Moser-Trudinger and logarithmic HLS inequalities for systems

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Abstract

We prove several optimal Moser-Trudinger and logarithmic Hardy-Littlewood-Sobolev inequalities for systems in two dimensions. These include inequalities on the sphere S^2, on a bounded domain $\Omega \subset \mathbb{R}^2$ and on all of $\mathbb{R}^2$. In some cases we also address the question of existence of minimizers.

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1 Introduction

The Moser-Trudinger (MT) inequality (see [13]) on the two-sphere S^2 reads:

$$f^M(u) := \int_{S^2} \frac{1}{2}|\nabla u|^2\, d\omega - M \log\Big(\int_{S^2} e^u \frac{d\omega}{4\pi}\Big) \ge -C, \quad \forall u \in H^1(S^2)\ , \int_{S^2} u\, d\omega = 0\ , \tag{1.1}$$

where $0 < M \le 8\pi$. The MT inequality plays an important role in problems of prescribing Gauss curvature, see Aubin [1], Chang-Yang [7] and the references therein. A sharp version of (1.1), which is due to Onofri [15], states that (1.1) is valid with the optimal $C = 0$ on the right hand side. Moser's original proof of (1.1) was obtained as a corollary of his stronger inequality:

$$\int_{S^2} e^{4\pi u^2} \le C, \quad \forall u \in H^1(S^2) \text{ with } \int_{S^2} u = 0 \text{ and } \int_{S^2} |\nabla u|^2 = 1. \tag{1.2}$$

Onofri's proof of the sharp form of (1.1), i.e. with $C = 0$, used estimates of Aubin [1] and the conformal invariance of the functional. Other proofs of Onofri's result were later given by Hong [10] and by Osgood, Phillips and Sarnak [16]. Of a particular interest to us is the alternative derivation of Onofri's inequality by Beckner [2] (see also Carlen and Loss [5]), which is based on a duality principle and on Lieb's sharp form of the Hardy-Littlewood-Sobolev inequality. [1] On S^2 the dual inequality takes the form,

$$\psi_{S^2}(\rho) := \int_{S^2} \rho \log \rho + \frac{1}{4\pi} \int_{S^2} \int_{S^2} \rho(\omega_1) \log |\omega_1 - \omega_2| \rho(\omega_2)\, d\omega_1 d\omega_2 \ge -C, \tag{1.3}$$

for all $\rho \in \Gamma_M(S^2)$, if $M \le 8\pi$, where

$$\Gamma_M(S^2) := \left\{ \rho \ge 0 \text{ with } \int_{S^2} \rho \log \rho < \infty \ \text{ and } \int_{S^2} \rho = M \right\}, \tag{1.4}$$

[1] Actually Beckner's result generalizes Onofri's inequality to any dimension

and $|\omega_1 - \omega_2|$ stands for the Euclidean distance between ω_1 and ω_2 in $\mathbb{R}^3$.

The generalization of the functional ψ_{S^2} to the system case is the functional

$$\Psi_{S^2}(\boldsymbol{\rho}) = \sum_{i\in I} \int_{S^2} \rho_i \log \rho_i + \frac{1}{4\pi} \sum_{i,j\in I} a_{i,j} \int_{S^2}\int_{S^2} \rho_i(\omega_1) \log|\omega_1 - \omega_2| \rho_j(\omega_2)\, d\omega_1 d\omega_2 \,, \tag{1.5}$$

considered on the domain,

$$\boldsymbol{\Gamma_M}(S^2) = \left\{ \boldsymbol{\rho} = (\rho_i)_{i\in I} \text{ with } \rho_i \geq 0, \ \int_{S^2} \rho_i \log \rho_i < \infty \ \text{ and } \int_{S^2} \rho_i = M_i, \ \forall i \right\} , \tag{1.6}$$

where $I := \{1, 2, \ldots, n\}$ and $A := \{a_{i,j}\}$ is a symmetric n by n matrix. In the sequel we assume that $a_{i,j} \geq 0, \ \forall i, j$, but later we shall also study other classes of matrices. The duality relation between (1.1) and (1.3) can be extended, under the additional hypothesis that A is positive definite, to a duality between (1.5) and the functional

$$F^{\boldsymbol{M}}(\boldsymbol{u}) = \frac{1}{2} \sum_{i,j\in I} a_{i,j} \int_{S^2} \nabla u_i \nabla u_j - \sum_{i\in I} M_i \log\Big(\frac{1}{4\pi} \int_{S^2} \exp(\sum_{j\in I} a_{i,j} u_j)\Big) \tag{1.7}$$

over the class

$$\mathcal{H}_n(S^2) := \{\boldsymbol{u} \in (H^1(S^2))^n \text{ with } \int_{S^2} u_i = 0, \ \forall i\}.$$

Note that in the *scalar* case $n = 1$ it follows from the MT-inequality that a necessary and sufficient condition for the boundedness from below of f^M and ψ_{S^2} over $\mathcal{H}_1(S^2)$ and $\Gamma_M(S^2)$, respectively, is $M \leq 8\pi$. The analogue of this condition to the system case turns out to be a set of $2^n - 1$ inequalities involving the quadratic polynomials

$$\Lambda_J(\boldsymbol{M}) = 8\pi \sum_{i\in J} M_i - \sum_{i,j\in J} a_{i,j} M_i M_j = \sum_{i\in J} M_i(8\pi - \sum_{j\in J} a_{i,j} M_j), \tag{1.8}$$

for every nonempty subset $J \subseteq I$. The polynomial Λ_I was first introduced by Chanillo and Kiessling [6] in their study of entire solutions of Liouville systems in $\mathbb{R}^2$. A set of conditions ("sub-critical"),

$$\Lambda_J(\boldsymbol{M}) > 0, \quad \forall J \subseteq I, \ J \neq \emptyset. \tag{1.9}$$

was used in [4] for the study of a related variational problem on bounded domains in $\mathbb{R}^2$ and the associated minimizers, see also below. On the other hand, a simple rescaling argument (as in [4, Lemma 2.2]) shows that if for *some* J, $\Lambda_J(\boldsymbol{M}) < 0$, then $F^{\boldsymbol{M}}$ and Ψ_{S^2} are unbounded from below. Wang [20] proved an analogue result to that of [4] for compact surfaces, showing that the *subcritical* condition (1.9) is sufficient for the boundedness of (the analogue of) $F^{\boldsymbol{M}}$ in this case. A natural question that we address here is whether the bound still holds in the *critical case*, i.e. when we turn some, or all, of the inequalities in (1.9) into equalities to get the weaker condition

$$\Lambda_J(\boldsymbol{M}) \geq 0, \quad \forall J \subseteq I, \ J \neq \emptyset\,. \tag{1.10}$$

One of our main results, Theorem 2, asserts that the bound for Ψ_{S^2} (res. $F^{\boldsymbol{M}}$) indeed holds under assumption (1.10) if $a_{i,i} > 0, \ \forall i$, but a slightly stronger condition is needed if we allow zero diagonal elements (see (2.16) below). So far, results for the *critical case* were obtained only for very special systems by Wang [20] and by Jost and Wang [11], see (2.11) and below. We shall also consider analogous functionals on the all plane $\mathbb{R}^2$ as well as on bounded domains $\Omega \subset \mathbb{R}^2$. In certain cases, when bound is verified, we shall address the question of the existence of minimizers.

In the next section we introduce the full details of our main results. Some of them were announced in [18]. The proofs are given in the proceeding sections.

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2 Description of main results

Before stating our main results we want to focus on two important notions which demonstrate the similarities and differences between the scalar problem and the vectorial one. The first notion is *duality*.

In the *scalar case* $n = 1$, the functionals (1.1) on $\mathcal{H}_1(S^2)$ and (1.3) on $\Gamma_M(S^2)$ are dual in the sense that both are bounded or unbounded from below, simultaneously. Moreover, a minimizer u for f^M, if exists, is related to a minimizer ρ of ψ_{S^2} via $\rho = Me^u/\int_{S^2} e^u$ and $-\Delta u = \rho - \frac{M}{4\pi}$. This duality extends to the vector functionals Ψ_{S^2} (on $\mathbf{\Gamma_M}(S^2)$) and $F^{\boldsymbol{M}}$ (on $\mathcal{H}_n(S^2)$) provided the matrix $\{a_{i,j}\}$ is *positive definite* (see Section 3). However, our results for Ψ_{S^2} do not require this condition. Our basic assumption is

$$a_{i,j} \geq 0, \ \forall i, j \,. \tag{2.1}$$

but we shall also study other classes of matrices in Subsection 5.2.

Next we turn to the notion of *conformal invariance*. It is known that $\psi_{S^2}(\rho)$ is conformally invariant in the critical case $M = 8\pi$. By this we mean that the l.h.s. of (1.3) is invariant under the conformal action,

$$\rho \longmapsto \rho^\tau := (\rho \circ \tau) \cdot |\mathcal{J}_\tau|, \tag{2.2}$$

where τ is any conformal automorphism of S^2 and $\mathcal{J}_\tau$ is its Jacobian (here and in the sequel we do not distinguish between conformal and anti-conformal automorphisms). We shall often apply (2.2) for a special class of automorphisms, $\{\tau_{y,\alpha} : y \in S^2, \alpha \in \mathbb{R}_+\}$, defined as follows,

$$\tau_{y,\alpha}(x) = \mathcal{S}_y{}^{-1}\big(\alpha \mathcal{S}_y(x)\big), \quad \forall x \in S^2, \tag{2.3}$$

where $\mathcal{S}_y$ is the stereographic projection satisfying $\mathcal{S}_y(y) = \infty$. However, in the vectorial case (i.e. $n \geq 2$) it turns out that the analogue condition to $M = 8\pi$, namely $\Lambda_I(\boldsymbol{M}) = 0$, is *not* sufficient, in general, to ensure the conformal invariance of Ψ_{S^2} on $\mathbf{\Gamma_M}(S^2)$.

In fact, Ψ_{S^2} is clearly invariant w.r.t. the action (2.2) (applied to $\rho = \rho_i, \ i \in I$) when τ is an *isometry* of S^2. But the conformal group contains also other automorphisms whose action is more transparent when we use the stereographic projection to transform the problem to

$\mathbb{R}^2$. Using $\mathcal{S} = \mathcal{S}_{\mathcal{N}}$ ($\mathcal{N}$ denoting the north pole) we associate to each $\boldsymbol{\rho} : S^2 \to \mathbb{R}^n$ a function $\tilde{\boldsymbol{\rho}} : \mathbb{R}^2 \to \mathbb{R}^n$ via the transformation (see [2, 5])

$$\begin{cases} \tilde{\rho}_i \longleftrightarrow \rho_i = [\tilde{\rho}_i \cdot (1+|x|^2)^2/4] \circ \mathcal{S}, \\ \rho_i \longleftrightarrow \tilde{\rho}_i = \dfrac{4}{(1+|x|^2)^2} \cdot (\rho_i \circ \mathcal{S}^{-1}), \end{cases} \quad \forall i \in I. \tag{2.4}$$

Note that

$$|\mathcal{S}^{-1}(x) - \mathcal{S}^{-1}(y)| = \frac{2|x-y|}{(1+|x|^2)^{1/2}(1+|y|^2)^{1/2}}, \quad \forall x, y \in \mathbb{R}^2. \tag{2.5}$$

By a simple computation, using (2.5), we obtain for $\boldsymbol{\rho}$ and $\tilde{\boldsymbol{\rho}}$ which are related by (2.4),

$$\begin{aligned} \Psi_{S^2}(\boldsymbol{\rho}) = \widetilde{\Psi}_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}}) := \sum_{i\in I} \int_{\mathbb{R}^2} \tilde{\rho}_i \log \tilde{\rho}_i \, dx + \frac{1}{4\pi} \sum_{i,j\in I} a_{i,j} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \tilde{\rho}_i(x) \log|x-y| \tilde{\rho}_j(y) \, dx \, dy \\ + \sum_{i\in I} \nu_i \int_{\mathbb{R}^2} \tilde{\rho}_i \log(1+|x|^2) \, dx - \frac{1}{4\pi} \Lambda_I(\boldsymbol{M}) \log 2, \end{aligned} \tag{2.6}$$

with

$$\nu_i = 2 - \frac{1}{4\pi} \sum_{j\in I} a_{i,j} M_j, \quad \forall i \in I. \tag{2.7}$$

It is clear that,

$$\int_{\mathbb{R}^2} \tilde{\rho}_i = \int_{S^2} \rho_i, \quad \forall i \in I.$$

Moreover, using the arguments of [5] it can be shown that Ψ_{S^2} is bounded below over $\boldsymbol{\Gamma_M}(S^2)$ (see (1.6)) if and only if $\widetilde{\Psi}_{\mathbb{R}^2}$ is bounded below on

$$\boldsymbol{\Gamma_M}(\mathbb{R}^2) = \{\tilde{\boldsymbol{\rho}} \mid \tilde{\rho}_i \geq 0, \int_{\mathbb{R}^2} \tilde{\rho}_i |\log \tilde{\rho}_i| < \infty, \int_{\mathbb{R}^2} \tilde{\rho}_i = M_i, \int_{\mathbb{R}^2} \tilde{\rho}_i \log(1+|x|^2) < \infty, \forall i \in I\}. \tag{2.8}$$

Now we can observe that the functional,

$$\Psi_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}}) = \sum_{i\in I} \int_{\mathbb{R}^2} \tilde{\rho}_i \log \tilde{\rho}_i \, dx + \frac{1}{4\pi} \sum_{i,j\in I} a_{i,j} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \tilde{\rho}_i(x) \log|x-y| \tilde{\rho}_j(y) \, dx \, dy \tag{2.9}$$

is invariant w.r.t. tranlations, i.e. $\tilde{\rho}_i(x) \longmapsto \tilde{\rho}_i(x+v)$, $\forall i \in I$ (for any fixed $v \in \mathbb{R}^2$) and dilatations, i.e. $\tilde{\rho}_i(x) \longmapsto \alpha^2 \tilde{\rho}_i(\alpha x)$, $\forall i \in I$ (for any fixed $\alpha > 0$), provided that $\Lambda_I(\boldsymbol{M}) = 0$. But clearly the difference between the two functionals,

$$\widetilde{\Psi}_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}}) - \Psi_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}}) = \sum_{i\in I} \nu_i \int_{\mathbb{R}^2} \tilde{\rho}_i \log \frac{1+|x|^2}{2} \, dx, \tag{2.10}$$

is *not invariant* w.r.t. tranlations and dilatations *unless* $\nu_i = 0$, $\forall i \in I$. We shall call this last case, i.e. when

$$\sum_{j\in I} a_{i,j} M_j = 8\pi, \quad \forall i \in I, \tag{2.11}$$

the *conformal case*. Indeed, from the above we see that (2.11) is a necessary and sufficient condition for the *full invariance* of the functional Ψ_{S^2} w.r.t. the conformal group of the sphere. Evidently, (2.11) implies that $\Lambda_I(\boldsymbol{M}) = 0$, but as explained above, the converse is false in general. We should mention that Wang [20] studied a special case of the conformal case (2.11) in which the positive definite matrix A is stochastic, namely

$$\sum_{i\in I} a_{i,j} = 1, \quad \forall j \in I, \tag{2.12}$$

and the vector of masses $\boldsymbol{M}$ satisfies $M_i = 8\pi$, $\forall i \in I$. Under these assumptions he proved that the functional $F^{\boldsymbol{M}}$ is bounded below on $\mathcal{H}_n(S^2)$ (this clearly implies boundedness also when $M_i \le 8\pi$, $\forall i$). Actually, the result of Wang is more general since he studied a functional which is defined on any closed surface Σ (i.e. two dimensional compact Riemannian manifold without boundary). In Section 3 we shall prove the following *optimal* result for the conformal case on S^2, which can be viewed as the natural generalization of the results of Onofri and Beckner to the system case (since it gives the optimal *additive* constant). Here again we see the advantage of using the dual formulation: it allows us to deduce easily the system analogue from Beckner's scalar result.

We recall that a symmetric matrix A is called *irreducible* if for all $i, j \in I$, there exist $\{k_1, \ldots, k_l\} \in I$ with $k_1 = i$ and $k_l = j$, such that $a_{k_1,k_2} \cdot a_{k_2,k_3} \cdots a_{k_{l-1},k_l} \ne 0$. Equivalently, A is irreducible if there is no $\emptyset \ne J \subsetneqq I$ such that $a_{i,j} = 0$, $\forall i \in J$, $\forall j \notin J$. Any symmetric matrix A can be decomposed into a sum of irreducible matrices, inducing a decomposition of the functional Ψ_{S^2} to a sum of independent functionals, each corresponding to an irreducible factor. The assumption of irreducibility is useful for some uniqueness questions.

