,.\end{aligned} \tag{3.29}$$

Combining (3.29) with (3.28) we obtain that

$$\int_{G_{c_0\varepsilon^\alpha}\setminus G_\varepsilon} d_{0\varepsilon} \le C\varepsilon\,.$$

On the other hand, the inequality

$$\int_{G_\varepsilon} d_{0\varepsilon} \le C\varepsilon$$

is obvious since $|G_\varepsilon| = O(\varepsilon)$ and $d_{0\varepsilon} \le \lambda_1$ by (3.19) (if $\varepsilon \le \varepsilon_0$, otherwise the result is clear). This completes the proof of (3.26) and the result of the lemma follows. □

The next proposition establishes a lower-bound for the energy on $G_{c_0\varepsilon^\alpha}$ which is the basis for the proof of Proposition 3.1.

Proposition 3.2. *There exist positive constants $K = K(I)$ and $C_2 = C_2(I)$ such that for every $\varepsilon \in (0,1)$,$\alpha \in I$ and $c_0 \in J_{\alpha,\varepsilon}$ (see Lemma 3.2) there holds*

$$\int_{G_{c_0\varepsilon^\alpha}} |\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon^2} \ge \frac{2}{\varepsilon}\int_{\partial G} \Psi(g(\sigma))\,d\sigma + K\int_{\partial G} \frac{|u_\varepsilon(\sigma, c_0\varepsilon^\alpha) - \tilde{s}(g(\sigma))|^2}{\varepsilon^\alpha}\,d\sigma - C_2\,. \tag{3.30}$$

Proof. By Lemma 3.5 and (3.23) we have

$$\int_{G_{c_0\varepsilon^\alpha}} \frac{|\nabla\Psi(u_\varepsilon)|^2}{W(u_\varepsilon)} + \frac{W(u_\varepsilon)}{\varepsilon^2} \ge \frac{2}{\varepsilon}\int_{\partial G} \Psi(g(\sigma))\,d\sigma + \int_{G_{c_0\varepsilon^\alpha}} \frac{|\nabla d_{1\varepsilon}|^2}{W(u_\varepsilon)+\varepsilon^2} - C\,.$$

Combining it with (3.12)–(3.13) yields

$$\begin{aligned}\int_{G_{c_0\varepsilon^\alpha}} |\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon^2} \ge{}& \frac{2}{\varepsilon}\int_{\partial G} \Psi(g(\sigma))\,d\sigma \\ &+ \int_{G_{c_0\varepsilon^\alpha}} \frac{|\nabla d_{1\varepsilon}|^2}{W(u_\varepsilon)+\varepsilon^2} + \beta\int_{G_{c_0\varepsilon^\alpha}\cap G_0^\varepsilon} |\nabla(\tilde{s}(u_\varepsilon))|^2 - C\,.\end{aligned} \tag{3.31}$$

Fix any $\sigma_0 \in \partial G$. We distinguish two cases.

Case 1: For all $\delta \in (0, c_0\varepsilon^\alpha)$ we have $d_{1\varepsilon}(\sigma_0,\delta) \le \lambda_0 - \lambda_1$ (see (1.14)).

In this case, since $d_{0\varepsilon}(\sigma_0,\delta) \le \lambda_1$ by (3.19), we have $(\sigma_0,\delta) \in G_0^\varepsilon$ for every $\delta \in (0, c_0\varepsilon^\alpha)$ (see (3.11)). Using the Cauchy-Schwarz inequality we get

$$\beta\int_0^{c_0\varepsilon^\alpha} |\nabla(\tilde{s}(u_\varepsilon(\sigma_0,\delta)))|^2\,d\delta \ge C\frac{|\tilde{s}(g(\sigma_0)) - \tilde{s}(u_\varepsilon(\sigma_0, c_0\varepsilon^\alpha))|^2}{\varepsilon^\alpha}\,.$$

By (2.6),

$$|u_\varepsilon(\sigma_0, c_0\varepsilon^\alpha) - \tilde{s}(u_\varepsilon(\sigma_0, c_0\varepsilon^\alpha))|^2 = O(\Psi(u_\varepsilon(\sigma_0, c_0\varepsilon^\alpha))) = O(W(u_\varepsilon(\sigma_0, c_0\varepsilon^\alpha)))\,.$$

So in this case we obtain, for some constants $K_0, K_1 > 0$:

$$\beta \int_0^{c_0\varepsilon^\alpha} |\nabla(\tilde{s}(u_\varepsilon(\sigma_0,\delta)))|^2\, d\delta \geq K_0 \frac{|u_\varepsilon(\sigma_0, c_0\varepsilon^\alpha) - \tilde{s}(g(\sigma_0))|^2}{\varepsilon^\alpha} - K_1 \frac{W(u_\varepsilon(\sigma_0, c_0\varepsilon^\alpha))}{\varepsilon^\alpha}. \tag{3.32}$$

<u>Case 2:</u> There exists $\delta' \in (0, c_0\varepsilon^\alpha)$ such that $d_{1\varepsilon}(\sigma_0, \delta') > \lambda_0 - \lambda_1$.

In this case, since u_ε is bounded thanks to (3.1), we obtain again by the Cauchy-Schwarz inequality that

$$\begin{aligned} \int_0^{c_0\varepsilon^\alpha} \frac{|\nabla d_{1\varepsilon}(\sigma_0,\delta)|^2\, d\delta}{W(u_\varepsilon(\sigma_0,\delta)) + \varepsilon^2} &\geq c \int_0^{\delta'} |\nabla d_{1\varepsilon}(\sigma_0,\delta)|^2\, d\delta \geq c\, \frac{d_{1,\varepsilon}^2(\sigma_0,\delta')}{\delta'} \\ &\geq c\, \frac{(\lambda_0-\lambda_1)^2}{c_0\varepsilon^\alpha} \geq K_2\, \frac{|u_\varepsilon(\sigma_0, c_0\varepsilon^\alpha) - \tilde{s}(g(\sigma_0))|^2}{\varepsilon^\alpha}\,, \end{aligned} \tag{3.33}$$

for some $K_2 > 0$.

Integration over $\sigma_0 \in \partial G$ of either (3.32) or (3.33) yields for some positive constants K and $\tilde{K}_1$:

$$\begin{aligned} &\int_{G_{c_0\varepsilon^\alpha}} \frac{|\nabla d_{1\varepsilon}|^2}{W(u_\varepsilon) + \varepsilon^2} + \beta \int_{G_{c_0\varepsilon^\alpha} \cap G_0^\varepsilon} |\nabla(\tilde{s}(u_\varepsilon))|^2 \geq \\ &\qquad K \int_{\partial G} \frac{|u_\varepsilon(\sigma, c_0\varepsilon^\alpha) - \tilde{s}(g(\sigma))|^2}{\varepsilon^\alpha}\, d\sigma - \tilde{K}_1 \int_{\partial G} \frac{W(u_\varepsilon(\sigma, c_0\varepsilon^\alpha))}{\varepsilon^\alpha}\, d\sigma := I_1 - I_2\,. \end{aligned} \tag{3.34}$$

Since I_2 is bounded thanks to (3.6), the result of the lemma follows from (3.31) and (3.34). □

Proof of Proposition 3.1. By (1.1)–(1.3) we get that there exists a constant $\kappa > 0$ such that

$$W(z) \geq \kappa \tilde{\delta}^2(z), \quad \forall z \in \mathbb{R}^2\,. \tag{3.35}$$

Therefore, from Proposition 3.2 we obtain for any $\alpha \in I \subset\subset (1/2, 1)$ that

$$\begin{aligned} &E_\varepsilon(u_\varepsilon) - \frac{2}{\varepsilon} \int_{\partial G} \Psi(g(\sigma))\, d\sigma \geq \\ &\qquad \int_{G \setminus G_{c_0\varepsilon^\alpha}} |\nabla u_\varepsilon|^2 + \kappa\, \frac{\tilde{\delta}^2(u_\varepsilon)}{\varepsilon^2} + K \int_{\partial G} \frac{|u_\varepsilon(\sigma, c_0\varepsilon^\alpha) - \tilde{s}(g(\sigma))|^2}{\varepsilon^\alpha}\, d\sigma - C_2\,, \end{aligned} \tag{3.36}$$

with c_0, C_2 and K given by Proposition 3.2. By the same proof as that of [2, Proposition 5.1], which deals with the case of $W(u) = F(\tilde{\delta}(u))$ (here $F(t) = \kappa t^2$), we conclude that

$$\int_{G \setminus G_{c_0\varepsilon^\alpha}} |\nabla u_\varepsilon|^2 + \kappa\, \frac{\tilde{\delta}^2(u_\varepsilon)}{\varepsilon^2} + K \int_{\partial G} \frac{|u_\varepsilon(\sigma, c_0\varepsilon^\alpha) - \tilde{s}(g(\sigma))|^2}{\varepsilon^\alpha}\, d\sigma \geq D\, \frac{l^2(\Gamma)}{2\pi} \log \frac{1}{\varepsilon} - C\,. \tag{3.37}$$

The conclusion of Proposition 3.1 follows from (3.36) and (3.37). □

Proof of Theorem 1. It suffices to combine the upper-bound of Proposition 2.1 with the lower-bound of Proposition 3.1. □

4 Convergence of u_ε

The proof of the convergence result Theorem 2 is very similar to that of [2, Theorem 2]. Therefore we shall go over the main steps of the proof, underlying only the modifications of the arguments of [2] which are needed. The next lemma is analogous to [2, Lemma 6.1]. We shall use the notations of Lemma 3.2.