Theorem 1. *Let A be a symmetric matrix satisfying* (2.1) *and $\boldsymbol{M} \in \mathbb{R}^n_+$ such that* (2.11) *holds. Denoting $\boldsymbol{\rho}^0 = (\frac{M_1}{4\pi}, \ldots, \frac{M_n}{4\pi})$ and $\tilde{\boldsymbol{\rho}}^0 = (\frac{M_1}{\pi(1+|x|^2)^2}, \ldots, \frac{M_n}{\pi(1+|x|^2)^2})$ we have:*
(i)

$$\min_{\boldsymbol{\Gamma}_{\boldsymbol{M}}(S^2)} \Psi_{S^2} = \Psi_{S^2}(\boldsymbol{\rho}^0), \tag{2.13}$$

and

$$\min_{\boldsymbol{\Gamma}_{\boldsymbol{M}}(\mathbb{R}^2)} \Psi_{\mathbb{R}^2} = \Psi_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}}^0). \tag{2.14}$$

(ii) The conformal images (as defined in (2.2)*) of the constant vector $\boldsymbol{\rho}^0$ are minimizers in* (2.13)*.*
(iii) The conformal images of $\tilde{\boldsymbol{\rho}}^0$ (i.e. $(\tilde{\boldsymbol{\rho}}^0 \circ \tau) \cdot |\mathcal{J}_\tau|$, $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ conformal) are minimizers in (2.14)*.*
(iv) If, in addition, the matrix A is positive definite, then,

$$F^{\boldsymbol{M}}(\boldsymbol{u}) \ge 0, \quad \forall \boldsymbol{u} \in \mathcal{H}_n(S^2), \tag{2.15}$$

with equality if

$$\boldsymbol{u} = (\frac{M_1}{8\pi} \log |\mathcal{J}_\tau|, \ldots, \frac{M_n}{8\pi} \log |\mathcal{J}_\tau|) + \boldsymbol{c},$$

where $\boldsymbol{c}$ is a constant vector.
(v) Assume now that A is irreducible. Then, the minimizers given explicitly in (ii)–(iv) are

the unique minimizers. Also, in the subconformal case*, i.e. when* $\nu_i \geq 0$, $\forall i \in I$ *with at least one strict inequality,* $\boldsymbol{\rho}^0$ *is the unique minimizer in* (2.13) *and if* A *is positive definite, then* $\boldsymbol{u} = \boldsymbol{0}$ *is the unique minimizer in* (2.15).

As explained above, for $n \geq 2$ the conformal case is exceptional among the critical configurations of A and $\boldsymbol{M}$. Our next theorem provides an optimal criterion for boundedness from below of the functionals $\Psi_{S^2}(\boldsymbol{\rho})$ and $F^{\boldsymbol{M}}$ in the general case. As it turns out, this criterion requires a slightly stronger condition than (1.10), namely,

$$\begin{cases} \Lambda_J(\boldsymbol{M}) \geq 0 \text{ for all } \emptyset \neq J \subseteq I, \\ \text{if } \Lambda_J(\boldsymbol{M}) = 0 \text{ for some } J, \text{ then } a_{i,i} + \Lambda_{J \setminus \{i\}}(\boldsymbol{M}) > 0, \ \forall i \in J. \end{cases} \tag{2.16}$$

Remark 2.1. *Note that* (2.16) *is equivalent to* (1.10) *if the matrix* A *has a positive diagonal, namely* $a_{i,i} > 0$ *for all* $i \in I$.

Theorem 2. *Let* A *be a symmetric matrix satisfying* (2.1) *and* $\boldsymbol{M} \in \mathbb{R}^n_+$. *Then,*

(i) Condition (2.16) *is necessary and sufficient for the boundedness from below of* Ψ_{S^2} *on* $\boldsymbol{\Gamma}_{\boldsymbol{M}}(S^2)$.

(ii) If, in addition, the matrix A *is positive definite, then condition* (2.16) *is necessary and sufficient for the boundedness from below of* $F^{\boldsymbol{M}}$ *on* $\mathcal{H}_n(S^2)$.

Remark 2.2. *In the general non-conformal critical case, in contrast with the conformal case, we do not know whether minimizers exist, both in (i) and (ii). What we do know is that* $\boldsymbol{\rho}^0 = (\frac{M_1}{4\pi}, \ldots, \frac{M_n}{4\pi})$ *is not a minimizer for* Ψ_{S^2} *and that* $\boldsymbol{u}^0 \equiv 0$ *is not a minimizer for* $F^{\boldsymbol{M}}$ *(although both are solutions of the corresponding Euler-Lagrange equations), see Proposition 3.1 below. In [19] we obtained a generalization of Theorem 2(i) for compact manifolds in dimension* $N \geq 2$.

So far we considered only nonnegative matrices A (i.e. those satisfying (2.1)). However, there is interest in studying a more general class of systems, namely that of *collaborating systems*. These are systems associated with symmetric matrices A which have the following structure: there exists a decomposition of I as a disjoint union of K ($1 \leq K \leq n$) subsets $I_1, \ldots, I_K$, such that,

$$\begin{cases} a_{i,j} \geq 0, \quad \forall i,j \in I_l, \, l = 1, \ldots, K, \\ a_{i,j} \leq 0, \quad \forall i \in I_l, \ \forall j \in I_m, \forall l \neq m, \ 1 \leq l,m \leq K. \end{cases} \tag{2.17}$$

The case $K = 1$ corresponds of course to a nonnegative matrix, but for $K \geq 2$, an assumption that we shall make in the sequel, we obtain new types of matrices. Of a particular interest is the extreme case $K = n$. Here all the I_l's are singletons and we get the condition,

$$a_{i,i} \geq 0, \ \forall i \quad \text{and} \quad a_{i,j} \leq 0, \ \forall i \neq j. \tag{2.18}$$

In fact, Jost and Wang studied in [11] a special system of this type, the Toda system, which corresponds to the case where A is the Cartan matrix for $SU(n+1)$, i.e.

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & \ddots & 0 & \vdots \\ 0 & -1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 & 0 \\ \vdots & 0 & \ddots & -1 & 2 & -1 \\ 0 & \dots & \dots & 0 & -1 & 2 \end{pmatrix}. \tag{2.19}$$

For A in the class (2.18) the condition (1.10) (which is easily seen to be equivalent to (2.16) in this case) simplifies to

$$0 \leq \Lambda_{\{i\}}(\boldsymbol{M}) = 8\pi - a_{i,i}M_i, \quad \forall i. \tag{2.20}$$

Indeed, the validity of (1.10) for all singletons, $J = \{i\}$, $\forall i$, implies its validity for all $J \subseteq I$, since all the off-diagonal elements of A are non-positive. In fact, in [11] it was proved that for the Toda system condition (2.20), i.e. $M_i \leq 4\pi$, $\forall i$, is sufficient for the corresponding $F^{\boldsymbol{M}}$ to be bounded below on $\mathcal{H}_n(S^2)$ (necessity is known to hold true). Actually, Jost and Wang proved the analogue result on every 2-dimensional compact surface . Using the dual formulation we are able to obtain a very simple proof of the boundedness from below of Ψ_{S^2} on $\boldsymbol{\Gamma}_{\boldsymbol{M}}(S^2)$ in this case. Moreover, we are able to compute the exact value of the infimum and to prove that the infimum is not achieved. We summarize our results for general collaborating systems in the following theorem.

Theorem 3. *Let A be a symmetric matrix corresponding to a collaborating system with $K \geq 2$ blocks and let $\boldsymbol{M} \in \mathbb{R}^n_+$ be given. Then,*

(i) The validity of (2.16) *for each I_l, i.e.*

$$\forall l = 1, \dots, K: \quad \begin{cases} \Lambda_J(\boldsymbol{M}) \geq 0 \text{ for all } \emptyset \neq J \subseteq I_l, \\ \text{if } \Lambda_J(\boldsymbol{M}) = 0 \text{ for some } J \subseteq I_l, \text{ then } a_{i,i} + \Lambda_{J\setminus\{i\}}(\boldsymbol{M}) > 0, \ \forall i \in J, \end{cases} \tag{2.21}$$

is a necessary and sufficient condition for the boundedness from below of Ψ_{S^2} on $\boldsymbol{\Gamma}_{\boldsymbol{M}}(S^2)$, and when A is positive definite, for the boundedness from below of $F^{\boldsymbol{M}}$ on $\mathcal{H}_n(S^2)$.
(ii) If $K = n$, i.e. A satisfies (2.18), *and if the critical case conditions, $a_{i,i}M_i = 8\pi$, $\forall i$, are satisfied, then,*

$$\inf_{\boldsymbol{\Gamma}_{\boldsymbol{M}}(S^2)} \Psi_{S^2} = \sum_{i \in I} \left[M_i \log \frac{M_i}{4\pi} + (\frac{M_i}{2\pi}) c_0 \right] - \frac{1}{4\pi} \sup_{(S^2)^n} W(\boldsymbol{x}), \tag{2.22}$$

where

$$W(\boldsymbol{x}) := \sum_{i \neq j} (-a_{i,j}) M_i M_j, \log |x_i - x_j| \tag{2.23}$$

and

$$c_0 = \int_{S^2} \log|x-y| \, dy = 2\pi(\log 4 - 1). \tag{2.24}$$

(iii) Under the assumptions of (ii), if we suppose in addition that A does not have a row of zeros (ignoring the diagonal), then the infimum in (2.22) is not attained. Moreover, any weak limit (in the sense of measures) of a minimizing sequence is of the form $\boldsymbol{\rho} = (M_1\delta_{x_1}, \dots, M_n\delta_{x_n})$, where $\boldsymbol{x} = (x_1, \dots, x_n) \in \left(S^2\right)^n$ is a maximizer of W.

In Subsection 5.2 we shall present a variant of Theorem 3(ii),(iii) for a more general class of matrices than (2.18).

Remark 2.3. *Since the proof of part (i) of Theorem 3 uses only the scalar Moser-Trudinger inequality, which is known to be true on any two dimensional compact surface (see [9, 14]), it follows that the assertion in (i) is valid in this more general setting (as proved in [11] for the Toda system).*

Next we present two results on related variational problems, on $\mathbb{R}^2$ and on a bounded domain $\Omega \subset \mathbb{R}^2$. In the sequel we shall assume again that A satisfies (2.1). We shall first describe an entropy inequality which involves the functional $\Psi_{\mathbb{R}^2}$, already defined in (2.9). We claim that this functional is well defined on $\boldsymbol{\Gamma}_{\boldsymbol{M}}(\mathbb{R}^2)$ for every $\boldsymbol{M} \in \mathbb{R}^n_+$. Indeed, using (2.5), the stereographic projection and the obvious fact that the Euclidean distance between any two points on the unit sphere is less or equal to 2, we get the elementary inequality,

$$\log|x-y| \le \frac{1}{2}\log(1+|x|^2) + \frac{1}{2}\log(1+|y|^2), \quad \forall x, y \in \mathbb{R}^2. \tag{2.25}$$

By (2.25) we obtain for any $\tilde{\boldsymbol{\rho}} \in \boldsymbol{\Gamma}_{\boldsymbol{M}}(\mathbb{R}^2)$, as in [5], that

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \tilde{\rho}_i(x)\log|x-y|\tilde{\rho}_j(y)\,dxdy < \infty, \quad \forall i, j.$$

On the other hand, using (3.3) below, we get that

$$\int_{\{|y-x|\le 1\}} \tilde{\rho}_i(y)\log\big(\frac{1}{|x-y|}\big)\,dy \le C, \quad \forall x \in \mathbb{R}^2, \forall i.$$

Hence

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \tilde{\rho}_i(x)\log|x-y|\tilde{\rho}_j(y)\,dxdy > -\infty, \quad \forall i, j,$$

and our claim follows. From the discussion after (2.8) it follows that $\Psi_{\mathbb{R}^2}$ is invariant w.r.t. translations and dilatations, but, in general, not w.r.t. the other conformal actions of $\mathbb{R}^2$. The invariance w.r.t. the noncompact actions, translations and dilatations, implies that $\Lambda_I(\boldsymbol{M}) = 0$ is a necessary condition for boundedness below of $\Psi_{\mathbb{R}^2}$. Using similar techniques to those used in the proof of Theorem 2, we obtain an analogue result concerning the boundedness of the functional. In addition, we establish an existence result for minimizers.

Theorem 4. *Let A be a symmetric matrix satisfying (2.1) and $\boldsymbol{M} \in \mathbb{R}^n_+$. Then:*

(i) $\Lambda_I(\boldsymbol{M}) = 0$ and (2.16) are necessary and sufficient conditions for the boundedness from below of $\Psi_{\mathbb{R}^2}$ on $\boldsymbol{\Gamma}_{\boldsymbol{M}}(\mathbb{R}^2)$.
(ii) There exists a minimizer $\boldsymbol{\rho}$ for $\Psi_{\mathbb{R}^2}$ over $\boldsymbol{\Gamma}_{\boldsymbol{M}}(\mathbb{R}^2)$ if and only if

$$\Lambda_I(\boldsymbol{M}) = 0 \text{ and } \Lambda_J(\boldsymbol{M}) > 0, \ \forall J \subsetneq I. \tag{2.26}$$

Finally we turn to a version of Moser-Trudinger inequality for systems on bounded domains. By Moser's inequality [13],

$$\frac{1}{2}\int_\Omega |\nabla u|^2 - 8\pi \log\Big(\int_\Omega e^u\Big) \geq -C, \quad \forall u \in H_0^1(\Omega), \tag{2.27}$$

where Ω is a bounded domain in $\mathbb{R}^2$. The extension to systems is expected to take the form,

$$\frac{1}{2}\sum_{i,j\in I}\int_\Omega a_{i,j}\nabla u_i \nabla u_j - \sum_{i\in I} M_i \log\Big(\int_\Omega \exp(\sum_{j\in I} a_{i,j}u_j)\Big) \geq -C, \quad \forall \boldsymbol{u} \in (H_0^1(\Omega))^n, \tag{2.28}$$

where A is a matrix satisfying (2.1). In [4] it was shown that (2.28) holds in the *subcritical case* (1.9). The question whether the same result remains valid in the *critical case* was left open. Our last theorem provides a positive answer to that question. Here again we use a dual formulation, involving the Green function $G_\Omega(x,y)$ for the operator $-\Delta$ on Ω with Dirichlet boundary conditions.

Theorem 5. *Let Ω be a bounded domain in $\mathbb{R}^2$, A a symmetric matrix and $\boldsymbol{M} \in \mathbb{R}^n_+$. Then,*

(i) If A satisfies (2.1) *then condition* (2.16) *is necessary and sufficient for the boundedness from below of*

$$\Psi_\Omega(\boldsymbol{\rho}) = \sum_{i\in I}\int_\Omega \rho_i(x)\log\rho_i(x)\,dx - \frac{1}{2}\sum_{i,j\in I} a_{i,j}\int_\Omega\int_\Omega \rho_i(x)G_\Omega(x,y)\rho_j(y)\,dx\,dy\,, \tag{2.29}$$

over

$$\boldsymbol{\Gamma}_{\boldsymbol{M}}(\Omega) = \Big\{\boldsymbol{\rho} = (\rho_i)_{i\in I} \text{ with } \rho_i \geq 0, \ \int_\Omega \rho_i \log \rho_i < \infty \ \text{ and } \int_\Omega \rho_i = M_i, \ \forall i\Big\}. \tag{2.30}$$

If, in addition, the matrix A is positive definite, then condition (2.16) *is necessary and sufficient for* (2.28) *to hold.*

(ii) More generally, if A corresponds to a collaborating system with K blocks, then (2.21) *is a necessary and sufficient condition for the boundedness from below of Ψ_Ω on $\boldsymbol{\Gamma}_{\boldsymbol{M}}(\Omega)$, and when A is positive definite, for the validity of* (2.28).

3 On duality and conformal invariance

In this section we explore the two important notions of duality and conformal invariance. In particular, we shall prove Theorem 1 on the conformal case.

We begin by presenting a duality principle which connects the Moser-Trudinger functional $F^{\boldsymbol{M}}$ with Ψ_{S^2}. An analogue statement for the problem on a bounded domain was proved in [4, Proposition 2.1]. A simple adaptation of the argument yields the result for our context too, but we prefer to present a slightly different approach which involves the functional

$$\Phi(\boldsymbol{\rho},\boldsymbol{u}) = \sum_{i\in I}\int_{S^2}\rho_i\log\rho_i + \sum_{i,j\in I} a_{i,j}\Big[\int_{S^2}\frac{1}{2}\nabla u_i\cdot\nabla u_j - \rho_i u_j\Big],$$

defined on $\boldsymbol{\Gamma}_{\boldsymbol{M}}(S^2)\times\mathcal{H}_n(S^2)$.

Lemma 3.1. *Let A be a symmetric $n\times n$ matrix. Then, for any fixed $\boldsymbol{u}\in\mathcal{H}_n(S^2)$, $\Phi(\cdot,\boldsymbol{u})$ is bounded from below on $\boldsymbol{\Gamma_M}(S^2)$ and*

$$\inf_{\boldsymbol{\rho}\in\boldsymbol{\Gamma_M}(S^2)}\Phi(\boldsymbol{\rho},\boldsymbol{u})=F^{\boldsymbol{M}}(\boldsymbol{u})+\sum_{i\in I}M_i\log\frac{M_i}{4\pi}\,. \tag{3.1}$$

Moreover, the infimum in (3.1) *is uniquely attained at $\bar{\boldsymbol{\rho}}$ given by,*

$$\bar\rho_i=M_i\frac{\exp(\sum_{j\in I}a_{i,j}u_j)}{\int_{S^2}\exp(\sum_{j\in I}a_{i,j}u_j)}\,,\quad\forall i\in I. \tag{3.2}$$

Proof. Applying the elementary inequality,

$$ab\le b\log b-b+e^a,\quad\forall a\in\mathbb{R},\,\forall b\in\mathbb{R}_+, \tag{3.3}$$

with $a=u_j(x)/\gamma$ and $b=\gamma\rho_i(x)$, and using (1.1) we obtain that for all $i,j\in I$ and $\gamma>0$ there holds,

$$\int_{S^2}\rho_iu_j\le\gamma\int_{S^2}\rho_i\log\rho_i+\int_{S^2}e^{u_j/\gamma}+\gamma M_i\log(\gamma/e)\le\gamma\int_{S^2}\rho_i\log\rho_i+C(u_j,M_i,\gamma).$$

Therefore, for some $\varepsilon>0$ and $C=C(\boldsymbol{u},\boldsymbol{M},\varepsilon)$ we have,

$$\Phi(\boldsymbol{\rho},\boldsymbol{u})\ge\varepsilon\sum_{i\in I}\int_{S^2}\rho_i\log\rho_i-C.$$

Hence, for each fixed $\boldsymbol{u}\in\mathcal{H}_n(S^2)$ the functional $\Phi(\cdot,\boldsymbol{u})$, which is continuous and strictly convex on the closed and convex subset $\boldsymbol{\Gamma_M}(S^2)$ of the reflexive Banach space (Orlicz space),

$$X=\{\boldsymbol{\rho}=(\rho_i)_{i\in I}\text{ with }\int_{S^2}(1+|\rho_i|)\log(1+|\rho_i|)-|\rho_i|<\infty,\ \forall i\},$$

satisfies the coercivity condition,

$$\Phi(\boldsymbol{\rho},\boldsymbol{u})\to\infty\text{ as }\sum_{i\in I}\int_{S^2}\rho_i\log\rho_i\to\infty,\quad\text{for }\boldsymbol{\rho}\in\boldsymbol{\Gamma_M}(S^2).$$

It follows that the minimum of $\Phi(\cdot,\boldsymbol{u})$ over $\boldsymbol{\Gamma_M}(S^2)$ is attained, and the unique minimizer $\boldsymbol{\rho}$ must satisfy (3.2). Plugging it in Φ we are led to (3.1). □

Next we examine the infimum of $\Phi(\boldsymbol{\rho},\cdot)$ w.r.t. $\boldsymbol{u}$, for a fixed $\boldsymbol{\rho}$, under the additional assumption that A is positive definite.

Lemma 3.2. *Suppose A is positive definite. Then, for any fixed $\boldsymbol{\rho}\in\boldsymbol{\Gamma_M}(S^2)$, the infimum of $\Phi(\boldsymbol{\rho},\cdot)$ over $\mathcal{H}_n(S^2)$ is attained by $\bar{\boldsymbol{u}}$ given by,*

$$-\Delta\bar u_i=\rho_i-\frac{M_i}{4\pi}\quad\text{in }S^2,\quad\forall i\in I. \tag{3.4}$$

Moreover,

$$\min_{\boldsymbol{u}\in\mathcal{H}_n(S^2)}\Phi(\boldsymbol{\rho},\boldsymbol{u})=\sum_{i\in I}\int_{S^2}\rho_i\log\rho_i+\frac{1}{4\pi}\sum_{i,j\in I}a_{i,j}\int_{S^2}\int_{S^2}\big(\rho_i(x)-\frac{M_i}{4\pi}\big)\log|x-y|\big(\rho_j(y)-\frac{M_j}{4\pi}\big)\,dxdy. \tag{3.5}$$

Proof. Similarly to the proof of Lemma 3.1 we get that $\Phi(\boldsymbol{\rho},\cdot)$ is strictly convex and coercive over $\mathcal{H}_n(S^2)$ (here we use the assumption that A is positive definite), and it follows that the minimum is attained at a unique $\boldsymbol{u}\in\mathcal{H}_n(S^2)$. Taking the variation of $\Phi(\boldsymbol{\rho},\boldsymbol{u})$ with respect to each u_j yields (3.4). The Green function $G(x,y)$ for $-\Delta$ on S^2 is by definition a solution to,

$$-\Delta_y G(x,y)=\delta_x-\frac{1}{4\pi}\quad \text{on } S^2. \tag{3.6}$$

The solution to (3.6) is unique up to an additive constant, and we can choose as a representative $G(x,y)=-\frac{1}{2\pi}\log|x-y|$ (here again $|x-y|$ stands for the Euclidean distance in $\mathbb{R}^3$). Therefore, we may rewrite (3.4) as

$$\bar{u}_i=G*(\rho_i-\frac{M_i}{4\pi})\quad \forall i\in I. \tag{3.7}$$

Plugging (3.7) in Φ and using the equality,

$$\begin{aligned}\int_{S^2}\frac{1}{2}\nabla\bar{u}_i\cdot\nabla\bar{u}_j-\rho_i\bar{u}_j&=\int_{S^2}\frac{1}{2}(\rho_i-\frac{M_i}{4\pi})\bar{u}_j-\rho_i\bar{u}_j\\&=-\frac{1}{2}\int_{S^2}\int_{S^2}\big(\rho_i(x)-\frac{M_i}{4\pi}\big)G(x,y)\big(\rho_j(y)-\frac{M_j}{4\pi}\big)\,dxdy\,,\end{aligned}$$

we are led to (3.5). □

A simple corollary of the above is the following analogue to [4, Proposition 2.1].

Corollary 3.1. *Let $A=(a_{i,j})$ be a positive definite matrix. Then, Ψ_{S^2} is bounded from below on $\boldsymbol{\Gamma}_{\boldsymbol{M}}(S^2)$ iff $F^{\boldsymbol{M}}$ is bounded from below on $\mathcal{H}_n(S^2)$ and*

$$\inf_{\boldsymbol{\rho}\in\boldsymbol{\Gamma}_{\boldsymbol{M}}(S^2)}\Psi_{S^2}(\boldsymbol{\rho})=\inf_{\boldsymbol{u}\in\mathcal{H}_n(S^2)}F^{\boldsymbol{M}}(\boldsymbol{u})+\sum_{i\in I}M_i\log\frac{M_i}{4\pi}+\frac{c_0}{16\pi^2}\sum_{i,j\in I}a_{i,j}M_iM_j\,, \tag{3.8}$$

where c_0 is defined in (2.24). Moreover, existence of minimizers for the two problems is equivalent, and the minimizers are related via (3.2) and (3.4).

Proof. The result follows immediately from Lemma 3.1 and Lemma 3.2 using,

$$\inf_{\boldsymbol{u}\in\mathcal{H}_n(S^2)}\inf_{\boldsymbol{\rho}\in\boldsymbol{\Gamma}_{\boldsymbol{M}}(S^2)}\Phi(\boldsymbol{\rho},\boldsymbol{u})=\inf_{\boldsymbol{\rho}\in\boldsymbol{\Gamma}_{\boldsymbol{M}}(S^2)}\inf_{\boldsymbol{u}\in\mathcal{H}_n(S^2)}\Phi(\boldsymbol{\rho},\boldsymbol{u})\,.$$

□

Next we turn to the notion of conformal invariance and present the simple proof of Theorem 1 concerning the conformal and subconformal cases.

Proof of Theorem 1. By the duality principle of Corollary 3.1 it follows that, for positive definite A, (2.13) is equivalent to (2.15). Indeed, it suffices to note that the function $w=\log|\mathcal{J}_\tau|$ satisfies the equation

$$-\Delta w=2(e^w-1)=2(|\mathcal{J}_\tau|-1)\,.$$

Hence, for $\rho_i = (\frac{M_i}{4\pi})|\mathcal{J}_\tau|$, (3.4) gives $\bar{u}_i = (\frac{M_i}{8\pi})\log|\mathcal{J}_\tau| + c_i$. As explained in Section 2, (2.13) is equivalent to (2.14) in the conformal case. It suffices thus to prove the assertions about Ψ_{S^2}. We recall the following form of the logarithmic Hardy-Littlewood-Sobolev inequality, see [2, Theorem 2] (it is equivalent to the critical case of (1.3)):

$$\int_{S^2} F\log F + \int_{S^2} G\log G + \frac{1}{\pi}\int_{S^2}\int_{S^2} F(x)\log|x-y|G(y)\,dxdy \geq 4c_0, \ \forall F, G \in \Gamma_{4\pi}(S^2), \tag{3.9}$$

with equality iff F and G both equal the same conformal image of the constant function $H \equiv 1$. Applying (3.9) to $F = (\frac{4\pi}{M_i})\rho_i$ and $G = (\frac{4\pi}{M_j})\rho_j$ yields, for each $i, j \in I$,

$$M_j \int_{S^2} \rho_i \log\rho_i + M_i \int_{S^2} \rho_j \log\rho_j + 4\int_{S^2}\int_{S^2} \rho_i(x)\log|x-y|\rho_j(y)\,dxdy \geq \Big(\frac{M_i M_j}{\pi}\Big)c_0 + M_i M_j \log\Big(\frac{M_i M_j}{16\pi^2}\Big). \tag{3.10}$$

Multiplying (3.10) by $\frac{a_{i,j}}{16\pi}$, and summing on i, j yields,

$$\sum_{i\in I}\Big(\frac{\sum_{j\in I} a_{i,j}M_j}{8\pi}\Big)\int_{S^2}\rho_i\log\rho_i + \frac{1}{4\pi}\sum_{i,j\in I} a_{i,j}\int_{S^2}\int_{S^2}\rho_i(x)\log|x-y|\rho_j(y)\,dxdy \geq \Big(\frac{c_0}{16\pi^2}\Big)\sum_{i,j\in I} a_{i,j}M_iM_j + \frac{1}{16\pi}\sum_{i,j\in I} a_{i,j}M_iM_j\log\Big(\frac{M_iM_j}{16\pi^2}\Big)$$
$$= \Big(\frac{c_0}{16\pi^2}\Big)\sum_{i,j\in I} a_{i,j}M_iM_j + \sum_{i\in I}(1-\frac{\nu_i}{2})M_i\log\big(\frac{M_i}{4\pi}\big), \tag{3.11}$$

which may be rewritten as,

$$\Psi_{S^2}(\boldsymbol{\rho}) \geq \Psi_{S^2}(\boldsymbol{\rho}^0) + \sum_{i\in I}\frac{\nu_i}{2}\int_{S^2}\rho_i\log(4\pi\rho_i/M_i), \tag{3.12}$$

where ν_i is defined in (2.7). In the conformal case, $\nu_i = 0$, $\forall i \in I$, and (2.13) follows from (3.12). Moreover, (ii) follows from conformal invariance, and the uniqueness assertion for this case in (v) follows from the characterization of uniqueness in (3.9).

In the subconformal case ($\nu_i \geq 0$, $\forall i$ and $\nu_{i_0} > 0$ for some i_0) we apply Jensen inequality,

$$\int_{S^2}(4\pi\rho_i/M_i)\log(4\pi\rho_i/M_i) \geq 0, \tag{3.13}$$

to the r.h.s. of (3.12) to infer that

$$\Psi_{S^2}(\boldsymbol{\rho}) \geq \Psi_{S^2}(\boldsymbol{\rho}^0). \tag{3.14}$$

For $\boldsymbol{\rho}^0$ there is an equality in (3.14). Therefore, $\boldsymbol{\rho}^0$ is a minimizer. Moreover, if $\boldsymbol{\rho}$ is any minimizer, then equality must hold in (3.10) for all $i, j \in I$ such that $a_{i,j} > 0$. Under the assumption that A is irreducible, it follows that $\boldsymbol{\rho}$ is a conformal image of $\boldsymbol{\rho}^0$. We claim that actually $\boldsymbol{\rho} = \boldsymbol{\rho}^0$. Indeed, assume by negation that $\boldsymbol{\rho}$ is a nontrivial conformal image of $\boldsymbol{\rho}^0$, and thus all its components are nonconstant. This yields strict inequality in (3.13) for all $i \in I$. By assumption there is an $i_0 \in I$ for which $\nu_{i_0} > 0$, and therefore, by (3.12) the inequality in (3.14) too is strict. Contradiction. □

Our next result shows that the minimizing property of the constant configuration $\boldsymbol{\rho}^0$ characterizes the conformal case among the critical cases. More precisely we have,

Proposition 3.1. *Let A be a symmetric matrix satisfying* (2.1). *Assume that $\Lambda_I(\boldsymbol{M}) = 0$ but that* (2.11) *is* not *satisfied, i.e., $\exists i_0$ such that $\nu_{i_0} \neq 0$. Then, $\boldsymbol{\rho}^0$ is* not *a minimizer for Ψ_{S^2} over $\boldsymbol{\Gamma_M}(S^2)$.*

Proof. Assume by negation that $\boldsymbol{\rho}^0$ is a minimizer. Then also $\tilde{\boldsymbol{\rho}}^0$ is a minimizer for $\widetilde{\Psi}_{\mathbb{R}^2}$ over $\boldsymbol{\Gamma_M}(\mathbb{R}^2)$. Using our assumption $\Lambda_I(\boldsymbol{M}) = 0$ (i.e. $\sum_{i\in I} \nu_i M_i = 0$) and the fact that the components of $\tilde{\boldsymbol{\rho}}^0$ are proportional to each other in (2.10) yields,

$$\Psi_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}}^0) = \widetilde{\Psi}_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}}^0). \tag{3.15}$$

Moreover, whenever $\Lambda_I(\boldsymbol{M}) = 0$ we also have,

$$\inf_{\boldsymbol{\Gamma_M}(\mathbb{R}^2)} \widetilde{\Psi}_{\mathbb{R}^2} \leq \inf_{\boldsymbol{\Gamma_M}(\mathbb{R}^2)} \Psi_{\mathbb{R}^2}\,. \tag{3.16}$$

Indeed, fix any $\tilde{\boldsymbol{\rho}} \in \boldsymbol{\Gamma_M}(\mathbb{R}^2)$. Then, the functional $\Psi_{\mathbb{R}^2}$ is invariant w.r.t. dilatations, i.e. $\Psi_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}}^{(\alpha)}) = \Psi_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}})$ for all $\alpha > 0$, where $\tilde{\boldsymbol{\rho}}^{(\alpha)}(x) = \alpha^2 \tilde{\boldsymbol{\rho}}(\alpha x)$. Moreover,

$$\lim_{\alpha\to\infty} \int_{\mathbb{R}^2} \tilde{\rho}_i^{(\alpha)} \log(1 + |x|^2)\, dx = \lim_{\alpha\to\infty} \int_{\mathbb{R}^2} \tilde{\rho}_i \log(1 + |x/\alpha|^2)\, dx = 0, \quad \forall i \in I,$$

and thus $\lim_{\alpha\to\infty} \widetilde{\Psi}_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}}^{(\alpha)}) = \Psi_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}})$. This clearly implies (3.16). Combining (3.16) with (3.15) it follows that $\tilde{\boldsymbol{\rho}}^0$ is also a minimizer of $\Psi_{\mathbb{R}^2}$, over $\Gamma_{\boldsymbol{M}}(\mathbb{R}^2)$. However, every minimizer $\tilde{\boldsymbol{\rho}}$ of $\widetilde{\Psi}_{\mathbb{R}^2}$ over $\Gamma_{\boldsymbol{M}}(\mathbb{R}^2)$ satisfies the Euler-Lagrange equations,

$$\tilde{\rho}_i(x) = \lambda_i (1 + |x|^2)^{-\nu_i} \exp\big(-\frac{1}{2\pi} \sum_{j\in I} a_{i,j} \int_{\mathbb{R}^2} \rho_j(y) \log|x - y|\, dy \big), \quad \forall i \in I, \tag{3.17}$$

for some positive constants $\lambda_1, \ldots, \lambda_n$. On the other hand, any minimizer of $\Psi_{\mathbb{R}^2}$ satisfies (3.17) with $\nu_i = 0$, $\forall i \in I$. Since $\tilde{\boldsymbol{\rho}}^0$ is a minimizer for both functionals, we get a contradiction for $i = i_0$. □

4 A basic estimate

Proposition 4.1 below provides the main tool for the proofs of Theorem 2, Theorem 4 and Theorem 5. Since in all these results condition (2.16) plays an important role, we begin with an interpretation of it. The proof of the following elementary lemma requires a simple modification of the proof of [4, Lemma 5.1].