Lemma 4.1. *Let I be a compact subinterval of $(\frac{1}{2},1)$. Then, there exists a constant $C_0>0$ such that for every $\varepsilon\in(0,1)$, $\alpha\in I$ and $c_0\in J_{\alpha,\varepsilon}$ we have*

$$\int_{\partial G}\frac{|u_\varepsilon(\sigma,c_0\varepsilon^\alpha)-\tilde{s}(g(\sigma))|^2}{\varepsilon^\alpha}\,d\sigma\le C_0\,. \tag{4.1}$$

Furthermore, setting

$$c_1=c_1(\alpha,\varepsilon)=\inf\{c_0\in J_{\alpha,\varepsilon}\}\quad\text{and}\quad c_2=c_2(\alpha,\varepsilon)=\sup\{c_0\in J_{\alpha,\varepsilon}\}\,, \tag{4.2}$$

we have

$$\int_{G_{c_2\varepsilon^\alpha}\setminus G_{c_1\varepsilon^\alpha}}|\nabla u_\varepsilon|^2+\int_{G\setminus G_{c_1\varepsilon^\alpha}}W(u_\varepsilon)\le C_1\,. \tag{4.3}$$

Proof. By the upper-bound (2.12) and (3.36) we get that

$$\int_{G\setminus G_{c_0\varepsilon^\alpha}}|\nabla u_\varepsilon|^2+\kappa\frac{\tilde{\delta}^2(u_\varepsilon)}{\varepsilon^2}+K\int_{\partial G}\frac{|u_\varepsilon(\sigma,c_0\varepsilon^\alpha)-\tilde{s}(g(\sigma))|^2}{\varepsilon^\alpha}\,d\sigma\le D\,\frac{l^2(\Gamma)}{2\pi}\log\frac{1}{\varepsilon}+C,$$
$$\forall c_0\in J_{\alpha,\varepsilon},\ \forall\varepsilon\in(0,1)\,. \tag{4.4}$$

The proof of (3.37) gives the same estimate with K being replaced by $K/2$, namely

$$\int_{G\setminus G_{c_0\varepsilon^\alpha}}|\nabla u_\varepsilon|^2+\kappa\frac{\tilde{\delta}^2(u_\varepsilon)}{\varepsilon^2}+\frac{K}{2}\int_{\partial G}\frac{|u_\varepsilon(\sigma,c_0\varepsilon^\alpha)-\tilde{s}(g(\sigma))|^2}{\varepsilon^\alpha}\,d\sigma\ge D\,\frac{l^2(\Gamma)}{2\pi}\log\frac{1}{\varepsilon}-C\,. \tag{4.5}$$

Combining (4.4) with (4.5) we are led to (4.1).

Similarly, since (3.37) remains valid when κ is replaced by $\kappa/2$, we conclude that

$$\int_{G\setminus G_{c_1\varepsilon^\alpha}}\frac{W(u_\varepsilon)}{\varepsilon^2}\le c\int_{G\setminus G_{c_1\varepsilon^\alpha}}\frac{\tilde{\delta}^2(u_\varepsilon)}{\varepsilon^2}\le C\,,$$

with c_1 given by (4.2) (we used (4.1),(3.35) and (3.1)). Finally, to prove the estimate for the first term on the l.h.s. of (4.3), it suffices to notice that the above arguments imply that

$$D\,\frac{l^2(\Gamma)}{2\pi}\log\frac{1}{\varepsilon}-C\le\int_{G\setminus G_{c_2\varepsilon^\alpha}}|\nabla u_\varepsilon|^2\le\int_{G\setminus G_{c_1\varepsilon^\alpha}}|\nabla u_\varepsilon|^2\le D\,\frac{l^2(\Gamma)}{2\pi}\log\frac{1}{\varepsilon}+C\,.$$

□

Fix any interval $I\subset\subset(1/2,1)$ and then choose any $\alpha\in I$. Since for every $\varepsilon\in(0,1)$ we have $c_2-c_1\ge\mu$ (see (4.2) and Lemma 3.2), it follows from (4.3) and Fubini's theorem that there exists $t_\varepsilon\in(c_1\varepsilon^\alpha,c_2\varepsilon^\alpha)$ such that

$$\frac{t_\varepsilon}{\varepsilon^\alpha}\in J_{\alpha,\varepsilon}\ \text{ and }\ \int_{\Sigma_{t_\varepsilon}}|\nabla u_\varepsilon|^2\le\frac{C}{\varepsilon^\alpha}\,. \tag{4.6}$$

Following [2], we shall next define a set of "bad points" $\widetilde{S}_\varepsilon$ (a notion that was introduced originally in [5]). We fix a small positive m satisfying

$$\{z\in\mathbb{R}^2,\ |\tilde\delta(z)|<m\}\subset\Omega_{\lambda_0}\ \text{(see (1.11))}\,,$$

and set

$$\widetilde{S}_\varepsilon=\{x\in G\setminus G_{t_\varepsilon}:|\tilde\delta(u_\varepsilon(x))|>m\}\cup\{x\in\Sigma_{t_\varepsilon}\ :\ |u_\varepsilon(x)-\tilde s(g(\sigma(x)))|>m\}$$
$$:=\widetilde{S}_{\varepsilon,1}\cup\widetilde{S}_{\varepsilon,2}\,. \tag{4.7}$$

Our next objective is to cover the set $\widetilde{S}_{\varepsilon,1}$ by a finite collection of discs/"half discs" with radii of the order ε, and to cover the set $\widetilde{S}_{\varepsilon,2}$ by a finite collection of "half discs" with radii of the order ε^α. A "half disc" is by definition a set of the form $B^+(\sigma,r)=B(\sigma,r)\cap(G\setminus G_{t_\varepsilon})$, for some $\sigma\in\Sigma_{t_\varepsilon}$ and $r>0$. We shall denote for $x\in G\setminus G_{t_\varepsilon}$: $\delta_1(x)=\operatorname{dist}(x,\Sigma_{t_\varepsilon})$. As in [2, Lemma 6.2] we have

Lemma 4.2. *There exist a constant $k_1>0$ and an integer N_1, both independent of ε, such that, the collection of mutually disjoint discs/half discs $\{B(x_i^\varepsilon,\lambda_\varepsilon\varepsilon)\cap(G\setminus G_{t_\varepsilon})\}_{i=1}^{N_1(\varepsilon)}$ satisfies*

$$\widetilde{S}_{\varepsilon,1}\subset\bigcup_{i=1}^{N_1(\varepsilon)}B(x_i^\varepsilon,\lambda_\varepsilon\varepsilon)\cap(G\setminus G_{t_\varepsilon}), \tag{4.8}$$

with $\lambda_\varepsilon\le k_1$ and $N_1(\varepsilon)\le N_1$, such that for each i either $\delta_1(x_i^\varepsilon)\ge2\lambda_\varepsilon\varepsilon$ or $x_i^\varepsilon\in\partial\Sigma_{t_\varepsilon}$. Moreover, there exist k_2 and N_2 (independent of ε), and for each $\varepsilon>0$, a collection of *mutually disjoint half bad discs* $\{B^+(y_j^\varepsilon,\gamma_\varepsilon t_\varepsilon)\}_{j=1}^{N_2(\varepsilon)}$*, with* $y_j^\varepsilon\in\Sigma_{t_\varepsilon}$*, that satisfies*

$$\widetilde{S}_{\varepsilon,2}\subset\bigcup_{j=1}^{N_2(\varepsilon)}B^+(y_j^\varepsilon,\gamma_\varepsilon t_\varepsilon), \tag{4.9}$$

with $\gamma_\varepsilon\le k_2$ *and* $N_2(\varepsilon)\le N_2$*. Further, we can construct such collections with the property that for each* $1\le i\le N_1(\varepsilon)$ *and* $1\le j\le N_2(\varepsilon)$*,*

$$\text{either } B(x_i^\varepsilon,\lambda_\varepsilon\varepsilon)\cap B^+(y_j^\varepsilon,\gamma_\varepsilon t_\varepsilon)=\emptyset \quad\text{or}\quad B(x_i^\varepsilon,\lambda_\varepsilon\varepsilon)\cap(G\setminus G_{t_\varepsilon})\subset B^+(y_j^\varepsilon,\gamma_\varepsilon t_\varepsilon). \tag{4.10}$$