Lemma 4.1. *Let A be a symmetric matrix and let $\boldsymbol{M} \in \mathbb{R}_+^n$ satisfy $\Lambda_I(\boldsymbol{M}) = 0$. Then, for each $i \in I$ the following two conditions are equivalent:*

$$\frac{\partial \Lambda_I}{\partial M_i}(\boldsymbol{M}) < 0, \tag{4.1}$$

$$\Lambda_{I\setminus\{i\}}(\boldsymbol{M}) + a_{i,i} M_i^2 > 0. \tag{4.2}$$

Proof. Since $\frac{\partial \Lambda_I}{\partial M_i}(\boldsymbol{M}) = 8\pi - 2\sum_{j\in I} a_{i,j} M_j$, we have,

$$0 = \Lambda_I(\boldsymbol{M}) = \Lambda_{I\setminus\{i\}}(\boldsymbol{M}) + a_{i,i}M_i^2 + M_i(8\pi - 2\sum_{j\in I} a_{i,j}M_j)$$
$$= \Lambda_{I\setminus\{i\}}(\boldsymbol{M}) + a_{i,i}M_i^2 + M_i \frac{\partial \Lambda_I}{\partial M_i}(\boldsymbol{M}),$$

and the result follows. □

The next lemma explains the significance of condition (2.16).

Lemma 4.2. *Let A be a symmetric matrix satisfying $a_{i,i} \geq 0$, $\forall i \in i$ and let $\boldsymbol{M} \in \mathbb{R}_+^n$ satisfy* (1.10)*. Then* (2.16) *is satisfied if and only if there is no edge of the box*

$$\mathcal{B}(\boldsymbol{M}) = \{\boldsymbol{N} \in \mathbb{R}_+^n \ : \ \boldsymbol{N} \leq \boldsymbol{M}\}.$$

on which Λ_I is identically zero

Proof. The 2^n vertices of $\mathcal{B}(\boldsymbol{M})$ are $\mathbf{0}$ and $\{\boldsymbol{M}_J\}_{\emptyset\neq J\subseteq I}$ where

$$(\boldsymbol{M}_J)_j = \begin{cases} M_j & \text{if } j \in J, \\ 0 & \text{otherwise.} \end{cases}$$

Assume first that (2.16) is not satisfied. Then, there exist $J \subseteq I$ and $i \in J$ such that $\Lambda_J(\boldsymbol{M}) = 0$, $a_{i,i} = 0$ and $\Lambda_K(\boldsymbol{M}) = 0$, for $K = J \setminus \{i\}$. Since $a_{j,j} \geq 0$, $\forall j \in I$, and $\Lambda_I \geq 0$ on all the vertices of $\mathcal{B}(\boldsymbol{M})$ (by (1.10)), it follows from the maximum principle that $\Lambda_I \geq 0$ on $\mathcal{B}(\boldsymbol{M})$. Let $f_J : [0, M_i] \to [0, \infty)$ denote the restriction of Λ_I to the edge connecting $\boldsymbol{M}_K$ to $\boldsymbol{M}_J$. Since $a_{i,i} = 0$, f_J is a linear function on $[0, M_i]$ satisfying $f_J(M_i) = \Lambda_J(\boldsymbol{M}) = 0$ and $f_J'(M_i) = 0$ (by Lemma 4.1). Therefore $f_J \equiv 0$ on $[0, M_i]$, i.e. $\Lambda_I = 0$ on the edge connecting $\boldsymbol{M}_K$ to $\boldsymbol{M}_J$.

Assume next that Λ_I is identically zero on some edge connecting $\boldsymbol{M}_K$ to $\boldsymbol{M}_J$ with $J = K \cup \{i\}$. Clearly $\Lambda_I(\boldsymbol{M}_J) = \Lambda_I(\boldsymbol{M}_K) = 0$ and $\frac{\partial \Lambda_I}{\partial M_i}(\boldsymbol{M}_J) = 0$. Therefore, by Lemma 4.1 we get that $\Lambda_K(\boldsymbol{M}) = a_{i,i} = 0$ and (2.16) fails to hold. □

Now we are in position to present the main result of this section.

Proposition 4.1. *Let A be a symmetric matrix with $a_{i,i} \geq 0$, $\forall i \in I$, and let $\boldsymbol{M} \in \mathbb{R}_+^n$. Then, there exists a constant C_0 such that*

$$\int_{-\infty}^{0} \sum_{i\in I} w_i' \log w_i' \, ds + \frac{1}{4\pi}\int_{-\infty}^{0} \Lambda_I(\boldsymbol{w}) \, ds \geq -C_0, \tag{4.3}$$

for all $\boldsymbol{w} = (w_1, \ldots, w_n)$ whose components are absolutely continuous on $\mathbb{R}_-$ and satisfy

$$w_i' \geq 0 \text{ on } (-\infty, 0), \lim_{s\to-\infty} w_i(s) = 0 \text{ and } w_i(0) = M_i, \ \forall i, \tag{4.4}$$

if and only if $\boldsymbol{M}$ satisfies condition (2.16)*.*

Proof. (i) We first prove the sufficiency of condition (2.16). Put

$$F(\boldsymbol{w}, \boldsymbol{w}') = \sum_{i \in I} w_i' \log w_i' + \frac{1}{4\pi} \Lambda_I(\boldsymbol{w}).$$

Applying the elementary inequality (3.3), with $a = \log(\frac{\Lambda_I(\boldsymbol{w})}{4\pi n})$ and $b = w_i'$, yields for each $s \in (-\infty, 0)$,

$$w_i' \log w_i' + \frac{\Lambda_I(\boldsymbol{w})}{4\pi n} \geq w_i' \big[\log \Lambda_I(\boldsymbol{w}) - \log 4\pi n + 1 \big] \,.$$

Therefore,

$$\int_{-\infty}^{0} F(\boldsymbol{w}, \boldsymbol{w}')\, ds \geq \int_{-\infty}^{0} \big(\sum_{i \in I} w_i' \big) \log \Lambda_I(\boldsymbol{w})\, ds - \log(\frac{4\pi n}{e}) \sum_{i \in I} M_i. \tag{4.5}$$

For each i we have,

$$\int_{-\infty}^{0} w_i' \log \Lambda_I(\boldsymbol{w})\, ds \geq \int_{-\infty}^{0} w_i' \log \lambda_i(w_i)\, ds = \int_{0}^{M_i} \log \lambda_i(m)\, dm \,, \tag{4.6}$$

where

$$\lambda_i(m) := \inf\{\Lambda_I(\boldsymbol{N}) \,:\, \boldsymbol{N} = (N_1, \ldots, N_n) \in \mathcal{B}(\boldsymbol{M}),\, N_i = m\}. \tag{4.7}$$

Since $a_{j,j} \geq 0$, $\forall j$, the minimum in (4.7) is attained at one of the vertices of the $n-1$-dimensional box $\{\boldsymbol{N} \in \mathcal{B}(\boldsymbol{M}) \,:\, N_i = m\}$.

Let us fix any $i \in I$. Setting for each $J \subsetneqq I$ ($J = \emptyset$ is allowed, and we denote $\boldsymbol{M}_\emptyset = \boldsymbol{0}$) and $j \notin J$,

$$g_{J,j}(m) = \Lambda_I\big((1 - m/M_j) \boldsymbol{M}_J + (m/M_j) \boldsymbol{M}_{J \cup \{j\}} \big), \quad \text{for } m \in [0, M_j],$$

we have then that

$$\lambda_i(m) = \min\{g_{J,i}(m) \,:\, J \subseteq I \setminus \{i\}\}, \quad \text{for } m \in [0, M_i]. \tag{4.8}$$

For each $J \subseteq I \setminus \{i\}$, $g_{J,i}$ is a concave quadratic polynomial in the variable m (since $a_{i,i} \geq 0$) which is nonnegative on $[0, M_i]$. If $g_{J,i}(0) = 0$ then we must have $g_{J,i}'(0) > 0$. Indeed, $g_{J,i}'(0) = 0$ would imply that Λ_I is identically zero on the edge joining M_J to $M_{J \cup \{i\}}$, contradicting (2.16) and Lemma 4.2. By the same argument we have: either $g_{J,i}(M_i) > 0$, or, if $g_{J,i}(M_i) = 0$, then $g_{J,i}'(M_i) < 0$. We conclude then that there exists $\alpha_{J,i} > 0$ such that,

$$g_{J,i}(m) \geq \alpha_{J,i} m (M_i - m) \quad \text{on } [0, M_i].$$

In view of (4.8), we obtain for $\bar{\alpha}_i := \min\{\alpha_{J,i} \,:\, J \subseteq I \setminus \{i\}\}$,

$$\lambda_i(m) \geq \bar{\alpha}_i m (M_i - m) \quad \text{on } [0, M_i]. \tag{4.9}$$

Clearly,

$$\int_{0}^{M_i} \log\big(m (M_i - m) \big)\, dm > -\infty, \tag{4.10}$$

and (4.3) follows from (4.5), (4.6), (4.9) and (4.10).

(ii) Next we prove the necessity of (2.16). Assume that (2.16) is not satisfied. If $\Lambda_J(\boldsymbol{M}) < 0$ for some $\emptyset \neq J \subseteq I$, then by the argument of [4, Lemma 2.2] it follows that (4.3) cannot hold. Assume then that (1.10) is satisfied, but for some $i \in I$ and $\emptyset \neq K \subseteq I \setminus \{i\}$ we have, for $J = K \cup \{i\}$,

$$\Lambda_I(\boldsymbol{M}_K) = \Lambda_I(\boldsymbol{M}_J) = a_{i,i} = 0\,. \tag{4.11}$$

Note that $\Lambda_I(\boldsymbol{M}_J) = a_{i,i} = 0$ implies that J is not a singleton, i.e. $K \neq \emptyset$. Then, by Lemma 4.2 it follows that Λ_I is identically zero on the edge connecting $\boldsymbol{M}_K$ to $\boldsymbol{M}_J$. For each m we are going to construct a path $\boldsymbol{z}_m : (-\infty, 0) \to \mathcal{B}(\boldsymbol{M})$ connecting $\mathbf{0}$ to $\boldsymbol{M}$ as follows. For simplicity we omit the subscript m. First, on $(-\infty, -m-1)$ we set,

$$z_j(t) = \begin{cases} M_j e^{t+m+1} & \text{if } j \in K, \\ 0 & \text{if } j \notin K. \end{cases}$$

Then,

$$\int_{-\infty}^{-m-1} F(\boldsymbol{z}, \boldsymbol{z}')\,dt = c_1, \tag{4.12}$$

for some constant c_1 independent of m. On $[-m-1, -1]$ we connect $\boldsymbol{M}_K$ to $\boldsymbol{M}_J$ by setting,

$$z_j(t) = \begin{cases} M_j & \text{if } j \in K, \\ M_i \frac{t}{m} & \text{if } j = i, \\ 0 & \text{if } j \notin J. \end{cases}$$

Then,

$$\int_{-m-1}^{-1} F(\boldsymbol{z}, \boldsymbol{z}')\,dt = \int_{-m-1}^{-1} z_i' \log z_i'\,dt = M_i \log(\frac{M_i}{m}) \to -\infty, \text{ as } m \to \infty. \tag{4.13}$$

Finally, on $[-1, 0]$ we connect $\boldsymbol{M}_J$ to $\boldsymbol{M}$ by $\boldsymbol{z}(t) = (-t)\boldsymbol{M}_J + (1+t)\boldsymbol{M}$ which gives,

$$\int_{-1}^{0} F(\boldsymbol{z}, \boldsymbol{z}')\,dt = c_2. \tag{4.14}$$

Combining (4.12)–(4.14) yields,

$$\lim_{m\to\infty} \int_{-\infty}^{0} F(\boldsymbol{z}_m, \boldsymbol{z}_m')\,dt = -\infty,$$

and thus (4.3) does not hold. □

5 Proofs of the main results

In this section we prove our main results Theorems 2–5. We divide the assumptions on A into two cases. The first case, of a nonnegative A (i.e. satisfying (2.1)) will be treated in Subsection 5.1. The second case, studied in Subsection 5.2, is of a *multi-block collaborating system*, i.e. when, up to a permutation of the indices of I, A consists of $K \geq 2$ nonnegative blocks on the diagonal and outside these blocks all the elements are nonpositive (see (2.17)). The case of a nonnegative A, corresponding to the case $K = 1$, is entitled then: *single-block collaborating system.*

5.1 Single-block collaborating systems

We begin with the proof of Theorem 5(i), which is concerned with the problem on a bounded domain $\Omega \subset \mathbb{R}^2$. It extends [4, Lemma 2.1] to the critical case.

Proof of Theorem 5(i). By the duality principle of [4, Proposition 2.1] proving boundedness of Ψ_Ω will imply (2.28). Using Schwarz symmetrization as in the proof of [4, Lemma 2.1] we see that it suffices to consider the case where $\Omega = B_R$, the disc centered at 0 with radius R, and where each ρ_i is radially symmetric and nonincreasing. We next sketch the argument from [4, Lemma 2.1] in order to obtain an equivalent expression for $\Psi_\Omega(\boldsymbol{\rho})$. Denoting, for each i, by u_i the solution to $-\Delta u_i = \rho_i$ in B_R with zero boundary condition we have

$$\Psi_\Omega(\boldsymbol{\rho}) = \sum_{i\in I}\int_{B_R} \rho_i \log \rho_i \, dx - \frac{1}{2}\sum_{i,j\in I} a_{i,j}\int_{B_R} \rho_i u_j \, dx \,. \tag{5.1}$$

Put $m_i(r) = \int_{B_r} \rho_i \, dx = 2\pi \int_0^r \tau \rho_i(\tau)\, d\tau$ so that $u_i'(r) = -\frac{m_i(r)}{2\pi r}$. Then,

$$\begin{aligned}\int_{B_R} \rho_i \log \rho_i \, dx = 2\pi \int_0^R \rho_i \log \rho_i \, r dr &= \int_0^R m_i' \log m_i' \, dr - \int_0^R m_i' \log r dr - m_i(R)\log(2\pi) \\ &= \int_0^R m_i' \log m_i' \, dr + \int_0^R \frac{m_i}{r}\, dr - m_i(R)\log(2\pi R)\end{aligned} \tag{5.2}$$

and

$$\int_{B_R} \rho_i u_j \, dx = \int_0^R m_i' u_j \, dr = \frac{1}{2\pi}\int_0^R \frac{m_i m_j}{r}\, dr + m_i(R) u_j(R) = \frac{1}{2\pi}\int_0^R \frac{m_i m_j}{r}\, dr \,. \tag{5.3}$$

Plugging (5.2)–(5.3) in (5.1) yields,

$$\Psi_\Omega(\boldsymbol{\rho}) = \int_0^R \sum_{i\in I} m_i' \log m_i' \, dr + \int_0^R \Big[\sum_{i\in I}\frac{m_i}{r} - \frac{1}{4\pi}\sum_{i,j\in I} a_{i,j}\frac{m_i m_j}{r}\Big]\, dr - \log(2\pi R)\sum_{i\in I} M_i \,. \tag{5.4}$$

Finally, setting $w_i(s) = m_i(e^s)$, $\forall s \in (-\infty, \log R]$, $\forall i \in I$, we may rewrite (5.4) as

$$\Psi_\Omega(\boldsymbol{\rho}) = \int_{-\infty}^{\log R} \sum_{i\in I} w_i' \log w_i' \, ds + \int_{-\infty}^{\log R} \Big[2\sum_{i\in I} w_i - \frac{1}{4\pi}\sum_{i,j\in I} a_{i,j} w_i w_j\Big]\, ds - \log(2\pi R)\sum_{i\in I} M_i,$$

and the result then follows directly from Proposition 4.1. □

Next we present the proof of the entropy inequality of Theorem 4.

Proof of Theorem 4(i). We begin by proving sufficiency of the conditions. First we show that it is enough to consider radially symmetric $\tilde{\boldsymbol{\rho}}$. In fact, setting $\tilde{\boldsymbol{\rho}}^* = (\tilde{\rho}_i^*)_{i=1}^n$, where for each i, $\tilde{\rho}_i^*$ is the symmetric decreasing rearrangement of $\tilde{\rho}_i$, we have clearly,

$$\int_{\mathbb{R}^2} \tilde{\rho}_i^* \log \tilde{\rho}_i^* \, dx = \int_{\mathbb{R}^2} \tilde{\rho}_i \log \tilde{\rho}_i \, dx, \quad \int_{\mathbb{R}^2} \tilde{\rho}_i^* |\log \tilde{\rho}_i^*| \, dx = \int_{\mathbb{R}^2} \tilde{\rho}_i |\log \tilde{\rho}_i| \, dx \,,$$

and

$$\int_{\mathbb{R}^2} \tilde{\rho}_i^* \log(1+|x|^2)\,dx \le \int_{\mathbb{R}^2} \tilde{\rho}_i \log(1+|x|^2)\,dx.$$

In particular we deduce that $\tilde{\boldsymbol{\rho}}^* \in \boldsymbol{\Gamma}_{\boldsymbol{M}}(\mathbb{R}^2)$ whenever $\tilde{\boldsymbol{\rho}} \in \boldsymbol{\Gamma}_{\boldsymbol{M}}(\mathbb{R}^2)$. By a variant of Riesz rearrangement inequality (see [5, Lemma 2]) we have,

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \tilde{\rho}_i(x)\log|x-y|\tilde{\rho}_j(y)\,dx\,dy \ge \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \tilde{\rho}_i^*(x)\log|x-y|\tilde{\rho}_j^*(y)\,dx\,dy, \quad \forall i,j.$$

Thus $\Psi_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}}^*) \le \Psi_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}})$, and we may assume in the sequel that each $\tilde{\rho}_i$ is a radially symmetric and decreasing function of $r=|x|$. For each i let $-u_i$ denote the *Newtonian Potential* of $\tilde{\rho}_i$, i.e.

$$u_i(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \tilde{\rho}_i(y)\log|x-y|\,dy.$$

Thus $u_i(x)=u_i(r)$ is a radial function satisfying $-\Delta u_i = \tilde{\rho}_i$ in $\mathbb{R}^2$. Our assumption that $\int_{\mathbb{R}^2}\tilde{\rho}_i(x)\log(1+|x|^2)\,dx<\infty$ (c.f. (2.8)) implies that $\tilde{\rho}_i$ is *regular at infinity* (see [8, Ch. II,§3]), and in particular we have, since $\tilde{\rho}_i$ is radial, that

$$-u_i(r) = \left(\frac{\log r}{2\pi}\right)\int_{B_r}\tilde{\rho}_i(y)\,dy + \frac{1}{2\pi}\int_{\mathbb{R}^2\setminus B_r}\log|y|\,\tilde{\rho}_i(y)\,dy, \tag{5.5}$$

see [8, Ch. II,§3, Lemma 9]. By (5.5) we deduce easily that,

$$\lim_{R\to\infty} u_i(R) + \frac{M_i}{2\pi}\log R = 0\,. \tag{5.6}$$

We denote $m_i(r) = 2\pi r\int_0^r \tilde{\rho}_i(\tau)\,d\tau$ as above. Since for $R>1$,

$$0 \le (\log R)(M_i - m_i(R)) = (\log R)\int_{\mathbb{R}^2\setminus B_R}\tilde{\rho}_i(y)\,dy \le \int_{\mathbb{R}^2\setminus B_R}\tilde{\rho}_i(y)\log|y|\,dy\,,$$

we obtain that,

$$\lim_{R\to\infty}(M_i - m_i(R))\cdot\log R = 0, \tag{5.7}$$

Clearly,

$$\Psi_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}}) = \lim_{R\to\infty}\overline{\Psi}_R(\tilde{\boldsymbol{\rho}}), \tag{5.8}$$

where

$$\overline{\Psi}_R(\tilde{\boldsymbol{\rho}}) = \sum_{i\in I}\int_{B_R}\tilde{\rho}_i\log\tilde{\rho}_i\,dx - \frac{1}{2}\sum_{i,j\in I}a_{i,j}\int_{B_R}\tilde{\rho}_i u_j\,dx. \tag{5.9}$$

Using (5.2) and the first two equalities in (5.3) yields that $\overline{\Psi}_R(\tilde{\boldsymbol{\rho}}) = G_R(\boldsymbol{m}) - \log(2\pi)\sum_{i\in I}m_i(R)$ with

$$\begin{aligned} G_R(\boldsymbol{m}) = \int_0^R \sum_{i\in I} m_i'\log m_i'\,dr + \int_0^R\Big[\sum_{i\in I}\frac{m_i}{r} - \frac{1}{4\pi}\sum_{i,j\in I}a_{i,j}\frac{m_i m_j}{r}\Big]\,dr \\ - \sum_{i\in I} m_i(R)\big(\log R + \frac{1}{2}\sum_{j\in I}a_{i,j}u_j(R)\big). \end{aligned} \tag{5.10}$$

Next, setting $w_i(s) = m_i(e^s)$ as above we get that,

$$\begin{aligned} G_R(\boldsymbol{m}) = &\int_{-\infty}^{\log R} \sum_{i\in I} w_i' \log w_i' \, ds + \int_{-\infty}^{\log R} \big[2\sum_{i\in I} w_i - \frac{1}{4\pi}\sum_{i,j\in I} a_{i,j} w_i w_j\big]\, ds \\ &- \sum_{i\in I} m_i(R)\big(2\log R + \frac{1}{2}\sum_{j\in I} a_{i,j} u_j(R)\big). \end{aligned} \tag{5.11}$$

Further, by (5.6) and (5.7) we obtain that,

$$\begin{aligned} &\lim_{R\to\infty} \sum_{i\in I} m_i(R)\big(2\log R + \frac{1}{2}\sum_{j\in I} a_{i,j}u_j(R)\big) - \frac{1}{4\pi}\Lambda_I(\boldsymbol{M})\log R \\ &\qquad = \lim_{R\to\infty} (\log R)\cdot \sum_{i\in I}(m_i(R) - M_i)\big(2 - \frac{1}{4\pi}\sum_{j\in I} a_{i,j}M_j\big) = 0. \end{aligned} \tag{5.12}$$

By (5.8)–(5.12) we finally conclude, using our assumption $\Lambda_I(\boldsymbol{M}) = 0$, that

$$\Psi_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}}) = \lim_{R\to\infty} \int_{-\infty}^{\log R} \sum_{i\in I} w_i' \log w_i' \, ds + \int_{-\infty}^{\log R} \big[2\sum_{i\in I} w_i - \frac{1}{4\pi}\sum_{i,j\in I} a_{i,j}w_iw_j\big]\, ds - \log(2\pi)\sum_{i\in I} M_i. \tag{5.13}$$

The result follows from (5.13) and Proposition 4.1, granted (2.16), where we replace $w_i(s)$ by $w_i(s+\log R)$, $\forall i$.

Next we turn to the necessity of the conditions. We assume then that $\Psi_{\mathbb{R}^2}$ is bounded below on $\boldsymbol{\Gamma_M}(\mathbb{R}^2)$. Fix $\hat{\boldsymbol{\rho}} \in \boldsymbol{\Gamma_M}(\mathbb{R}^2)$ with compact support in B_1. Define for each $\alpha > 0$, $\hat{\boldsymbol{\rho}}_\alpha(x) = \alpha^2\hat{\boldsymbol{\rho}}(\alpha x)$. It is easy to verify that $\hat{\boldsymbol{\rho}}_\alpha \in \boldsymbol{\Gamma_M}(\mathbb{R}^2)$ and that

$$\Psi_{\mathbb{R}^2}(\hat{\boldsymbol{\rho}}_\alpha) = \Psi_{\mathbb{R}^2}(\hat{\boldsymbol{\rho}}) + \frac{1}{4\pi}\Lambda_I(\boldsymbol{M})\log\alpha. \tag{5.14}$$

From (5.14) we get immediately the necessity of the condition $\Lambda_I(\boldsymbol{M}) = 0$. To see why (2.16) is necessary as well, we shall use Theorem 5 with $\Omega = B_1$. Extend $\boldsymbol{\rho} \in \boldsymbol{\Gamma_M}(\Omega)$ by $\boldsymbol{0}$ on $\mathbb{R}^2 \setminus B_1$. From (5.13), $\Psi_{\mathbb{R}^2}(\boldsymbol{\rho}) = \Psi_\Omega(\boldsymbol{\rho}) - \log(2\pi)\sum_{i\in I} M_i$, and the necessity of (2.16) follows from the analogue result in Theorem 5.

□

Proof of Theorem 4(ii). We shall prove existence of a minimizer using the construction of an entire solution to a Liouville system in [4, Theorem 1.4]. Below is a short description of this construction. First, using our assumption (2.26) we can find a sequence $\{\boldsymbol{M}^{(m)}\}$ such that $\boldsymbol{M}^{(m)} \to \boldsymbol{M}$, and $\Lambda_J(\boldsymbol{M}^{(m)}) > 0$, $\forall J \subseteq I$. By the results of [4] it follows that for each m there exists a radially symmetric and decreasing minimizer, $\boldsymbol{\rho}^{(m)}$, for Ψ_{B_1} over $\boldsymbol{\Gamma}_{\boldsymbol{M}^{(m)}}(B_1)$. It was shown in the proof of [4, Theorem 1.4], that for an appropriate choice of a sequence $R^{(m)} \to \infty$, the rescaled sequence, $\tilde{\boldsymbol{\rho}}^{(m)}(x) = \frac{1}{(R^{(m)})^2}\boldsymbol{\rho}^{(m)}(\frac{x}{R^{(m)}})$, satisfies,

$$\lim_{m\to\infty} \tilde{\boldsymbol{\rho}}^{(m)} = \tilde{\boldsymbol{\rho}}^{(\infty)} \quad \text{locally uniformly on } \mathbb{R}^2, \tag{5.15}$$

with $\tilde\rho_i^{(\infty)} > 0$ and $\int_{\mathbb{R}^2} \rho_i^{(\infty)} = M_i,\ \forall i \in I$. Moreover, setting

$$\tilde u_i^{(\infty)}(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|\tilde\rho_i^{(\infty)}(y)\,dy, \quad \forall i \in I,$$

we obtain an entire solution to the Liouville system,

$$-\Delta \tilde u_i^{(\infty)} = \lambda_i^{(\infty)} \exp(\sum_{j\in I} a_{i,j}\tilde u_j^{(\infty)}), \quad \forall i \in I,\ \text{on } \mathbb{R}^2, \tag{5.16}$$

for some positive constants $\lambda_i^{(\infty)},\ i = 1,\dots,n$. Clearly, each $\tilde{\boldsymbol\rho}^{(m)}$ is a minimizer for $\Psi_{B_{R^{(m)}}}$ over $\boldsymbol\Gamma_{\boldsymbol M^{(m)}}(B_{R^{(m)}})$. It will be also useful to consider, as in [4], $\tilde v_i^{(m)} = \log\tilde\rho_i^{(m)},\ \forall i \in I$, which satisfies,

$$-\Delta \tilde v_i^{(m)} = \sum_{j\in I} a_{i,j}\exp(\tilde v_j^{(m)}), \quad \text{on } B_{R^{(m)}}. \tag{5.17}$$

We want to prove that $\boldsymbol\rho^{(\infty)}$ is a minimizer for $\Psi_{\mathbb{R}^2}$ over $\boldsymbol\Gamma_{\boldsymbol M}(\mathbb{R}^2)$.

Note first that each $\boldsymbol\rho \in \boldsymbol\Gamma_{\boldsymbol M^{(m)}}(B_{R^{(m)}})$ can be considered also as a member in $\boldsymbol\Gamma_{\boldsymbol M^{(m)}}(\mathbb{R}^2)$, by extending it by zero outside $B_{R^{(m)}}$. For such $\boldsymbol\rho$ we have (see (5.9)),

$$\Psi_{\mathbb{R}^2}(\boldsymbol\rho) = \overline\Psi_{R^{(m)}}(\boldsymbol\rho) = \sum_{i\in I}\int_{B_{R^{(m)}}} \rho_i\log\rho_i\,dx - \frac12\sum_{i,j\in I} a_{i,j}\int_{B_{R^{(m)}}} \rho_i u_j\,dx,$$

with

$$u_i(x) = -\frac{1}{2\pi}\int_{B_{R^{(m)}}} \rho_i(y)\log|x-y|\,dy, \quad \forall i \in I.$$

Define also,

$$v_i(x) = \int_{\mathbb{R}^2} \rho_i(y) G_{B_{R^{(m)}}}(x,y)\,dy,\ \forall i \in I.$$

Then, $u_i - v_i$ is a radial harmonic function on $B_{R^{(m)}}$, which must be identically equal to the constant

$$u_i(R^{(m)}) - v_i(R^{(m)}) = u_i(R^{(m)}) = -\Big(\frac{M_i^{(m)}}{2\pi}\Big)\log R^{(m)} \quad \text{(using (5.5))}.$$

Therefore, by definition of $\Psi_{B_{R^{(m)}}}$,

$$\Psi_{\mathbb{R}^2}(\boldsymbol\rho) - \Psi_{B_{R^{(m)}}}(\boldsymbol\rho) = \frac12\sum_{i,j\in I} a_{i,j}\int_{B_{R^{(m)}}} \rho_i(v_j - u_j)\,dx = \frac{1}{4\pi}\sum_{i,j\in I} a_{i,j}M_i^{(m)}M_j^{(m)}\log R^{(m)}. \tag{5.18}$$

It follows from (5.18) that for each m, $\tilde{\boldsymbol\rho}^{(m)}$ is a minimizer for $\Psi_{\mathbb{R}^2}$ (equivalently, of $\overline\Psi_{R^{(m)}}$) over $\boldsymbol\Gamma_{\boldsymbol M^{(m)}}(B_{R^{(m)}})$.

Put

$$u_i^{(m)}(x) = -\frac{1}{2\pi}\int_{B_{R^{(m)}}} \tilde\rho_i^{(m)}(y)\log|x-y|\,dy, \quad \forall i \in I. \tag{5.19}$$

A simple but important consequence of our assumption (2.26) is

$$\frac{1}{4\pi}\frac{\partial \Lambda_I}{\partial M_i}(\boldsymbol{M}) = 2 - \frac{1}{2\pi}\sum_{j\in I} a_{i,j}M_j < 0, \ \forall i \in I \quad \text{(see [4, Lemma 5.1])}, \tag{5.20}$$

which implies the existence of $\varepsilon_0 > 0$ and $R_0 > 0$ such that,

$$\frac{1}{2\pi}\int_{B_{R_0}} \sum_{j\in I} a_{i,j}\tilde{\rho}_i^{(\infty)}(x)\,dx \geq 2 + 2\varepsilon_0, \quad \forall i \in I.$$

Using (5.15) we deduce that, for m_0 large enough,

$$\frac{1}{2\pi}\int_{B_{R_0}} \sum_{j\in I} a_{i,j}\tilde{\rho}_i^{(m)}(x)\,dx \geq 2 + \varepsilon_0, \quad \forall i \in I, \forall m \geq m_0. \tag{5.21}$$

By (5.17) and (5.21) we obtain that,

$$-\frac{\sum_{j\in I} a_{i,j}M_j}{2\pi r} \leq \frac{\partial \tilde{v}_i^{(m)}}{\partial r}(r) \leq -\frac{2+\varepsilon_0}{r}, \quad \forall r \in [R_0, R^{(m)}], \ \forall i \in I. \tag{5.22}$$

An immediate consequence of (5.22) and (5.15) is that for some constants c and c_1,

$$c_1 - \Big(\frac{\sum_{j\in I} a_{i,j}M_j}{2\pi}\Big)\log r \leq \tilde{v}_i^{(m)}(r) \leq c - (2+\varepsilon_0)\log r, \quad \forall r \in [R_0, R^{(m)}], \ \forall i \in I, \tag{5.23}$$

and therefore

$$\tilde{\rho}_i^{(m)}(r) \leq \frac{e^c}{r^{2+\varepsilon_0}}, \quad \forall r \geq R_0, \ \forall i \in I. \tag{5.24}$$

From (5.24),(5.23),(5.15) and dominated convergence we obtain,

$$\lim_{m\to\infty} \Psi_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}}^{(m)}) = \Psi_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}}^{(\infty)})\,. \tag{5.25}$$

Fix now any $\tilde{\boldsymbol{\rho}} \in \boldsymbol{\Gamma}_{\boldsymbol{M}}(\mathbb{R}^2)$. As explained in the proof of assertion (i), it is enough to consider $\tilde{\boldsymbol{\rho}}$ whose components are radially symmetric and decreasing. Fix $\alpha > 0$ and $\delta > 0$ such that

$$\tilde{\rho}_i(x) \geq \delta, \quad \forall x \in B_\alpha, \ \forall i \in I. \tag{5.26}$$

Let the function ζ be defined by $\zeta = (\frac{1}{\pi\alpha^2})\chi_{B_\alpha}$. Define $\widetilde{\boldsymbol{M}}$ by setting $\widetilde{M}_i^{(m)} = \int_{B_{R^{(m)}}} \tilde{\rho}_i$, $\forall i \in I$, and then,

$$\hat{\boldsymbol{\rho}}^{(m)} = \chi_{B_{R^{(m)}}}\tilde{\boldsymbol{\rho}} + \zeta(\boldsymbol{M}^{(m)} - \widetilde{\boldsymbol{M}}^{(m)}).$$

For m large enough $\hat{\boldsymbol{\rho}}^{(m)} \in \boldsymbol{\Gamma}_{\boldsymbol{M}^{(m)}}(B_{R^{(m)}})$, and since $\tilde{\boldsymbol{\rho}}^{(m)}$ is a minimizer for $\overline{\Psi}_{R^{(m)}}$ over $\boldsymbol{\Gamma}_{\boldsymbol{M}^{(m)}}(B_{R^{(m)}})$, we have,

$$\Psi_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}}^{(m)}) \leq \Psi_{\mathbb{R}^2}(\hat{\boldsymbol{\rho}}^{(m)}). \tag{5.27}$$

By (5.27), dominated convergence and (5.25) we infer that,

$$\Psi_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}}) = \lim_{m\to\infty} \Psi_{\mathbb{R}^2}(\chi_{B_{R^{(m)}}}\tilde{\boldsymbol{\rho}}) = \lim_{m\to\infty} \Psi_{\mathbb{R}^2}(\hat{\boldsymbol{\rho}}^{(m)}) \geq \lim_{m\to\infty} \Psi_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}}^{(m)}) = \Psi_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}}^{(\infty)})\,,$$

and the result follows.