The proof is identical to the one given in [2], so we omit it. We just mention that in order to prove that $\widetilde{S}_{\varepsilon,2}$ can be covered by a bounded number (i.e. uniformly in ε) of half bad discs $\{B^+(y_j^\varepsilon,\gamma_\varepsilon t_\varepsilon)\}_{j=1}^{N_2(\varepsilon)}$ we need to know that there exists $\mu_2>0$ such that

$$x\in\widetilde{S}_{\varepsilon,2}\implies |u_\varepsilon(y)-\tilde{g}(y)|>m/2,\quad \forall y\in\Sigma_{t_\varepsilon}\text{ with }|y-x|\le\mu_2 t_\varepsilon. \tag{4.11}$$

The proof of (4.11) follows from the estimate

$$|u_\varepsilon(x)-u_\varepsilon(y)|^2\le C\Big(\int_{\Sigma_{t_\varepsilon}}|\nabla u_\varepsilon|^2\Big)|x-y|\le\Big(\frac{C}{t_\varepsilon}\Big)|x-y|,\quad \forall x,y\in\Sigma_{t_\varepsilon}\,,$$

which is a consequence of the Cauchy-Schwarz inequality and the choice of t_ε in (4.6).

Next, as in [2] we show that the degree of u_ε on the boundary of each of the discs/half discs is bounded, uniformly in ε (see [2] for definition of the degree on the boundary of a half-disc). Then, we can identify the limits of the centers of the discs/half discs (for a subsequence $\varepsilon_n\to 0$) and obtain the following result, whose proof is identical to that of [2, Proposition 6.1] (see in particular (6.30) in [2]).

Proposition 4.1. *For a subsequence* $\varepsilon_n\to 0$*, there exist* D *distinct points* $a_1,\dots,a_D\in G$ *such that the degree of* u_{ε_n} *around each* a_j *is* 1 *and for every* r *satisfying*

$$0<r<\frac{1}{2}\min\big(\min\{|a_i-a_j|\,:\,i\ne j\},\min\{\operatorname{dist}(a_j,\partial G)\,:\,j=1,\dots,D\}\big)\,, \tag{4.12}$$

we have

$$E_{\varepsilon_n}\Big(u_{\varepsilon_n},G\setminus\big(G_r\cup\bigcup_{j=1}^{D}B(a_j,r)\big)\Big)\le C(r)\,. \tag{4.13}$$

Now we are ready to complete the proof of Theorem 2.

Proof of Theorem 2. By (4.13) we get (possibly after passing to a further subsequence) that

$$u_{\varepsilon_n} \rightharpoonup u_* \text{ weakly in } H^1_{\rm loc}(G \setminus \{a_1, \dots, a_D\}) \,, \tag{4.14}$$

for some $u_* \in H^1_{\rm loc}(G \setminus \{a_1, \dots, a_D\}, \mathit{\Gamma})$. In order to obtain strong convergence in $H^1_{\rm loc}(G \setminus \{a_1, \dots, a_D\})$ we fix any disc $B(x_0, r) \subset\subset G \setminus \{a_1, \dots, a_D\}$. By (4.13) we have $E_{\varepsilon_n}(u_{\varepsilon_n}, B(x_0, r)) \le C$. Using Fubini's theorem we can find $r_0 \in (r/2, r)$ such that (by passing to a subsequence if necessary),

$$\int_{\partial B(x_0,r_0)} |\nabla u_{\varepsilon_n}|^2 + \frac{W(u_{\varepsilon_n})}{\varepsilon_n^2} \le C, \ \forall n \,.$$

Applying the argument of [4] yields that

$$u_{\varepsilon_n} \to u_* \text{ in } H^1(B(x_0, r_0)) \quad \text{and} \quad \tilde{\delta}(u_{\varepsilon_n}) \to 0 \text{ uniformly on } \overline{B(x_0, r_0)}. \tag{4.15}$$

Moreover, u_* is a $\mathit{\Gamma}$-valued map which is a local minimizer of the Dirichlet energy among such maps in $G \setminus \{a_1, \dots, a_D\}$. We can write then, locally in $G \setminus \{a_1, \dots, a_D\}$, $u_* = \tau(e^{i\phi_*})$ (see (1.17)) where ϕ_* is an harmonic function.

Next we shall apply the argument of Steps A.3 and A.4 from the proof of [4, Theorem 1], and more specifically the variant in [2], to show that

$$\{u_{\varepsilon_n}\} \text{ is bounded in } H^2_{\rm loc}(B(x_0, r_0)) \quad \text{and} \quad \{\nabla u_{\varepsilon_n}\} \text{ is bounded in } L^\infty_{\rm loc}(B(x_0, r_0)). \tag{4.16}$$

We first introduce a new coordinate system in a neighborhood of $\mathit{\Gamma}$. From our assumptions on $\mathit{\Gamma}$ and W it follows that for some small $\eta_1 > 0$, the set $\mathit{\Gamma}_{\eta_1}$ (see (2.2)) is covered by a system of non-intersecting gradient lines of W. For each $x \in \mathit{\Gamma}_{\eta_1}$ we associate the coordinate $\bar{\sigma} = \bar{\sigma}(x)$ which is the intersection of the gradient line passing through x with $\mathit{\Gamma}$. The second coordinate is by definition $\bar{\delta} = \bar{\delta}(x)$ satisfying:
(i) $|\bar{\delta}(x)| = \sqrt{W(x)}$,
(ii) $\operatorname{sgn} \bar{\delta}(x)$ is negative inside $\mathit{\Gamma}$ and positive outside.

Since $\mathit{\Gamma}$ and W are both of class C^4, it follows that $\bar{\sigma}$ and $\bar{\delta}$ are of class C^2 in $\mathit{\Gamma}_{\eta_1}$. Using $|\nabla \bar{\delta}| = \frac{1}{2} W^{-1/2} |\nabla W|$, together with (2.4) and (5.1) we obtain that

$$\alpha_1 \le |\nabla \bar{\delta}(x)| \le \alpha_2, \ \forall x \in \mathit{\Gamma}_{\eta_1}, \tag{4.17}$$

for some positive constants α_1, α_2. Note also that $W_{\bar{\delta}\bar{\delta}} = 2$ in a neighborhood of $\mathit{\Gamma}$.

Dropping for simplicity the subscript ε_n, we set $A = \frac{1}{2}|\nabla u|^2$, so that

$$\Delta A = |D^2 u|^2 + \sum_{i=1}^{2} u_{x_i} \Delta(u_{x_i})\,. \tag{4.18}$$

Writing $u = (u_1, u_2)$ we get by a direct computation on $B(x_0, r_0)$, for $i = 1, 2$ and ε_n small enough, that

$$\begin{aligned} u_{x_i}\Delta(u_{x_i}) =& \frac{W_{\bar\delta}(u)}{2\varepsilon_n^2}\left(\bar\delta_{u_1u_1}(u_1)_{x_i}^2 + 2\bar\delta_{u_1u_2}(u_1)_{x_i}(u_2)_{x_i} + \bar\delta_{u_2u_2}(u_2)_{x_i}^2\right) \\ &+ \frac{W_{\bar\delta\bar\delta}}{2\varepsilon_n^2}\left(\bar\delta_{u_1}(u_1)_{x_i} + \bar\delta_{u_2}(u_2)_{x_i}\right)^2 \\ &\geq \frac{W_{\bar\delta}(u)}{2\varepsilon_n^2}\left(\bar\delta_{u_1u_1}(u_1)_{x_i}^2 + 2\bar\delta_{u_1u_2}(u_1)_{x_i}(u_2)_{x_i} + \bar\delta_{u_2u_2}(u_2)_{x_i}^2\right)\,. \end{aligned} \tag{4.19}$$

Noting that by (3.3)

$$|\Delta u| = \frac{|\nabla W(u)|}{2\varepsilon_n^2} = \frac{|W_{\bar\delta}(u)||\nabla\bar\delta|}{2\varepsilon_n^2}\,,$$

we infer from (4.18),(4.19) and (4.17) that for some $c > 0$ we have

$$\Delta A \geq |D^2 u|^2 - c|\Delta u||\nabla u|^2. \tag{4.20}$$

Since $|\Delta u| \leq \sqrt{2}|D^2 u|$ we get by (4.20) and the Cauchy-Schwarz inequality that

$$-\Delta A + |D^2 u|^2 \leq \frac{1}{2}|D^2 u|^2 + 4cA^2. \tag{4.21}$$

From the Bochner-type inequality (4.21) we deduce (4.16) as in [4]. In particular we obtain that,

$$u_{\varepsilon_n} \to u_* \text{ in } C^\alpha_{\rm loc}(G \setminus \{a_1, \ldots, a_D\}). \tag{4.22}$$

Next we show that the trace of u_* on ∂G equals $\tilde s(g)$. Note first that thanks to (4.13), $u_* \in H^1(G_r)$ for r satisfying (4.12), so it has a trace in $H^{1/2}(\partial G, S^1)$. Fix a point $y \in \partial G$ which is *not* a limit point of the centers of discs/half discs given by Lemma 4.2. It is enough to show that the trace of u_* equals $\tilde s(g)$ in a small boundary interval around each such y. Fix next some $r > 0$ small enough so that $B(y, r) \cap G$ does not intersect any of the discs/half discs. For each n we shall now define a map w_{ε_n} on the domain $B(y, r) \cap G$ as follows. On $B(y, r) \cap (G \setminus G_{t_{\varepsilon_n}})$ we simply set $w_{\varepsilon_n} = u_{\varepsilon_n}$ (t_{ε_n} is given in (4.6)). On the remaining part, namely $B(y, r) \cap G_{t_{\varepsilon_n}}$ we set, using the (σ, δ)-coordinates (see the beginning of Section 2):

$$w_{\varepsilon_n}(\sigma, \delta) = (\delta/t_{\varepsilon_n})u_{\varepsilon_n}(\sigma, t_{\varepsilon_n}) + (1 - \delta/t_{\varepsilon_n})\tilde s(g(\sigma))\,.$$

A simple computation, using (4.1) and (4.6), gives that

$$\int_{B(y,r)\cap G_{t_{\varepsilon_n}}} |\nabla w_{\varepsilon_n}|^2 + |w_{\varepsilon_n}|^2 \le C, \text{ uniformly in } n.$$

By (4.14) we conclude then that $w_{\varepsilon_n} \rightharpoonup u_*$ in $H^1(B(y,r)\cap G)$. Since the trace of each w_{ε_n} on $B(y,r)\cap\partial G$ is $\tilde{s}(g)$, we conclude that the same is true for u_*.

In order to complete the proof of Theorem 2 we need to show that u_* is of the from given in (1.18). First, applying an argument of Struwe [12] we obtain that $u_* \in W^{1,p}(G,S^1)$ for every $p\in[1,2)$. Then, it follows from [5, Remark I.1] that $v_* := \tau^{-1}(u_*)$ must be of the form

$$v_*(z) = e^{i\phi_0}\prod_{j=1}^{D}\Big(\frac{z-a_j}{|z-a_j|}\Big)\, e^{i(\sum_{j=1}^{D} c_j \log|z-a_j|)}\,, \tag{4.23}$$

for some constants $c_1,\dots,c_D$, where ϕ_0 is a smooth harmonic function. Fix any r satisfying (4.12). Then by (4.23) we have

$$\int_{B(a_j,r)\setminus B(a_j,\eta)} |\nabla u_*|^2 = (1+c_j^2)\frac{l^2(\Gamma)}{2\pi}\log\frac{1}{\eta} + O(1),\ \forall \eta<r,\ j=1,\dots,D\,. \tag{4.24}$$

On the other hand, the argument leading to Proposition 4.1 gives

$$\int_{B(a_j,r)\setminus B(a_j,\eta)} |\nabla u_*|^2 \le \frac{l^2(\Gamma)}{2\pi}\log\frac{1}{\eta} + O(1),\ \forall \eta<r,\ j=1,\dots,D\,. \tag{4.25}$$

Combining (4.24) with (4.25) we get that $c_j = 0$, $\forall j$, and it follows that u_* has the desired form. □

Remark 4.1. Repeating the argument from [2], it is not difficult to prove that the limiting configuration $\vec{a} = (a_1,\dots,a_D)$ in Theorem 2 is a minimizing configuration for a certain renormalized energy, as defined in [5] (see [2] for details).

5 Appendix: Local regularity of Ψ

The main objective of this appendix is to derive a regularity result for Ψ in a neighborhood of Γ. Differentiating (2.4) gives the following formulas for the derivatives of W w.r.t. the variables $\tilde\sigma$ and $\tilde\delta$ (that were defined at the beginning of Section 2) in Γ_{η_0}:

$$\frac{\partial W}{\partial\tilde\sigma} = a_{\tilde\sigma}(\tilde\sigma,\tilde\delta)\tilde\delta^2 \quad\text{and}\quad \frac{\partial W}{\partial\tilde\delta} = a_{\tilde\delta}(\tilde\sigma,\tilde\delta)\tilde\delta^2 + 2a(\tilde\sigma,\tilde\delta)\tilde\delta\,. \tag{5.1}$$

We may write

$$\nabla W = \frac{\partial W}{\partial \tilde{\sigma}} \vec{\tau} + \frac{\partial W}{\partial \tilde{\delta}} \vec{n}, \tag{5.2}$$

with $\vec{n} = \nabla \tilde{\delta}$ and $\vec{\tau} = \nabla \tilde{\sigma}$. By our regularity assumptions on $\mathit{\Gamma}$ there exist continuous functions $c_1(x), c_2(x), c_3(x), c_4(x)$ such that

$$\frac{\partial \vec{n}}{\partial \tilde{\sigma}} = c_1(x)\vec{\tau}, \quad \frac{\partial \vec{\tau}}{\partial \tilde{\sigma}} = c_2(x)\vec{n} + c_3(x)\vec{\tau} \quad \text{and} \quad \frac{\partial \vec{\tau}}{\partial \tilde{\delta}} = c_4(x)\vec{\tau}\,.$$

Note that for $x \in \mathit{\Gamma}$ we have $c_1(x) = c(x)$ =the curvature of $\mathit{\Gamma}$ at x. From (5.1)–(5.2) we derive that

$$\frac{\partial}{\partial \tilde{\sigma}} \nabla W = \big(a_{\tilde{\sigma}\tilde{\sigma}}\tilde{~}\delta^2 + c_1(x)(a_{\tilde{\delta}}\tilde{\delta}^2 + 2a\tilde{\delta}) + c_3(x)a_{\tilde{\sigma}}\tilde{\delta}^2\big)\vec{\tau} + \big((a_{\tilde{\sigma}\tilde{\delta}} + c_2(x)a_{\tilde{\sigma}})\tilde{\delta}^2 + 2a_{\tilde{\sigma}}\tilde{\delta}\big)\vec{n} = O(\tilde{\delta}) \tag{5.3}$$

and

$$\frac{\partial}{\partial \tilde{\delta}} \nabla W = \Big(a_{\tilde{\sigma}\tilde{\delta}}\tilde{\delta}^2 + 2a_{\tilde{\sigma}}\tilde{\delta} + c_4(x)a_{\tilde{\sigma}}\tilde{\delta}^2\Big)\vec{\tau} + \Big(a_{\tilde{\delta}\tilde{\delta}}\tilde{\delta}^2 + 4a_{\tilde{\delta}}\tilde{\delta} + 2a\Big)\vec{n} = 2a\vec{n} + O(\tilde{\delta})\,. \tag{5.4}$$

The main result of this appendix is the following

Proposition 5.1. *There exists* $\lambda_0 > 0$ *such that the equation*

$$\begin{cases} |\nabla U|^2 = W\,, \\ U = 0 \text{ on } \mathit{\Gamma}\,, \end{cases} \tag{5.5}$$

has a unique positive C^2*-solution* U *in* $\overline{\Omega}_{\lambda_0}$. *Moreover, this solution coincides in* $\overline{\Omega}_{\lambda_0}$ *with* Ψ, *and* $\overline{\Omega}_{\lambda_0}$ *can be covered by a system of gradient lines of* Ψ, *given by the images of* $\{\gamma^{\pm}x_0\}_{x_0 \in \mathit{\Gamma}}$ *(see* (2.10)–(2.11)*).*