Finally, the necessity of condition (2.26) for the existence of a minimizer is an immediate consequence of [4, Theorem 1.4]. Indeed, an existence of a minimizer $\tilde{\boldsymbol{\rho}}^{(\infty)}$ implies the existence of an entire solution to the Liouville system (5.16). But it was shown in [4] that (2.26) is necessary for the latter to hold. □

Finally we turn to the proof of our main result on S^2, Theorem 2.

Proof of Theorem 2. We first remark that by the duality principle of Corollary 3.1, it is enough to prove assertion (i), which implies assertion (ii). Next we prove the sufficiency of condition (2.16). For each $i \in I$, let us denote by ρ_i^* the symmetric decreasing rearrangement of ρ_i (w.r.t. the north pole). Clearly $\int_{S^2} \rho_i^* \log \rho_i^* = \int_{S^2} \rho_i \log \rho_i$ for all i and by a result of Baernstein and Taylor [3, Theorem 2] (see also [2, 5]) we have

$$\int_{S^2}\int_{S^2} \rho_i(x) \log|x-y| \rho_j(y)\, dx\, dy \geq \int_{S^2}\int_{S^2} \rho_i^*(x) \log|x-y| \rho_j^*(y)\, dx\, dy, \quad \forall i, j.$$

Therefore, we may assume that each ρ_i is radially symmetrically decreasing from the north pole. Moreover, by a simple density argument, we may assume that the support of each ρ_i does not intersect a certain neighborhood of the north pole. Next we use stereographic projection in order to restate the variational problem in an equivalent form on $\mathbb{R}^2$. More precisely, defining $\tilde{\boldsymbol{\rho}}$ by (2.4) we have by (2.6) that $\Psi_{S^2}(\boldsymbol{\rho}) = \widetilde{\Psi}_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}})$. Moreover, our assumption on the support of $\boldsymbol{\rho}$ implies that $\tilde{\boldsymbol{\rho}}$ is supported in some disc B_{R_0}. Therefore, for any $R \geq R_0$ we have,

$$\begin{aligned}\widetilde{\Psi}_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}}) = \sum_{i\in I}\int_{B_R} \tilde{\rho}_i \log \tilde{\rho}_i\, dx + \frac{1}{4\pi}\sum_{i,j\in I} a_{i,j} \int_{B_R}\int_{B_R} \tilde{\rho}_i(x) \log|x-y| \tilde{\rho}_j(y)\, dx\, dy \\ + \sum_{i\in I} \nu_i \int_{B_R} \tilde{\rho}_i \log(1+|x|^2)\, dx.\end{aligned}$$

As in the proof of Theorem 4 we shall use,

$$u_i(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \tilde{\rho}_i(y) \log|x-y|\, dy, \quad m_i(r) = 2\pi\int_0^r s\rho_i(s)\, ds \quad \text{and } w_i(s) = m_i(e^s)\,.$$

Using

$$\begin{aligned}\int_{B_R} \tilde{\rho}_i \log(1+|x|^2)\, dx &= \int_0^R m_i'(r) \log(1+r^2)\, dr \\ &= M_i \log(1+R^2) - \int_0^R \frac{2m_i(r)r}{1+r^2}\, dr \\ &= M_i \log(1+R^2) - \int_{-\infty}^{\log R} \frac{2w_i(s)e^{2s}}{1+e^{2s}}\, ds\,,\end{aligned} \tag{5.28}$$

and (5.10)–(5.12) we get,

$$
\begin{aligned}
\widetilde{\Psi}_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}}) = &\int_{-\infty}^{\log R} \sum_{i\in I} w_i' \log w_i' \, ds + \int_{-\infty}^{\log R} \big[2\sum_{i\in I} w_i - \frac{1}{4\pi}\sum_{i,j\in I} a_{i,j} w_i w_j\big]\, ds \\
&- \sum_{i\in I} \big[2M_i \log(2\pi R) + \frac{1}{2} M_i \sum_{j\in I} a_{i,j} u_j(R)\big] \\
&+ \sum_{i\in I} \nu_i \big[M_i \log(1+R^2) - \int_{-\infty}^{\log R} \frac{2w_i e^{2s}}{1+e^{2s}}\, ds\big].
\end{aligned}
\tag{5.29}
$$

Using (5.6) and the identity $\frac{\Lambda_I(\boldsymbol{M})}{4\pi} = \sum_{i\in I} \nu_i M_i$ we obtain that

$$
\begin{aligned}
-\sum_{i\in I} \big[2M_i \log R + \frac{1}{2} M_i \sum_{j\in I} a_{i,j} u_j(R)\big] &+ \sum_{i\in I} \nu_i M_i \log(1+R^2) \\
&= \big[-\frac{\Lambda_I(\boldsymbol{M})}{4\pi} + 2\sum_{i\in I} \nu_i M_i\big] \log R + o(1) \\
&= \frac{\Lambda_I(\boldsymbol{M})}{4\pi} \log R + o(1),
\end{aligned}
$$

with $o(1)$ denoting a quantity which goes to 0 as $R \to \infty$. Therefore we may write

$$
\widetilde{\Psi}_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}}) = J_{-\infty}(\boldsymbol{w}) + J_\infty^R(\boldsymbol{w}) - 2\log(2\pi)\sum_{i\in I} M_i + o(1), \tag{5.30}
$$

where

$$
\begin{aligned}
J_{-\infty}(\boldsymbol{w}) = &\int_{-\infty}^{0} \sum_{i\in I} w_i' \log w_i' \, ds + \int_{-\infty}^{0} \big[2\sum_{i\in I} w_i - \frac{1}{4\pi}\sum_{i,j\in I} a_{i,j} w_i w_j\big]\, ds \\
&- \sum_{i\in I} \nu_i \int_{-\infty}^{0} \frac{2w_i e^{2s}}{1+e^{2s}}\, ds\,,
\end{aligned}
\tag{5.31}
$$

and

$$
\begin{aligned}
J_\infty^R(\boldsymbol{w}) = &\int_{0}^{\log R} \sum_{i\in I} w_i' \log w_i' \, ds + \int_{0}^{\log R} \big[2\sum_{i\in I} w_i - \frac{1}{4\pi}\sum_{i,j\in I} a_{i,j} w_i w_j + \frac{\Lambda_I(\boldsymbol{M})}{4\pi}\big]\, ds \\
&- \sum_{i\in I} \nu_i \int_{0}^{\log R} \frac{2w_i e^{2s}}{1+e^{2s}}\, ds\,.
\end{aligned}
\tag{5.32}
$$

Since clearly,

$$
\int_{-\infty}^{0} \frac{2w_i e^{2s}}{1+e^{2s}} \le 2M_i \int_{-\infty}^{0} e^{2s}\, ds = M_i,
$$

it follows from Proposition 4.1 that $J_{-\infty}(\boldsymbol{w}) \ge -C$ for some constant C. Hence it remains to find a lower bound for $J_\infty^R(\boldsymbol{w})$, uniformly in $R \in [R_0, \infty)$.

Since

$$-\sum_{i\in I}\nu_i\int_0^{\log R}\frac{2w_ie^{2s}}{1+e^{2s}}\,ds=-2\nu_i\sum_{i\in I}\int_0^{\log R}w_i\,ds+\sum_{i\in I}\nu_i\int_0^{\log R}\frac{2w_i}{1+e^{2s}}\,ds$$
$$\geq -2\nu_i\sum_{i\in I}\int_0^{\log R}w_i\,ds-\sum_{i\in I}|\nu_i|M_i\,,$$

it suffices to prove that the functional

$$G^R_\infty(\boldsymbol{w}):=\int_0^{\log R}\sum_{i\in I}w_i'\log w_i'\,ds+\int_0^{\log R}\Big[\sum_{i\in I}2(1-\nu_i)w_i-\frac{1}{4\pi}\sum_{i,j\in I}a_{i,j}w_iw_j+\frac{\Lambda_I(\boldsymbol{M})}{4\pi}\Big]\,ds, \tag{5.33}$$

is bounded below. A simple computation shows that

$$\sum_{i\in I}2(1-\nu_i)w_i-\frac{1}{4\pi}\sum_{i,j\in I}a_{i,j}w_iw_j+\frac{\Lambda_I(\boldsymbol{M})}{4\pi}=\sum_{i\in I}2(M_i-w_i)-\frac{1}{4\pi}\sum_{i,j\in I}a_{i,j}(M_i-w_i)(M_j-w_j).$$

Therefore, setting for each $i\in I$, $z_i(t)=M_i-w_i(-t)$ for $t\in[-\log R,0]$ and $z_i(t)=0$ for $t\in(-\infty,-\log R)$, we infer from (5.33) that

$$G^R_\infty(\boldsymbol{w})=\int_{-\infty}^0\Big[\sum_{i\in I}z_i'\log z_i'+\frac{1}{4\pi}\Lambda_I(\boldsymbol{z})\Big]\,dt. \tag{5.34}$$

Since $z_i(-\infty)=0$ and $z_i(0)=M_i-w_i(0)\leq M_i$, $\forall i$, we can apply Proposition 4.1 to conclude that the RHS of (5.34) is bounded from below, completing the proof of the sufficiency assertion.

For the proof of necessity of (2.16), we consider $\widetilde{\Psi}_{\mathbb{R}^2}(\tilde{\boldsymbol{\rho}})$ for $\tilde{\boldsymbol{\rho}}$ with support in B_1. Since $J^1_\infty(\boldsymbol{w})=0$, for the corresponding $\boldsymbol{w}$, we conclude from (5.30) that $\widetilde{\Psi}_{\mathbb{R}^2}$ is bounded below on the class of such $\tilde{\boldsymbol{\rho}}$'s iff $J_{-\infty}$ is bounded below on the corresponding class of $\boldsymbol{w}$'s. But the necessity assertion of Proposition 4.1 implies that (2.16) is necessary for the later to hold. $\square$

5.2 Multi-block systems

We start with the proof of assertion (i) of Theorem 3, dealing with a general collaborating system with K blocks.

Proof of Theorem 3(i). (i) Setting for each $l=1,\ldots,K$,

$$\Psi^{(I_l)}_{S^2}(\boldsymbol{\rho})=\sum_{i\in I_l}\int_{S^2}\rho_i\log\rho_i+\frac{1}{4\pi}\sum_{i,j\in I_l}a_{i,j}\int_{S^2}\int_{S^2}\rho_i(x)\log|x-y|\rho_j(y)\,dxdy, \tag{5.35}$$

we may write

$$\Psi_{S^2}(\boldsymbol{\rho})=\sum_{l=1}^K\Psi^{(I_l)}_{S^2}(\boldsymbol{\rho})+\frac{1}{4\pi}\sum_{l_1\neq l_2}\sum_{i\in I_{l_1}}\sum_{j\in I_{l_2}}a_{i,j}\int_{S^2}\int_{S^2}\rho_i(x)\log|x-y|\rho_j(y)\,dxdy\,. \tag{5.36}$$

Using the inequality, $|x-y| \le 2$, $\forall x, y \in S^2$, and the assumption (2.17) we get,

$$\sum_{l_1 \ne l_2} \sum_{i \in I_{l_1}} \sum_{j \in I_{l_2}} a_{i,j} \int_{S^2} \int_{S^2} \rho_i(x) \log|x-y| \rho_j(y)\, dxdy \ge (\log 2) \cdot \sum_{l_1 \ne l_2} \sum_{i \in I_{l_1}} \sum_{j \in I_{l_2}} a_{i,j} M_i M_j. \quad (5.37)$$

Therefore, if condition (2.21) is satisfied, then from Theorem 2 it follows that

$$\Psi_{S^2}^{(I_l)}(\boldsymbol{\rho}) \ge -C, \quad \forall \boldsymbol{\rho} \in \boldsymbol{\Gamma}_{\boldsymbol{M}}(S^2),\ l = 1, \dots, K,$$

and by (5.36) and (5.37) we obtain that

$$\Psi_{S^2}(\boldsymbol{\rho}) \ge -C, \quad \forall \boldsymbol{\rho} \in \boldsymbol{\Gamma}_{\boldsymbol{M}}(S^2).$$

To prove the necessity of condition (2.21), assume by negation that for some $1 \le l \le K$, condition (2.21) is violated. Then, by Theorem 2 there exists a sequence

$$\{\boldsymbol{\rho}^m = \left(\rho_i^m\right)_{i \in I_l}\} \subset \boldsymbol{\Gamma}_{\boldsymbol{M}^{(l)}}(S^2),$$

with $\boldsymbol{\Gamma}_{\boldsymbol{M}^{(l)}}(S^2)$ denoting the restriction of $\boldsymbol{\Gamma}_{\boldsymbol{M}}(S^2)$ to the coordinates of I_l, such that, $\lim_{m\to\infty} \Psi_{S^2}^{(I_l)}(\boldsymbol{\rho}^m) = -\infty$. Extend each $\boldsymbol{\rho}^m$ to $\boldsymbol{\Gamma}_{\boldsymbol{M}}(S^2)$ by setting,

$$\rho_i^m(x) = \begin{cases} \rho_i^m(x) & \text{if } i \in I_l, \\ M_i f(x) & \text{if } i \notin I_l, \end{cases}$$

where f is a smooth positive function on S^2 with $\int_{S^2} f(x)\, dx = 1$. Then, it is easy to verify, as in the proof of the necessity part of Theorem 2, that $\lim_{m\to\infty} \Psi_{S^2}(\boldsymbol{\rho}^m) = -\infty$. □

We next give the proof of assertion (ii) of Theorem 5 which deals with the multi-block case for a system on a bounded domain.

Proof of Theorem 5(ii). The proof uses the same argument as in Theorem 3(i). It suffices to note that $G_\Omega(x,y) = -\frac{1}{2\pi}\log|x-y| + R_y(x)$ with $R_y(x)$ an harmonic function on Ω, which is bounded above by $\frac{1}{2\pi}\log(\text{diam}(\Omega))$, for all $y \in \Omega$. □

For the proof of assertion (ii) of Theorem 3 we shall need the following lemma. For a symmetric n by n matrix A satisfying $a_{i,j} \le 0$ for all $i \ne j$ (here the diagonal elements play no role) consider the functional,

$$J(\boldsymbol{\mu}) = \sum_{i \ne j} (-a_{i,j}) \int_{S^2} \int_{S^2} \log|x-y|\, d\mu_i(x) d\mu_j(y)\,, \quad (5.38)$$

defined over the following set of n-vectors of finite Borel measures,

$$\mathcal{G}_{\boldsymbol{M}}(S^2) = \{\boldsymbol{\mu} = (\mu_1, \dots, \mu_n) \,:\, \mu_i \ge 0 \text{ and } \int_{S^2} d\mu_i = M_i, \forall i\}. \quad (5.39)$$

Note that J is well defined on $\mathcal{G}_{\boldsymbol{M}}(S^2)$ if we allow it to take the value $-\infty$, since the kernel $\log|x-y|$ is bounded above and $a_{i,j} \le 0$ for $i \ne j$.

Lemma 5.1. *Let A be a symmetric matrix, with $a_{i,j} \le 0$ for all $i \neq j$ and let $\boldsymbol{M} \in \mathbb{R}^n_+$. Then,*

$$\sup_{\boldsymbol{\mu} \in \mathcal{G}_{\boldsymbol{M}}(S^2)} J(\boldsymbol{\mu}) = \max_{\boldsymbol{y} \in (S^2)^n} W(\boldsymbol{y}) \quad \textit{(see (2.23))}, \tag{5.40}$$

and the supremum is attained at measures of the form

$$\boldsymbol{\mu} = (M_1\delta_{x_1}, \dots, M_n\delta_{x_n}), \quad \textit{with } \boldsymbol{x} = (x_1, \dots, x_n) \in (S^2)^n \textit{ a maximizer of } W. \tag{5.41}$$

Moreover, if A does not contain a row of zeros (ignoring the diagonal elements), then all the maxima of J and all the weak limits, in the sense of measures, of maximizing sequences are of the form (5.41).