In the *non-degenerate* case, the proof of such result is classical via the characteristic method, see [6, Section 3.2]. We shall use a variant of this method in order to overcome the difficulty caused by the degeneracy of the problem. Define an Hamiltonian

$$H(X, P) = |P|^2 - W(X), \tag{5.6}$$

so that the first equation in (5.5) can be written as $H(x, \nabla U) = 0$. We are looking for a solution $(X, P) : \mathit{\Gamma} \times (-\infty, c) \to \mathit{\Gamma}_\eta \times \mathbb{R}^2$, for some small $\eta > 0$ (see (2.2)) of the characteristics system

$$\begin{cases} X(x_0, -\infty) = x_0, \ \forall x_0 \in \mathit{\Gamma}, \\ \dot{X} = \frac{\partial H}{\partial P} = 2P, \\ \dot{P} = -\frac{\partial H}{\partial X} = \nabla W(X), \end{cases} \tag{5.7}$$

where dot represents derivative w.r.t. the variable t. The construction of a solution U from (X, P) is then standard, see (5.36) below. In order to define a problem on a bounded domain we make the change of variables $r = e^{\alpha t}$, where $\alpha = \alpha(x_0) = 2\sqrt{a(x_0)}$ (see (2.4)). Using this new variable, (5.7) becomes

$$\begin{cases} X(x_0, 0) = x_0 \text{ and } P(x_0, 0) = 0, \ \forall x_0 \in \Gamma\,, \\ \frac{\partial X}{\partial r} = \frac{2P}{\alpha(x_0) r}\,, \\ \frac{\partial P}{\partial r} = \frac{\nabla W(X)}{\alpha(x_0) r}\,. \end{cases} \tag{5.8}$$

We shall construct a solution X of (5.8) with image in a one-sided neighborhood of Γ, of the form $\Gamma_\eta^+ = \{x \in \mathbb{R}^2 \,:\, \tilde{\delta}(x) \in [0, \eta)\}$, but an analogous argument will give a solution also on the other side of Γ, namely in $\Gamma_\eta^- = \{x \in \mathbb{R}^2 \,:\, \tilde{\delta}(x) \in (-\eta, 0]\}$.

Integrating the equations in (5.8) yields an equivalent form:

$$X(x_0, r) - x_0 = \int_0^r \frac{2P(x_0, s)}{\alpha(x_0) s}\, ds\,, \quad P(x_0, r) = \int_0^r \frac{\nabla W(X(x_0, s))}{\alpha(x_0) s}\, ds\,. \tag{5.9}$$

Let Y and Q be defined by

$$Y(x_0, r) = \frac{X(x_0, r) - x_0}{r} \quad \text{and} \quad Q(x_0, r) = \frac{P(x_0, r)}{r}\,. \tag{5.10}$$

If Q and Y are associated with a solution (X, P) to (5.9), then by (5.3)–(5.4) it follows that

$$\begin{aligned} Q(x_0, r) &= \frac{1}{r} \int_0^r \frac{\nabla W(x_0 + sY(x_0, s))}{\alpha(x_0) s}\, ds \\ &= \frac{1}{\alpha(x_0) r} \int_0^r \frac{1}{s} \big(D^2 W(x_0)(sY(x_0, s) + s\, o(|Y(x_0, s)|) \big)\, ds \\ &= \frac{\vec{n}}{\alpha(x_0) r} \int_0^r 2a(x_0) Y(x_0, s) \cdot \vec{n}\, ds + o(1) \end{aligned} \tag{5.11}$$

and

$$Y(x_0, r) = \frac{1}{\alpha(x_0) r} \int_0^r 2Q(x_0, s)\, ds\,. \tag{5.12}$$

From (5.11)–(5.12) we deduce that

$$\begin{cases} Q(x_0, 0) = \sqrt{a(x_0)} (Y(x_0, 0) \cdot \vec{n})\, \vec{n}\,, \\ Y(x_0, 0) = \frac{Q(x_0, 0)}{\sqrt{a(x_0)}}\,. \end{cases} \tag{5.13}$$

Thus (5.13) implies a *compatibility* condition on the initial values $Q(x_0, 0)$, $Y(x_0, 0)$. We deduce in particular that $Y(x_0, 0)$ must be parallel to $\vec{n}(x_0)$, and we may choose $Y(x_0, 0) = \vec{n}(x_0)$ (and then necessarily $Q(x_0, 0) = \sqrt{a(x_0)}\vec{n}(x_0)$).

Next we introduce yet another pair of unknowns, $\widetilde{Y}, \widetilde{Q}$, by

$$\begin{cases} Y(x_0, r) = \vec{n}(x_0) + r\widetilde{Y}(x_0, r)\,, \\ Q(x_0, r) = \sqrt{a(x_0)}\,\big(\vec{n}(x_0) + r\widetilde{Q}(x_0, r)\big)\,. \end{cases} \tag{5.14}$$

From (5.11)–(5.12) we obtain the following system of equations that must be satisfied by $\widetilde{Y}$ and $\widetilde{Q}$:

$$\widetilde{Y}(x_0, r) = \frac{1}{r^2}\int_0^r \Big(\frac{1}{\sqrt{a(x_0)}}Q(x_0, s) - \vec{n}(x_0)\Big)\,ds = \frac{1}{r^2}\int_0^r s\,\widetilde{Q}(x_0, s)\,ds\,, \tag{5.15}$$

and

$$\begin{aligned} \widetilde{Q}(x_0, r) &= \frac{1}{r\sqrt{a(x_0)}}\big(Q(x_0, r) - \sqrt{a(x_0)}\vec{n}(x_0)\big) \\ &= \frac{1}{2a(x_0)r^2}\int_0^r \Big(\frac{1}{s}\nabla W(x_0 + s\vec{n}(x_0) + s^2\widetilde{Y}(x_0, s)) - 2a(x_0)\vec{n}(x_0)\Big)\,ds\,. \end{aligned} \tag{5.16}$$

Denoting by $T_1(\widetilde{Y}, \widetilde{Q})$ the r.h.s. of (5.15) and by $T_2(\widetilde{Y}, \widetilde{Q})$ the r.h.s. of (5.16), we define a map $T = (T_1, T_2)$ from $\big(C(\Gamma\times[0, R], \mathbb{R}^2))\big)^2$ to itself by $T(\widetilde{Y}, \widetilde{Q}) = (T_1(\widetilde{Y}, \widetilde{Q}), T_2(\widetilde{Y}, \widetilde{Q}))$. Clearly $(\widetilde{Y}, \widetilde{Q})$ is a solution to (5.15)–(5.16) if and only if it is a fixed point of T. Next we claim

Lemma 5.1. *For $R > 0$ small enough, T is a strict contraction w.r.t. the $\|\cdot\|_\infty$-norm (and therefore has a unique fixed point).*

Proof. Consider any $(\widetilde{Y}, \widetilde{Q}), (\widetilde{Y}', \widetilde{Q}') \in \big(C(\Gamma \times [0, R], \mathbb{R}^2)\big)^2$. Using (5.15) we get, for R small enough,

$$\begin{aligned} \|T_1(\widetilde{Y}, \widetilde{Q}) - T_1(\widetilde{Y}', \widetilde{Q}')\|_\infty &= \sup_{\substack{0<r<R\\ x_0\in\Gamma}} \frac{1}{r^2}\Big|\int_0^r s(\widetilde{Q} - \widetilde{Q}')(x_0, s)\,ds\Big| \le \frac{1}{2}\|\widetilde{Q} - \widetilde{Q}'\|_\infty \\ &\le \frac{1}{2}\|(\widetilde{Y}, \widetilde{Q}) - (\widetilde{Y}', \widetilde{Q}')\|_\infty\,. \end{aligned} \tag{5.17}$$

From (5.16) we obtain that

$$\|T_2(\widetilde{Y}, \widetilde{Q}) - T_2(\widetilde{Y}', \widetilde{Q}')\|_\infty = \\ \sup_{\substack{0<r<R\\ x_0\in\Gamma}} \frac{1}{2a(x_0)r^2}\Big|\int_0^r\Big(\nabla W(x_0{+}s\vec{n}(x_0){+}s^2\widetilde{Y}(x_0, s)){-}\nabla W(x_0{+}s\vec{n}(x_0){+}s^2\widetilde{Y}'(x_0, s))\Big)\frac{ds}{s}\Big|. \tag{5.18}$$

By (5.3)–(5.4) we get that

$$|\nabla W(x_0 + s\vec{n}(x_0) + s^2\widetilde{Y}) - \nabla W(x_0 + s\vec{n}(x_0) + s^2\widetilde{Y}')| \le (2a(x_0)s^2 + O(s^3)) \cdot |\widetilde{Y} - \widetilde{Y}'| \,.$$

Integration on s gives, for every $r \in (0, R)$,

$$\frac{1}{2a(x_0)r^2}\int_0^r \frac{1}{s}|\nabla W(x_0 + s\vec{n}(x_0) + s^2\widetilde{Y}) - \nabla W(x_0 + s\vec{n}(x_0) + s^2\widetilde{Y}')|\,ds \\ \le \Big(\frac{1}{2} + O(R)\Big)\|\widetilde{Y} - \widetilde{Y}'\|_\infty \,,$$

which plugged in (5.18) yields for R small enough

$$\|T_2(\widetilde{Y}, \widetilde{Q}) - T_2(\widetilde{Y}', \widetilde{Q}')\|_\infty \le \frac{3}{4}\|(\widetilde{Y}, \widetilde{Q}) - (\widetilde{Y}', \widetilde{Q}')\|_\infty \,. \tag{5.19}$$

The result of the lemma clearly follows from (5.17) and (5.19). □

Fix $R_0 > 0$ for which the conclusion of Lemma 5.1 holds and let $(\widetilde{Y}_\infty, \widetilde{Q}_\infty) \in \big(C(\mathit{\Gamma} \times [0, R_0], \mathbb{R}^2)\big)^2$ be the fixed point of T given by Lemma 5.1. The next lemma establishes improved regularity for $\widetilde{Y}_\infty$ and $\widetilde{Q}_\infty$. To study the regularity w.r.t. the first variable we examine the derivative w.r.t. the variable $\tilde{\sigma}_0$ ($x_0 = \gamma(\tilde{\sigma}_0)$, see the definition after (2.3)).

Lemma 5.2. *There exists $0 < R_1 \le R_0$ such that the functions $\widetilde{Y}_\infty$ and $\widetilde{Q}_\infty$ have continuous derivatives in the variable $\tilde{\sigma}_0$ on $\mathit{\Gamma} \times [0, R_1]$.*

Proof. Consider $\widetilde{Y}_0, \widetilde{Q}_0 \in C^1(\mathit{\Gamma} \times [0, R_0], \mathbb{R}^2)$ and the sequence of iterates $\{(\widetilde{Y}_n, \widetilde{Q}_n)\}_{n=0}^\infty = \{T^n(\widetilde{Y}_0, \widetilde{Q}_0)\}_{n=0}^\infty$ which converges uniformly to the fixed point $(\widetilde{Y}_\infty, \widetilde{Q}_\infty)$ of T. We will show that the sequence $\{(\frac{\partial \widetilde{Y}_n}{\partial \tilde{\sigma}_0}, \frac{\partial \widetilde{Q}_n}{\partial \tilde{\sigma}_0})\}_{n=0}^\infty$ is uniformly converging too. We next prove a uniform bound for the $\tilde{\sigma}_0$-derivative of the sequence, i.e., that there exists a constant $K > 0$ such that, on $\mathit{\Gamma} \times [0, R_0']$, with $0 < R_0' \le R_0$, we have

$$\|\frac{\partial \widetilde{Y}_n}{\partial \tilde{\sigma}_0}\|_\infty, \|\frac{\partial \widetilde{Q}_n}{\partial \tilde{\sigma}_0}\|_\infty \le K \,,\ \forall n \,. \tag{5.20}$$

Note first that

$$\frac{\partial \widetilde{Y}_{n+1}}{\partial \tilde{\sigma}_0} = \frac{\partial}{\partial \tilde{\sigma}_0} T_1(\widetilde{Y}_n, \widetilde{Q}_n) = \frac{1}{r^2}\int_0^r s\frac{\partial \widetilde{Q}(\gamma(\tilde{\sigma}_0), s)}{\partial \tilde{\sigma}_0}\,ds$$

implies that

$$\|\frac{\partial \widetilde{Y}_{n+1}}{\partial \tilde{\sigma}_0}\|_\infty \le \frac{1}{2}\|\frac{\partial \widetilde{Q}_n}{\partial \tilde{\sigma}_0}\|_\infty \,. \tag{5.21}$$

Let us write for short $\widetilde{Y} = \widetilde{Y}_n$ and $\widetilde{Q} = \widetilde{Q}_n$, and set

$$X = X(x_0, s) = x_0 + s\vec{n}(x_0) + s^2\widetilde{Y}(x_0, s)\,. \tag{5.22}$$

By (5.16),

$$\begin{aligned}\frac{\partial}{\partial\tilde{\sigma}_0}(T_2(\widetilde{Y},\widetilde{Q}))(x_0,r) = &- T_2(\widetilde{Y},\widetilde{Q})\frac{a_{\tilde{\sigma}_0}(x_0)}{a(x_0)} \\ &+ \frac{1}{2a(x_0)r^2}\int_0^r \frac{1}{s}\frac{\partial}{\partial\tilde{\sigma}_0}\Big(\nabla W(X(x_0,s)) - 2a(x_0)s\vec{n}(x_0)\Big)\,ds \\ :=& A_n + B_n\,.\end{aligned} \tag{5.23}$$

Since the sequence $\{(\widetilde{Y}_n, \widetilde{Q}_n)\}_{n=0}^{\infty}$ converges uniformly, by Lemma 5.1, it follows that

$$|A_n| \le K_1,\ \forall n. \tag{5.24}$$

Next we compute

$$\begin{aligned}\frac{\partial}{\partial\tilde{\sigma}_0}\nabla W(X(x_0,s)) &= \frac{\partial\nabla W}{\partial\tilde{\sigma}}(X(x_0,s))\frac{\partial\tilde{\sigma}(X(x_0,s))}{\partial\tilde{\sigma}_0} + \frac{\partial\nabla W}{\partial\tilde{\delta}}(X(x_0,s))\frac{\partial\tilde{\delta}(X(x_0,s))}{\partial\tilde{\sigma}_0} \\ &= \Big(\frac{\partial\nabla W}{\partial\tilde{\sigma}}\nabla\tilde{\sigma} + \frac{\partial\nabla W}{\partial\tilde{\delta}}\nabla\tilde{\delta}\Big)(X(x_0,s))\cdot\Big(\big(1+sc(x_0)\big)\vec{\tau}(x_0) + s^2\frac{\partial\widetilde{Y}}{\partial\tilde{\sigma}_0}\Big)\,.\end{aligned} \tag{5.25}$$

Denoting

$$Z = Z(x_0,s) = \big(\frac{1}{s} + c(x_0)\big)\vec{\tau}(x_0) + s\frac{\partial\widetilde{Y}}{\partial\tilde{\sigma}_0}\,, \tag{5.26}$$

we get from (5.25) and (5.3)–(5.4):

$$\begin{aligned}&\frac{1}{s}\frac{\partial}{\partial\tilde{\sigma}_0}\big(\nabla W(X)\big) - \frac{\partial}{\partial\tilde{\sigma}_0}\big(2a(x_0)\vec{n}(x_0)\big) = \\ &\Big[\big(a_{\tilde{\sigma}\tilde{\sigma}}\tilde{\delta}^2 + c_1(X)(a_{\tilde{\delta}}\tilde{\delta}^2 + 2a\tilde{\delta}) + c_3(X)a_{\tilde{\sigma}}\tilde{\delta}^2\big)\vec{\tau}(X) + \big((a_{\tilde{\sigma}\tilde{\delta}} + c_2(X)a_{\tilde{\sigma}})\tilde{\delta}^2 + 2a_{\tilde{\sigma}}\tilde{\delta}\big)\vec{n}(X)\Big]\big(Z\cdot\vec{\tau}(X)\big) \\ &\quad + \Big[\big(a_{\tilde{\sigma}\tilde{\delta}}\tilde{\delta}^2 + 2a_{\tilde{\sigma}}\tilde{\delta} + c_4(X)a_{\tilde{\sigma}}\tilde{\delta}^2\big)\vec{\tau}(X) + \big(a_{\tilde{\delta}\tilde{\delta}}\tilde{\delta}^2 + 4a_{\tilde{\delta}}\tilde{\delta} + 2a\big)\vec{n}(X)\Big]\big(Z\cdot\vec{n}(X)\big) \\ &\quad - 2a_{\tilde{\sigma}_0}(x_0)\vec{n}(x_0) - 2a(x_0)c(x_0)\vec{\tau}(x_0)\,.\end{aligned} \tag{5.27}$$