Proof. Consider any $\boldsymbol{\mu} \in \mathcal{G}_{\boldsymbol{M}}(S^2)$. For each $i \in I$ set

$$U_i^{\boldsymbol{\mu}}(x) = 2\sum_{j\neq i}(-a_{i,j}) \int_{S^2} \log|x-y|\, d\mu_j(y)\,.$$

Then we have,

$$J(\boldsymbol{\mu}) = \sum_{\{i\neq j\neq k\neq i\}} (-a_{j,k}) \int_{S^2}\int_{S^2} \log|x-y|\, d\mu_j(x) d\mu_k(y) + \int_{S^2} U_i^{\boldsymbol{\mu}}(x)\, d\mu_i(x)\,.$$

It is known that $U_i^{\boldsymbol{\mu}}$ is upper semicontinuous (see [17]), and therefore its maximum on S^2 is attained. For any $i \in I$ define $T_{i,y_i}(\boldsymbol{\mu}) \in \mathcal{G}_{\boldsymbol{M}}(S^2)$ by,

$$(T_{i,y_i}(\boldsymbol{\mu}))_j = \begin{cases} \mu_j & \text{for } j \neq i, \\ M_i\delta_{y_i} & \text{for } j = i, \end{cases} \tag{5.42}$$

where $y_i \in S^2$ is any maximum point of $U_i^{\boldsymbol{\mu}}$. It is clear that $J(T_{i,y_i}(\boldsymbol{\mu})) \ge J(\boldsymbol{\mu})$. Setting

$$\bar{\boldsymbol{\mu}} = T_{n,y_n} \circ T_{n-1,y_{n-1}} \circ \cdots \circ T_{1,y_1}(\boldsymbol{\mu}) = (M_1\delta_{y_1}, \dots, M_n\delta_{y_n})\,,$$

we have

$$W(\boldsymbol{y}) = J(\bar{\boldsymbol{\mu}}) \ge J(\boldsymbol{\mu})\,, \tag{5.43}$$

and (5.40) follows.

To prove the last claim, let $\boldsymbol{\mu}$ be a maximizer for J over $\mathcal{G}_{\boldsymbol{M}}(S^2)$. Assume by negation that for one of the components of $\boldsymbol{\mu}$, say μ_1, $\operatorname{supp}(\mu_1)$ is not a singleton. Consider then,

$$\bar{\boldsymbol{\mu}} = T_{n,y_n} \circ T_{n-1,y_{n-1}} \circ \cdots \circ T_{2,y_2}(\boldsymbol{\mu}) = (\mu_1, M_2\delta_{y_2}, \dots, M_n\delta_{y_n})\,.$$

By construction, $\bar{\boldsymbol{\mu}}$ is also a maximizer, for which $U_1^{\bar{\boldsymbol{\mu}}}$ must be constant on $\operatorname{supp}(\mu_1)$ (otherwise, we would have $J(T_{1,y_1}(\bar{\boldsymbol{\mu}})) > J(\bar{\boldsymbol{\mu}})$, for y_1 a maximum point of $U_1^{\bar{\boldsymbol{\mu}}}$). For two distinct points $x_1, z_1 \in \operatorname{supp}(\mu_1)$ put

$\boldsymbol{\mu}_{x_1} = T_{1,x_1}(\bar{\boldsymbol{\mu}}) = (M_1\delta_{x_1}, M_2\delta_{y_2}, \dots, M_n\delta_{y_n})$ and $\boldsymbol{\mu}_{z_1} = T_{1,z_1}(\bar{\boldsymbol{\mu}}) = (M_1\delta_{z_1}, M_2\delta_{y_2}, \dots, M_n\delta_{y_n})$.

Then, also $\boldsymbol{\mu}_{x_1}$ and $\boldsymbol{\mu}_{z_1}$ are maximizers for J, i.e. $(x_1, y_2, \ldots, y_n)$ and $(z_1, y_2, \ldots, y_n)$ are maximizers for W. We must have then for each $j \neq 1$,

$$\frac{\partial W}{\partial y_j}(x_1, y_2, \ldots, y_n) \times y_j = \frac{\partial W}{\partial y_j}(z_1, y_2, \ldots, y_n) \times y_j = 0, \quad \text{i.e.}$$

$$\begin{aligned} 0 &= (-a_{1,j}) M_1 M_j \frac{(y_j - x_1) \times y_j}{|y_j - x_1|^2} + \sum_{i \neq j,1} (-a_{i,j}) M_i M_j \frac{(y_j - y_i) \times y_j}{|y_j - y_i|^2} \\ &= (-a_{1,j}) M_1 M_j \frac{(y_j - z_1) \times y_j}{|y_j - z_1|^2} + \sum_{i \neq j,1} (-a_{i,j}) M_i M_j \frac{(y_j - y_i) \times y_j}{|y_j - y_i|^2} \,. \end{aligned} \tag{5.44}$$

By assumption, there exists $j_0 \neq 1$ with $a_{1,j_0} < 0$. For this j_0 we deduce from (5.44) that $\frac{(y_{j_0} - x_1) \times y_{j_0}}{|y_{j_0} - x_1|^2} = \frac{(y_{j_0} - z_1) \times y_{j_0}}{|y_{j_0} - z_1|^2}$. This last equality forces $x_1 = z_1$. Contradiction. Finally, the statement about the weak limits of maximizing sequences follows from the upper semi-continuity w.r.t. weak convergence of measures and the characterization of the maxima.

□

Proof of Theorem 3 completed. (ii) Since $K = n$, (5.35) takes the form,

$$\Psi_{S^2}^{(I_l)}(\boldsymbol{\rho}) = \psi_l(\rho_l) := \int_{S^2} \rho_l \log \rho_l + \frac{a_{l,l}}{4\pi} \int_{S^2} \int_{S^2} \rho_l(x) \log|x - y| \rho_l(y), \quad \forall l \in I.$$

Here, in the critical case, $a_{i,i} M_i = 8\pi, \ \forall i$, we have by Theorem 2 (see (3.10)) that

$$\psi_i(\rho_i) \geq \Big(\frac{M_i}{2\pi}\Big) c_0 + M_i \log\big(\frac{M_i}{4\pi}\big), \quad \forall i \in I. \tag{5.45}$$

Moreover, by Lemma 5.1,

$$\frac{1}{4\pi} \sum_{i \neq j} a_{i,j} \int_{S^2} \int_{S^2} \rho_i(x) \log|x - y| \rho_j(y) \, dx dy \geq -\frac{1}{4\pi} \sup_{(S^2)^n} W(\boldsymbol{x}). \tag{5.46}$$

Plugging (5.45)–(5.46) in (5.36) we are led to,

$$\inf_{\boldsymbol{\Gamma}_{\boldsymbol{M}}(S^2)} \Psi_{S^2} \geq \sum_{i \in I} \big[M_i \log \frac{M_i}{4\pi} + (\frac{M_i}{2\pi}) c_0\big] - \frac{1}{4\pi} \sup_{(S^2)^n} W(\boldsymbol{x}) \,. \tag{5.47}$$

Let $\boldsymbol{z} \in (S^2)^n$ be a maximizer for W. For each $\alpha > 0$ define $\boldsymbol{\rho}^{(\alpha)} = (\boldsymbol{\rho}^0)^{\tau_{x_i,\alpha}}$, applying (2.2) componentwise with $\boldsymbol{\rho}^0 = (\frac{M_1}{4\pi}, \ldots, \frac{M_n}{4\pi})$. By Theorem 1, for each $i \in I$, ${\rho_i}^{(\alpha)}$ gives equality in (5.45). Combining it with

$$\lim_{\alpha \to \infty} - \sum_{i \neq j} a_{i,j} \int_{S^2} \int_{S^2} \rho_i^{(\alpha)}(x) \log|x - y| \rho_j^{(\alpha)}(y) \, dx dy = W(\boldsymbol{z})$$

we are led to,

$$\lim_{\alpha \to \infty} \Psi_{S^2}(\boldsymbol{\rho}^{(\alpha)}) = \sum_{i \in I} \big[M_i \log \frac{M_i}{4\pi} + (\frac{M_i}{2\pi}) c_0\big] - \frac{1}{4\pi} \sup_{(S^2)^n} W(\boldsymbol{x}) \,.$$

This together with (5.47) leads to (2.22).

(iii) Suppose by negation that there exists $\boldsymbol{\rho} \in \boldsymbol{\Gamma_M}(S^2)$ which realizes the infimum in (2.22). It follows from the above that for each $i \in I$, ρ_i is a minimizer of ψ_i over $\Gamma_{M_i}(S^2)$, and that $\boldsymbol{\mu} = (\rho_1 dx, \dots, \rho_n dx)$ is a maximizer for J over $\mathcal{G}_{\boldsymbol{M}}(S^2)$ (see (5.38),(5.39)). But this contradicts the description of the maxima of J given by Lemma 5.1. The statement about the weak limits of minimizing sequences follows similarly. □

Remark 5.1. *For the special case of the Toda system with A given by* (2.19)*, the critical case is $M_i = 4\pi$, $\forall i \in I$. Here we find,*

$$W(\boldsymbol{x}) = 2\sum_{i=1}^{n-1} \log|x_{i+1} - x_i|,$$

which achieves its maximum only at configurations of the form:

$$x_j = \begin{cases} y_1, & \text{if } j \text{ is odd}, \\ y_2, & \text{if } j \text{ is even}, \end{cases}$$

where y_1 and y_2 are antipodal points in S^2.

We present now a generalization of Theorem 3(ii),(iii) to a larger class of systems than (2.18). Consider a symmetric matrix A with the following properties:

(P1) The set I is a disjoint union of sets $I_1, \dots, I_K$ where each sub-matrix $A[I_l, I_l]$ is a *conformal block*, namely, for each $l = 1, \dots, K$:
$a_{i,j} \geq 0$, $\forall i, j \in I_l$ and $\sum_{j \in I_l} a_{i,j} M_j = 8\pi$, $\forall i \in I_l$.

(P2) If $l \neq m$ then $a_{i,j} = a_{j,i} \leq 0$, for all $i \in I_l$, $j \in I_m$.

Let us define a "renormalization" of the system in the following sense. We define $\widehat{\boldsymbol{M}} \in \mathbb{R}_+^K$ by

$$\widehat{M}_l = \sum_{i \in I_l} M_i, \quad \forall l = 1, \dots, K, \tag{5.48}$$

and $\widehat{A} = \{\widehat{a}_{l,m}\}$, $l, m = 1, \dots, K$, by

$$\widehat{a}_{l,m} = \frac{\sum_{i \in I_l,\, j \in I_m} a_{i,j} M_i M_j}{\widehat{M}_l \widehat{M}_m}. \tag{5.49}$$

The associated function $\widehat{W}$ is defined by

$$\widehat{W}(\widehat{\boldsymbol{x}}) := \sum_{l \neq m} (-\widehat{a}_{l,m}) \widehat{M}_l \widehat{M}_m \log|\widehat{x}_l - \widehat{x}_m|, \quad \forall \widehat{\boldsymbol{x}} \in (S^2)^K. \tag{5.50}$$

The new problem consists of minimizing the functional $\widehat{\Psi}_{S^2}$, associated with $\widehat{A}$, over $\boldsymbol{\Gamma}_{\widehat{\boldsymbol{M}}}(S^2)$. Note that for each $1 \leq l \leq K$ we have by condition (P1),

$$\widehat{a}_{l,l} \widehat{M}_l = \Big(\sum_{i,j \in I_l} a_{i,j} M_i M_j \Big) / \widehat{M}_l = 8\pi.$$

Therefore, the conditions of Theorem 3(ii) are satisfied by the new system. We do not know however whether analogue results to Theorem 3(ii),(iii) hold, in general, for the original problem, of minimizing Ψ_{S^2} over $\mathbf{\Gamma}_{\boldsymbol{M}}(S^2)$. We were able to achieve it only in two special cases given below. The first case is $K = 2$. Note that in this case, for each $\widehat{\boldsymbol{x}} = (\widehat{x}_1, \widehat{x}_2) \in (S^2)^2$ we have,

$$\widehat{W}(\widehat{\boldsymbol{x}}) = 2\log|\widehat{x}_1 - \widehat{x}_2| \sum_{i\in I_1, j\in I_2} (-a_{i,j})M_iM_j. \tag{5.51}$$

Therefore,

$$\sup_{\widehat{\boldsymbol{x}}\in(S^2)^2} \widehat{W}(\widehat{\boldsymbol{x}}) = 2(\log 2) \sum_{i\in I_1, j\in I_2} (-a_{i,j})M_iM_j. \tag{5.52}$$

Proposition 5.1. *Assume A and $\boldsymbol{M}$ are such that conditions (P1)–(P2) are satisfied with $K = 2$. Then,*

$$\inf_{\mathbf{\Gamma}_{\boldsymbol{M}}(S^2)} \Psi_{S^2} = \sum_{l=1}^{2}\sum_{i\in I_l} \big[M_i \log\frac{M_i}{4\pi} + (\frac{M_i}{2\pi})c_0\big] - \Big(\frac{\log 2}{2\pi}\Big) \sum_{i\in I_1, j\in I_2} (-a_{i,j})M_iM_j\,. \tag{5.53}$$

If, in addition, there exist $i_1 \in I_1$ and $j_1 \in I_2$ with $a_{i_1,j_1} \neq 0$, then the infimum in (5.53) is not attained and any weak limit $\boldsymbol{\mu}$ of a minimizing sequence is of the form:

$$\mu_i = \begin{cases} M_i\delta_{\widehat{x}_1}, & \text{if } i \in I_1, \\ M_i\delta_{\widehat{x}_2}, & \text{if } i \in I_2, \end{cases}$$

where $\widehat{x}_1$ and $\widehat{x}_2$ are antipodal points.

Remark 5.2. *The Toda system, which was already seen to be a special case of Theorem 3, is also a special case of Proposition 5.1. Indeed, we can write $I = I_1 \cup I_2$ with I_1 and I_2 the even and odd indices respectively.*

For the proof we shall need the following lemma.