Using the estimates,

$$\vec{\tau}(X)\cdot\vec{\tau}(x_0) = 1 + O(s^2),\ \vec{n}(X)\cdot\vec{\tau}(x_0) = O(s^2),\ \tilde{\delta}(X) = s + O(s^2),$$

together with the regularity of the functions $a(x)$ (C^2) and $c(x)$ (C^1) in (5.27) yields

$$\Big|\frac{1}{s}\frac{\partial}{\partial\tilde\sigma_0}\big(\nabla W(X)\big)-\frac{\partial}{\partial\tilde\sigma_0}\big(2a(x_0)\vec n(x_0)\big)\Big|\le K_2 s+\big\|\frac{\partial\widetilde Y}{\partial\tilde\sigma_0}\big\|_\infty(2a(X)s+K_3s^2).$$

It follows that for $R\le R_0'$ there holds

$$\begin{aligned}|B_n|&\le\frac{1}{2a(x_0)r^2}\int_0^r K_2 s\,ds+\big\|\frac{\partial\widetilde Y}{\partial\tilde\sigma_0}\big\|_\infty\frac{1}{2a(x_0)r^2}\int_0^r(2a(X)s+K_3s^2)ds\\&\le K_4+\frac{3}{4}\big\|\frac{\partial\widetilde Y}{\partial\tilde\sigma_0}\big\|_\infty,\end{aligned}\tag{5.28}$$

for some positive constant K_4. Combining the estimate (5.28) with (5.24),(5.21) and (5.23) we obtain that

$$\max\Big(\big\|\frac{\partial\widetilde Y_{n+1}}{\partial\tilde\sigma_0}\big\|_\infty,\big\|\frac{\partial\widetilde Q_{n+1}}{\partial\tilde\sigma_0}\big\|_\infty\Big)\le\max(K_1,K_4)+\frac{3}{4}\max\Big(\big\|\frac{\partial\widetilde Y_{n}}{\partial\tilde\sigma_0}\big\|_\infty,\big\|\frac{\partial\widetilde Q_{n}}{\partial\tilde\sigma_0}\big\|_\infty\Big),$$

which clearly implies (5.20).

Next, let $(\widetilde Y,\widetilde Q)$ and $(\widetilde Y',\widetilde Q')$ be C^1 functions defined on $\Gamma\times[0,R]$ for R small enough, to be determined later, and let X and X' be associated to $\widetilde Y$ and $\widetilde Y'$, respectively. Let the corresponding Z and Z' be defined by (5.26). Note first that the equality

$$\frac{\partial}{\partial\tilde\sigma_0}(T_1(\widetilde Y,\widetilde Q))=\frac{1}{r^2}\int_0^r s\frac{\partial\widetilde Q}{\partial\tilde\sigma_0}(x_0,s)\,ds\,,$$

implies that

$$\big\|\frac{\partial}{\partial\tilde\sigma_0}T_1(\widetilde Y,\widetilde Q)-\frac{\partial}{\partial\tilde\sigma_0}T_1(\widetilde Y',\widetilde Q')\big\|_\infty\le\frac{1}{2}\big\|\frac{\partial\widetilde Q}{\partial\tilde\sigma_0}-\frac{\partial\widetilde Q'}{\partial\tilde\sigma_0}\big\|_\infty.\tag{5.29}$$

From (5.23) we get that

$$\begin{aligned}\big\|\frac{\partial}{\partial\tilde\sigma_0}T_2(\widetilde Y,\widetilde Q)-\frac{\partial}{\partial\tilde\sigma_0}T_2(\widetilde Y',\widetilde Q')\big\|_\infty&\le c_0\|\widetilde Q-\widetilde Q'\|_\infty\\&+\frac{1}{2a(x_0)r^2}\int_0^r\frac{1}{s}\Big\|\frac{\partial}{\partial\tilde\sigma_0}\nabla W(X(x_0,s))-\frac{\partial}{\partial\tilde\sigma_0}\nabla W(X'(x_0,s))\Big\|_\infty ds\,.\end{aligned}\tag{5.30}$$

Next we estimate for each $s\in(0,r)$,

$$\begin{aligned}&\frac{1}{s}\Big\|\frac{\partial}{\partial\tilde\sigma_0}\nabla W(X)-\frac{\partial}{\partial\tilde\sigma_0}\nabla W(X')\Big\|_\infty=\\&\Big\|\big(\frac{\partial\nabla W}{\partial\tilde\sigma}\nabla\tilde\sigma+\frac{\partial\nabla W}{\partial\tilde\delta}\nabla\tilde\delta\big)(X)\cdot Z-\big(\frac{\partial\nabla W}{\partial\tilde\sigma}\nabla\tilde\sigma+\frac{\partial\nabla W}{\partial\tilde\delta}\nabla\tilde\delta\big)(X')\cdot Z'\Big\|_\infty\le\\&\Big\|\big(\frac{\partial\nabla W}{\partial\tilde\sigma}\nabla\tilde\sigma+\frac{\partial\nabla W}{\partial\tilde\delta}\nabla\tilde\delta\big)(X)-\big(\frac{\partial\nabla W}{\partial\tilde\sigma}\nabla\tilde\sigma+\frac{\partial\nabla W}{\partial\tilde\delta}\nabla\tilde\delta\big)(X')\Big\|_\infty\|Z\|_\infty\\&\qquad+\Big\|\big(\frac{\partial\nabla W}{\partial\tilde\sigma}\nabla\tilde\sigma+\frac{\partial\nabla W}{\partial\tilde\delta}\nabla\tilde\delta\big)(X')\Big\|_\infty\|Z-Z'\|_\infty:=C_n+D_n.\end{aligned}$$

Since W is of class C^3 we get, for some constants c, c_1, c_2, C that

$$|C_n| \le c\|X - X'\|_\infty \|Z\|_\infty \le c_1 s^2 \Big\|\frac{\partial \tilde{Y}}{\partial \tilde{\sigma}_0} - \frac{\partial \tilde{Y}'}{\partial \tilde{\sigma}_0}\Big\|_\infty \Big(\frac{1}{s} + c_2 + s\Big\|\frac{\partial \tilde{Y}}{\partial \tilde{\sigma}_0}\Big\|_\infty\Big) \le$$
$$C s \Big\|\frac{\partial \tilde{Y}}{\partial \tilde{\sigma}_0} - \frac{\partial \tilde{Y}'}{\partial \tilde{\sigma}_0}\Big\|_\infty, \quad (5.31)$$

with C depending on $\big\|\frac{\partial \tilde{Y}}{\partial \tilde{\sigma}_0}\big\|_\infty$. Furthermore, by (5.3)–(5.4) we have

$$|D_n| \le (2a(X') + c_3 s)s\Big\|\frac{\partial \tilde{Y}}{\partial \tilde{\sigma}_0} - \frac{\partial \tilde{Y}'}{\partial \tilde{\sigma}_0}\Big\|_\infty \le (2a(x_0) + c_4 s)s\Big\|\frac{\partial \tilde{Y}}{\partial \tilde{\sigma}_0} - \frac{\partial \tilde{Y}'}{\partial \tilde{\sigma}_0}\Big\|_\infty, \quad (5.32)$$

with c_3, c_4 depending on the L^∞-norm of $\tilde{Y}'$. Using (5.31)–(5.32) in (5.30), we obtain that

$$\Big\|\frac{\partial}{\partial \tilde{\sigma}_0}\big(T_2(\widetilde{Y}, \widetilde{Q}) - T_2(\widetilde{Y}', \widetilde{Q}')\big)\Big\|_\infty \le c_0\|\widetilde{Q} - \widetilde{Q}'\|_\infty + (1/2 + O(R))\Big\|\frac{\partial \widetilde{Y}}{\partial \tilde{\sigma}_0} - \frac{\partial \widetilde{Y}'}{\partial \tilde{\sigma}_0}\Big\|_\infty. \quad (5.33)$$

Next we apply the above estimates for the choice $(\widetilde{Y}, \widetilde{Q}) = (\widetilde{Y}_n, \widetilde{Q}_n)$ and $(\widetilde{Y}', \widetilde{Q}') = (\widetilde{Y}_{n+1}, \widetilde{Q}_{n+1})$. Using Lemma 5.1 and (5.20) we conclude from (5.33) that for R small enough we have

$$\Big\|\frac{\partial}{\partial \tilde{\sigma}_0}\big(\tilde{Y}_{n+2} - \tilde{Y}_{n+1}\big)\Big\|_\infty \le c_0\|\widetilde{Q}_n - \widetilde{Q}_{n+1}\|_\infty + \frac{7}{8}\Big\|\frac{\partial}{\partial \tilde{\sigma}_0}\big(\widetilde{Y}_n - \widetilde{Y}_{n+1}\big)\Big\|_\infty. \quad (5.34)$$

On the other hand, the estimate (5.29) now reads

$$\Big\|\frac{\partial}{\partial \tilde{\sigma}_0}\big(\widetilde{Q}_{n+2} - \widetilde{Q}_{n+1}\big)\Big\|_\infty \le \frac{1}{2}\Big\|\frac{\partial}{\partial \tilde{\sigma}_0}\big(\widetilde{Q}_{n+1} - \widetilde{Q}_n\big)\Big\|_\infty. \quad (5.35)$$

Combining (5.34) with (5.35) and using (5.19) we conclude easily the convergence of the sequence $\{(\frac{\partial}{\partial \tilde{\sigma}_0}\widetilde{Y}_n, \frac{\partial}{\partial \tilde{\sigma}_0}\widetilde{Q}_n)\}$ in the L^∞-norm, and the result follows. $\square$

Using the above lemmata we can now present the proof of Proposition 5.1.