Lemma 5.2. *Let A be an irreducible symmetric n by n matrix satisfying (2.1) and $\boldsymbol{M} \in \mathbb{R}^n_+$ such that (2.11) holds. Suppose that $\boldsymbol{\rho}^{(m)}$ is a minimizing sequence for $\inf \Psi_{S^2}$ over $\mathbf{\Gamma}_{\boldsymbol{M}}(S^2)$ such that $\boldsymbol{\rho}^{(m)} \rightharpoonup \boldsymbol{\mu}$ weakly in the sense of measures. Then, $M_j\mu_i = M_i\mu_j,\ \forall i, j$.*

Proof. We use a similar argument to the one used in the proof of Theorem 1. By (2.11) we may write for any $\boldsymbol{\rho} \in \mathbf{\Gamma}_{\boldsymbol{M}}(S^2)$,

$$\Psi_{S^2}(\boldsymbol{\rho}) = \sum_{i,j} \frac{a_{i,j}}{16\pi}\big[M_j\int_{S^2} \rho_i \log\rho_i + M_i\int_{S^2} \rho_j \log\rho_j + 4\int_{S^2}\int_{S^2} \rho_i(x)\log|x-y|\rho_j(y)\,dxdy\big].$$

By (3.10) it follows that,

$$\lim_{m\to\infty} \big[M_j\int_{S^2} \rho_i^{(m)} \log\rho_i^{(m)} + M_i\int_{S^2} \rho_j^{(m)} \log\rho_j^{(m)} + 4\int_{S^2}\int_{S^2} \rho_i^{(m)}(x)\log|x-y|\rho_j^{(m)}(y)\,dxdy\big] = \Big(\frac{M_iM_j}{\pi}\Big)c_0 + M_iM_j\log\Big(\frac{M_iM_j}{16\pi^2}\Big),$$

for all i,j such that $a_{i,j} > 0$. Fixing any pair of such i,j and denoting $F_k^{(m)} = (\frac{4\pi}{M_k})\rho_k^{(m)}$, $\forall k$ (so that $\int_{S^2} F_k^{(m)} = 4\pi$) we conclude that

$$\lim_{m\to\infty} \Big[\int_{S^2} F_i^{(m)} \log F_i^{(m)} + \int_{S^2} F_j^{(m)} \log F_j^{(m)} + \frac{1}{\pi} \int_{S^2}\int_{S^2} F_i^{(m)}(x) \log|x-y| F_j^{(m)}(y)\, dxdy \Big] = 4c_0\,. \tag{5.54}$$

Note that by (3.9) we have,

$$\widetilde{\psi}(F_k^{(m)}) := \int_{S^2} F_k^{(m)} \log F_k^{(m)} + \frac{1}{2\pi} \int_{S^2}\int_{S^2} F_k^{(m)}(x) \log|x-y| F_k^{(m)}(y)\, dxdy \geq 2c_0, \quad \forall k. \tag{5.55}$$

Since,

$$\int_{S^2}\int_{S^2} (f(x)-1) \log|x-y| (f(y)-1)\, dxdy \leq 0, \quad \forall f \in \Gamma_{4\pi}(S^2),$$

we deduce that

$$\begin{aligned} \widetilde{\psi}(F_i^{(m)}) + \widetilde{\psi}(F_j^{(m)}) \leq & \int_{S^2} F_i^{(m)} \log F_i^{(m)} + \int_{S^2} F_j^{(m)} \log F_j^{(m)} \\ & + \frac{1}{\pi} \int_{S^2}\int_{S^2} F_i^{(m)}(x) \log|x-y| F_j^{(m)}(y)\, dxdy\,. \end{aligned} \tag{5.56}$$

Combining (5.56) with (5.54) and (5.55) we are led to,

$$\lim_{m\to\infty} \widetilde{\psi}(F_i^{(m)}) = \lim_{m\to\infty} \widetilde{\psi}(F_j^{(m)}) = 2c_0\,. \tag{5.57}$$

Plugging (5.57) in (5.54) yields,

$$\begin{aligned} \lim_{m\to\infty} \Big\{ & \int_{S^2}\int_{S^2} 2F_i^{(m)}(x) \log|x-y| F_j^{(m)}(y)\, dxdy - \\ & \int_{S^2}\int_{S^2} \big(F_i^{(m)}(x) \log|x-y| F_i^{(m)}(y) + F_j^{(m)}(x) \log|x-y| F_j^{(m)}(y) \big)\, dxdy \Big\} = 0. \end{aligned} \tag{5.58}$$

But setting,

$$U_k^{(m)} = -\frac{1}{2\pi} \int_{S^2} \log|x-y| F_k^{(m)}(y)\, dy, \quad \forall k,$$

we may rewrite (5.58) as

$$\lim_{m\to\infty} \int_{S^2} |\nabla (U_i^{(m)} - U_j^{(m)})|^2 = 0.$$

Therefore, for every $\varphi \in C^\infty(S^2)$ we have,

$$\frac{4\pi}{M_i} \int_{S^2} \varphi\, d\mu_i - \frac{4\pi}{M_j} \int_{S^2} \varphi\, d\mu_j = \lim_{m\to\infty} \int_{S^2} (F_i^{(m)} - F_j^{(m)}) \varphi\, dx = \lim_{m\to\infty} \int_{S^2} \nabla (U_i^{(m)} - U_j^{(m)}) \cdot \nabla \varphi = 0,$$

i.e. $M_j \mu_i = M_i \mu_j$. The result follows by the irreducibility of A. □

Proof of Proposition 5.1. By Theorem 1, applied to each $\Psi_{S^2}^{(I_l)}$ (see (5.35)), we have,

$$\Psi_{S^2}^{(I_l)}(\boldsymbol{\rho}) \geq \sum_{i\in I_l} \big(\frac{c_0}{2\pi}\big) M_i + M_i \log\Big(\frac{M_i}{4\pi}\Big), \quad \forall l. \tag{5.59}$$

Further, by (5.52)

$$2\sum_{i\in I_1, j\in I_2} (-a_{i,j}) \int_{S^2}\int_{S^2} \rho_i(x) \log|x-y| \rho_j(y)\, dxdy \leq 2(\log 2) \sum_{i\in I_1, j\in I_2} (-a_{i,j}) M_i M_j = \sup_{\widehat{\boldsymbol{x}}\in (S^2)^2} \widehat{W}(\widehat{\boldsymbol{x}}). \tag{5.60}$$

If $a_{i,j}=0,\ \forall i\in I_1, \forall j\in I_2$ then $\Psi_{S^2} = \Psi_{S^2}^{(I_1)} + \Psi_{S^2}^{(I_2)}$ and the result follows from (5.59) and Theorem 1. Assume then that at least one of these $a_{i,j}$'s is nonzero. Then the inequality in (5.60) is strict and combining it with (5.59) we obtain, for all $\boldsymbol{\rho}\in \boldsymbol{\Gamma}_{\boldsymbol{M}}(S^2)$,

$$\Psi_{S^2}(\boldsymbol{\rho}) > \sum_{l=1}^{2}\sum_{i\in I_l} \big[M_i \log\frac{M_i}{4\pi} + \big(\frac{M_i}{2\pi}\big)c_0\big] - \frac{1}{4\pi} \sup_{\widehat{\boldsymbol{x}}\in (S^2)^2} \widehat{W}(\widehat{\boldsymbol{x}})\,. \tag{5.61}$$

To finish the proof of (5.53) it is enough to show that $\Psi_{S^2}(\boldsymbol{\rho})$ can get as close as we wish to the value of the r.h.s. of (5.61). To this end, we fix a pair $\widehat{x}_1, \widehat{x}_2 \in S^2$ of antipodal points and define for every $\alpha>0$, $\boldsymbol{\rho}^{(\alpha)}\in \boldsymbol{\Gamma}_{\boldsymbol{M}}(S^2)$ as follows,

$$\rho_i^{(\alpha)} = (\rho_i^0)^{\tau_{\widehat{x}_m,\alpha}}, \quad \forall i\in I_m,\ m=1,2. \tag{5.62}$$

It is easy to see that the limit $\lim_{\alpha\to\infty}\Psi_{S^2}(\boldsymbol{\rho}^{(\alpha)})$ equals the r.h.s. of (5.61) and the proof of (5.53) is complete.

Next we turn to the proof of the statement about the weak limits of minimizing sequences. Let $\{\boldsymbol{\rho}^{(m)}\}$ be such a sequence with $\boldsymbol{\rho}^{(m)} \rightharpoonup \boldsymbol{\mu}$ weakly in the sense of measures. By the above,

$$\lim_{m\to\infty} \Psi_{S^2}^{(I_l)}(\boldsymbol{\rho}^{(m)}) = \sum_{i\in I_l} \big(\frac{c_0}{2\pi}\big) M_i + M_i \log\Big(\frac{M_i}{4\pi}\Big), \quad l=1,2, \tag{5.63}$$

i.e. the restriction of $\boldsymbol{\rho}^{(m)}$ to the indices of $I_l, l=1,2$, is a minimizing sequence for $\Psi_{S^2}^{(I_l)}$ over $\boldsymbol{\Gamma}_{\boldsymbol{M}^{(I_l)}}(S^2)$ ($\boldsymbol{M}^{(I_l)}$ denotes the restriction of $\boldsymbol{M}$ to the indices of I_l). Recall that A is irreducible, but the sub-matrix $A[I_1|I_1]$ may be reducible. So assume it is decomposable to $K_1\geq 1$ irreducible factors: $A[J_1|J_1],\dots,A[J_{K_1}|J_{K_1}]$. Similarly, $A[I_2|I_2]$ is decomposable to K_2 irreducible factors: $A[\tilde{J}_1|\tilde{J}_1],\dots,A[\tilde{J}_{K_2}|\tilde{J}_{K_2}]$. Fix any $1\leq k\leq K_1$. Since A is irreducible there exist $i_1\in J_k$ and $i_2\in I_2$ such that $a_{i_1,i_2}<0$. From (5.60) and Lemma 5.1 it follows that $\mu_{i_1} = M_{i_1}\delta_{x_{i_1}}$ and $\mu_{i_2} = M_{i_2}\delta_{x_{i_2}}$ with x_{i_1} and x_{i_2} a pair of antipodal points. By (5.63), the restriction of $\boldsymbol{\rho}^{(m)}$ to the indices of J_k is a minimizing sequence for $\Psi_{S^2}^{(J_k)}$ over $\boldsymbol{M}^{(J_k)}$, so that by Lemma 5.2 we deduce that $\mu_i = M_i\delta_{\widehat{x}_k}$ for all $i\in J_k$, for some $\widehat{x}_k\in S^2$. This holds for every $1\leq k\leq K_1$. Similarly, $\mu_i = M_i\delta_{\widehat{y}_k}$ if $i\in\tilde{J}_k$ for $1\leq k\leq K_2$.

Next we claim that $\widehat{x}_k=\widehat{x}$, $\forall k$ and $\widehat{y}_k=\widehat{y}$, $\forall k$, where $\widehat{x}$ and $\widehat{y}$ are antipodal, i.e. $\widehat{y}=-\widehat{x}$. We define a bipartite graph whose vertices are given by the points $\{\widehat{x}_k\}_{k=1}^{K_1}$ $\{\widehat{y}_k\}_{k=1}^{K_2}$. This graph contains an edge $[\widehat{x}_k,\widehat{y}_l]$ if and only if there exist $i\in J_k$, $j\in\tilde{J}_l$ with $a_{i,j}\neq 0$. We recall that if $[\widehat{x}_k,\widehat{y}_l]$ is an edge then $\widehat{x}_k = -\widehat{y}_l$. By assumption, this is a connected graph and the claim follows. □

The second case that we are able to treat allows for an arbitrary number of blocks, but requires a very particular structure of the matrix A.

Proposition 5.2. *Assume A and $\boldsymbol{M}$ are such that conditions (P1)–(P2) are satisfied. Suppose further that there exist n nonnegative numbers $\alpha_1, \dots, \alpha_n$ and a symmetric matrix $B = (b_{l,m})_{l,m=1}^K$ with nonpositive off-diagonal elements, such that $a_{i,j} = \alpha_i \alpha_j b_{l,m}$ for all $j \in I_m$, $i \in I_l$, whenever $m \neq l$. Then,*

$$\inf_{\boldsymbol{\Gamma}_{\boldsymbol{M}}(S^2)} \Psi_{S^2} = \sum_{l=1}^{K} \sum_{i \in I_l} \left[M_i \log \frac{M_i}{4\pi} + (\frac{M_i}{2\pi}) c_0 \right] - \frac{1}{4\pi} \sup_{\widehat{\boldsymbol{x}} \in (S^2)^K} \widehat{W}(\widehat{\boldsymbol{x}}) \,. \tag{5.64}$$

If, in addition, $\widehat{A}$ does not have a row of zeros (ignoring the diagonal), then the infimum in (5.64) *is not attained and any weak limit $\boldsymbol{\mu}$ of a minimizing sequence is of the form:*

$$\mu_i = M_i \delta_{\widehat{x}_l} \quad \text{if} \;\; i \in I_l, \; 1 \leq l \leq K,$$

where $\widehat{\boldsymbol{x}}$ is a maximizer of $\widehat{W}$.

Proof. We may rewrite (5.36) as

$$\Psi_{S^2}(\boldsymbol{\rho}) = \sum_{l=1}^{K} \Psi_{S^2}^{(I_l)}(\boldsymbol{\rho}) + \frac{1}{4\pi} \sum_{l \neq m} b_{l,m} \int_{S^2} \int_{S^2} \bar{\rho}_l(x) \log|x-y| \bar{\rho}_m(y) \, dxdy, \tag{5.65}$$

where

$$\bar{\rho}_l = \sum_{i \in I_l} \alpha_i \rho_i, \quad l = 1, \dots, K.$$

Applying Lemma 5.1 we get that,

$$\frac{1}{4\pi} \sum_{l \neq m} b_{l,m} \int_{S^2} \int_{S^2} \bar{\rho}_l(x) \log|x-y| \bar{\rho}_m(y) \geq -\frac{1}{4\pi} \sup_{\widehat{\boldsymbol{x}} \in (S^2)^K} \overline{W}(\widehat{\boldsymbol{x}}), \tag{5.66}$$

where

$$\overline{W}(\widehat{\boldsymbol{x}}) := \sum_{l \neq m} (-b_{l,m}) \overline{M}_l \overline{M}_m \log|\widehat{x}_l - \widehat{x}_m| \tag{5.67}$$

and $\overline{M}_l := \sum_{i \in I_l} \alpha_i M_i$, $l = 1, \dots, K$. Using the identity $\widehat{a}_{l,m} \widehat{M}_l \widehat{M}_m = b_{l,m} \overline{M}_l \overline{M}_m$ we see that $\overline{W} = \widehat{W}$. Hence, from (5.65), (5.66) and (5.59) we obtain that the r.h.s in (5.64) is a lower bound for Ψ_{S^2}. The optimality of this bound follows by considering the limit $\lim_{\alpha \to \infty} \Psi_{S^2}(\boldsymbol{\rho}^{(\alpha)})$ where,

$$\rho_i^{(\alpha)} = (\rho_i^0)^{\tau_{\widehat{x}_m, \alpha}}, \quad \forall i \in I_m, \; m = 1, \dots, K,$$

and $\widehat{\boldsymbol{x}} = (\widehat{x}_1, \dots, \widehat{x}_K)$ is a maximizer for $\widehat{W}$. If $\widehat{A}$ does not have a row of zeros (outside the diagonal) then the same holds for B. Applying Lemma 5.1 to

$$\bar{J}(\boldsymbol{\mu}) = \sum_{l \neq m} (-b_{l,m}) \int_{S^2} \int_{S^2} \log|x-y| \, d\mu_l(x) d\mu_m(y),$$

over

$$\mathcal{G}_{\overline{\boldsymbol{M}}}(S^2) = \{\boldsymbol{\mu} = (\mu_1, \ldots, \mu_K) \,:\, \mu_i \geq 0 \text{ and } \int_{S^2} d\mu_i = \overline{M}_i,\ i = 1, \ldots, K\}\,,$$

we deduce as in the proof of Theorem 3(iii) the nonexistence of a minimizer and the description of the weak limits of the minimizing sequences. □

References

[1] T. Aubin, *Meilleures constantes dans le théorème d'inclusion de Sobolev et un théorème de Fredholm non linéaire pour la transformation conforme de la courbure scalaire*, J. Funct. Anal. **32** (1979), 148–174.

[2] W. Beckner, *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality*, Ann. of Math. **128** (1993), 213–242.

[3] A. Baernstein and B.A. Taylor, *Spherical rearrangements, subharmonic functions, and ∗-functions in n-space*, Duke Math. J. **43** (1976), 245–268.

[4] M. Chipot, I. Shafrir and G. Wolansky, *On the solutions of Liouville systems*, J. Diff. Eq. **140** (1997), 59–105; Erratum loc. cit. **178** (2002), 630.

[5] E. Carlen and M. Loss, *Competing symmetries, the logarithmic HLS inequality and Onofri's inequality on S^n*, Geom. Func. Anal. **1** (1992), 90–104.

[6] S. Chanillo and M. K.-H. Kiessling, *Conformally invariant systems of nonlinear pde of Liouville type*, Geom. Func. Anal. **5** (1995), 924-947.

[7] S.-Y.A. Chang and P. Yang, *Prescribing Gaussian curvature on S^2*, Acta Math. **159** (1987), 215–259.

[8] R. Dautray and J.-L. Lions, Mathematical Analysis and Numerical Methods for Science and Technology Vol. 1, Berlin-Springer, 1988.

[9] L. Fontana, *Sharp borderline Sobolev inequalities on compact Riemannian manifolds*, Comment. Math. Helv. **68** (1993), 415–454.

[10] C. Hong, *A best constant and the Gaussian curvature*, Proc. Amer. Math. Soc. **97** (1986), 737–747.

[11] J. Jost and G. Wang, *Analytic aspects of the Toda system. I. A. Moser-Trudinger inequality*, Comm. Pure Appl. Math. **54** (2001), 1289–1319.

[12] E. H. Lieb, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. of Math. **118** (1983), 349–374.

[13] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1971), 1077-1092.

[14] M. Nolasco and G. Tarantello, *On a sharp Sobolev type inequality on two dimensional compact manifolds*, Arch. Ration. Mech. Anal. **145** (1998), 161–195.

[15] E. Onofri, *On the positivity of the effective action in a theory of random surfaces*, Comm. Math. Phys. **86** (1982), 321–326.

[16] B. Osgood, R. Phillips and P. Sarnak, *Extremals of determinants of Laplacians*, J. Funct. Anal. **80** (1988), 148–211.

[17] R. Ransford, Potential Theory in the Complex Plane, London Mathematical Society Student Texts **28**, 1995.

[18] I. Shafrir and G. Wolansky, *Moser-Trudinger type inequalities for systems in two dimensions*, C. R. Acad. Sci. Paris Sér. I Math. **333** (2001), 439–443.

[19] I. Shafrir and G. Wolansky, *The logarithmic HLS inequality for systems on compact manifolds*, to appear in J. Func. Anal..

[20] G. Wang, *Moser-Trudinger inequalities and Liouville systems*, C. R. Acad. Sci. Paris Sér. I Math. **328** (1999), 895–900.

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