Proof of Proposition 5.1. Denote by $(\widetilde{Y}, \widetilde{Q}) = (\widetilde{Y}_\infty, \widetilde{Q}_\infty)$ the solution constructed in Lemma 5.1. Put

$$\begin{aligned} X(x_0, s) &= x_0 + s\vec{n}(x_0) + s^2\widetilde{Y}(x_0, s), \\ P(x_0, s) &= \sqrt{a(x_0)}\big(s\vec{n}(x_0) + s^2\widetilde{Q}(x_0, s)\big), \end{aligned} \qquad \forall x_0 \in \Gamma,\ \forall s \in [-R, R].$$

Note that

$$\frac{\partial X}{\partial \tilde{\sigma}_0}(x_0, s) = \vec{\tau}(x_0) + sc(x_0)\vec{\tau}(x_0) + s^2\frac{\partial \widetilde{Y}}{\partial \tilde{\sigma}_0}(x_0, s)$$

and
$$\frac{\partial X}{\partial s}(x_0,s)=\frac{2P(x_0,s)}{2s\sqrt{a(x_0)}}=\vec{n}(x_0)+s\widetilde{Q}(x_0,s) \text{ (see (5.8))}$$

are continuous by Lemmata 5.1 and 5.2. In particular,

$$\frac{\partial X}{\partial \tilde{\sigma}_0}(x_0,0)=\vec{\tau}(x_0) \quad \text{and} \quad \frac{\partial X}{\partial s}(x_0,0)=\vec{n}(x_0)\,,$$

and it follows that $DX(x_0,0)$ is non-singular. Therefore, by the implicit function theorem there exists a neighborhood Ω_0 of Γ, and $0<R_2\le R_1$ (see Lemma 5.2) such that $X:\Gamma\times(-R_2,R_2)\to\Omega_0$ is a C^1-diffeomorphism. We finally define U in Ω_0 by

$$U(X)=U(X(x_0,r))=\int_0^r P(X(x_0,s))\cdot\frac{\partial X}{\partial s}(x_0,s)\,ds\,. \tag{5.36}$$

By (5.6) and (5.8) we have

$$\frac{\partial}{\partial r}H(X,P)=\frac{\partial H}{\partial X}\cdot\frac{\partial X}{\partial r}+\frac{\partial H}{\partial P}\cdot\frac{\partial P}{\partial r}=\frac{1}{\alpha r}\big(-\nabla W\cdot 2P+2P\cdot\nabla W\big)=0\,,$$

which implies that

$$H(X(x_0,r),P(x_0,r))=H(x_0,0)=0,\ \forall x_0,\ \forall r. \tag{5.37}$$

If we can show that

$$\nabla U=P, \tag{5.38}$$

it will follow from (5.37) that U solves (5.5). For that matter we differentiate (5.36) and get:

$$\frac{\partial U}{\partial r}=\nabla U\cdot\frac{\partial X}{\partial r}=P\cdot\frac{\partial X}{\partial r}\,, \tag{5.39}$$

and

$$\frac{\partial U}{\partial \tilde{\sigma}_0}=\nabla U\cdot\frac{\partial X}{\partial \tilde{\sigma}_0}=\int_0^r\Big[\frac{\partial P}{\partial \tilde{\sigma}_0}\cdot\frac{\partial X}{\partial s}+P\cdot\frac{\partial^2 X}{\partial s\partial \tilde{\sigma}_0}\Big]\,ds\,. \tag{5.40}$$

Differentiating (5.37) w.r.t. $\tilde{\sigma}_0$ yields

$$\frac{\partial H}{\partial X}\cdot\frac{\partial X}{\partial \tilde{\sigma}_0}+\frac{\partial H}{\partial P}\cdot\frac{\partial P}{\partial \tilde{\sigma}_0}=0\,. \tag{5.41}$$

Using (5.41) we get that

$$\frac{\partial P}{\partial \tilde{\sigma}_0}\cdot\frac{\partial X}{\partial s}=\frac{\partial P}{\partial \tilde{\sigma}_0}\cdot\frac{2P}{\alpha s}=\frac{1}{\alpha s}\frac{\partial H}{\partial P}\cdot\frac{\partial P}{\partial \tilde{\sigma}_0}=-\frac{1}{\alpha s}\frac{\partial H}{\partial X}\cdot\frac{\partial X}{\partial \tilde{\sigma}_0}=\frac{\nabla W}{\alpha s}\cdot\frac{\partial X}{\partial \tilde{\sigma}_0}=\frac{\partial P}{\partial s}\cdot\frac{\partial X}{\partial \tilde{\sigma}_0}\,. \tag{5.42}$$

Plugging (5.42) in (5.40) yields

$$\nabla U \cdot \frac{\partial X}{\partial \tilde{\sigma}_0} = \int_0^r \Big[\frac{\partial P}{\partial s} \cdot \frac{\partial X}{\partial \tilde{\sigma}_0} + P \cdot \frac{\partial^2 X}{\partial s \partial \tilde{\sigma}_0}\Big]\, ds = P \cdot \frac{\partial X}{\partial \tilde{\sigma}_0}\Big]_0^r = P(x_0, r) \cdot \frac{\partial X}{\partial \tilde{\sigma}_0}(x_0, r)\,. \quad (5.43)$$

In a neighborhood of Γ the vectors $\frac{\partial X}{\partial \tilde{\sigma}_0}$ and $\frac{\partial X}{\partial s}$ are linearly independent, so (5.38) follows from (5.39) and (5.43). We conclude that U is a solution of (5.5) in some neighborhood Γ_{η_1} of Γ and the C^2-regularity of U follows from the regularity of X and P.

Finally we prove that U coincides with Ψ in Γ_{η_1}. Fix a point $\zeta \in \Gamma_{\eta_1}$ and take any Lipschitz map $\gamma : [0, 1] \to \mathbb{R}^2$ such that $\gamma(0) \in \Gamma$ and $\gamma(1) = \zeta$. Then,

$$\begin{aligned}\int_0^1 \big(W(\gamma(t))\big)^{1/2} |\gamma'(t)|\, dt &= \int_0^1 |\nabla U(\gamma(t))|\, |\gamma'(t)|\, dt \\ &\geq \int_0^1 \nabla U(\gamma(t)) \cdot \gamma'(t)\, dt = U(\gamma(1)) - U(\gamma(0)) = U(\zeta)\,.\end{aligned}$$

Therefore, $\Psi(\zeta) \geq U(\zeta)$. On the other hand, there exists a unique $x_0 \in \Gamma$ such that $\zeta = X(x_0, r(\zeta))$ for some $r(\zeta) > 0$. Then we can define a path connecting x_0 to ζ by

$$\gamma_0(s) = X(x_0, s), \;\; s \in [0, r(\zeta)]\,.$$

Since $\nabla U(\gamma_0(s)) = P(x_0, s)$ by (5.38), $\sqrt{W(\gamma_0(s))} = |P(x_0, s)|$ by (5.37) and $\gamma_0'(s) = \frac{2P(x_0,s)}{\alpha s}$ by (5.8), we have

$$\begin{aligned}\int_0^{r(\zeta)} \big(W(\gamma_0(s))\big)^{1/2} |\gamma_0'(s)|\, ds &= \int_0^{r(\zeta)} |P(x_0, s)| \frac{|2P(x_0, s)|}{\alpha s}\, ds \\ &= \int_0^{r(\zeta)} \nabla U(\gamma_0(s)) \cdot \gamma_0'(s)\, ds = U(\gamma_0(r(\zeta))) - U(\gamma_0(0)) = U(\zeta)\,,\end{aligned}$$

which implies that $U(\zeta) \geq \Psi(\zeta)$. Combining the two inequalities yields $U(\zeta) = \Psi(\zeta)$ as claimed. □

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