Moser-Trudinger type inequalities for systems in two dimensions

Itai Shafrir and Gershon Wolansky

Abstract - We prove optimal Moser-Trudinger type inequalities for vector-valued functions on the sphere S^2 and on bounded domains in $\mathbb{R}^2$. We also prove entropy inequalities and logarithmic Hardy-Littlewood-Sobolev inequalities for systems on all of $\mathbb{R}^2$.

Inégalités de type Moser-Trudinger pour des systèmes en dimension deux

Résumé - *On démontre des inégalités optimales de type Moser-Trudinger pour des applications vectorielles sur la sphère S^2 et sur des domaines bornés dans $\mathbb{R}^2$. De plus, on démontre une inégalité d'entropie pour des systèmes sur $\mathbb{R}^2$ tout entier.*

Version française abrégée - L'inégalité suivante de Moser-Trudinger sur la sphère S^2 (voir [8]),

$$\int_{S^2} (\frac{1}{2}|\nabla u|^2 + 2u)\, d\omega - 8\pi \log\Big(\int_{S^2} e^u \,\frac{d\omega}{4\pi}\Big) \geq -C, \quad \forall u \in H^1(S^2), \tag{1}$$

joue un rôle important dans les problèmes de courbure de Gauss prescrite sur la sphère, voir Aubin [1], Chang-Yang [6]. Onofri [9] a démontré que (1) est valable avec la constante optimale $C = 0$ à droite. Une preuve différente de l'inégalité d'Onofri, qui permet une généralisation de l'inégalité pour S^N, $N \geq 3$, est due à Beckner [2], voir aussi Carlen et Loss [4]. Cette méthode utilise un argument de dualité et une inégalité logarithmique de type HLS (Hardy-Littlewood-Sobolev), déduite de la forme exacte de l'inégalité HLS de Lieb [7]. Dans cette Note on applique un argument de dualité pour généraliser l'inégalité (1) dans le cas de systèmes. On démontre aussi un résultat analogue pour un système sur un domaine borné de $\mathbb{R}^2$ et une inégalité d'entropie sur $\mathbb{R}^2$.

On suppose toujours que la matrice $n \times n, A = (a_{i,j})$ est une matrice symétrique dont tous les termes sont positifs ou nuls. Pour certains résultats on aura besoin de supposer en outre, que la matrice A est définie positive. Pour une telle matrice A on considère la fonctionnelle :

$$F^{\boldsymbol{M}}(\boldsymbol{u}) = \frac{1}{2}\sum_{i,j=1}^{n} a_{i,j}\int_{S^2}\Big(\nabla u_i \nabla u_j + (M_i/2\pi)u_j\Big) - \sum_{i=1}^{n} M_i \log\Big(\int_{S^2}\exp(\sum_{j=1}^{n} a_{i,j}u_j)\Big), \tag{2}$$

et on pose la question suivante: pour quels vecteurs $\boldsymbol{M} \in (\mathbb{R}_+)^n$ la fonctionnelle $F^{\boldsymbol{M}}$ est-elle bornée inférieurement sur l'espace $(H^1(S^2))^n$? La réponse dépend d'un système d'inégalités que doit satisfaire le vecteur $\boldsymbol{M}$. On définit

$$\Lambda_J(\boldsymbol{M}) = 8\pi \sum_{i\in J} M_i - \sum_{i,j\in J} a_{i,j}M_iM_j, \quad \emptyset \neq J \subseteq I \equiv \{1,\dots,n\}\,. \tag{3}$$

Théorème 1. *Soit A une matrice définie positive dont les termes sont positifs ou nuls. Alors la fonctionnelle (2) est bornée inférieurement sur $(H^1(S^2))^n$ si et seulement si les conditions suivantes sont satisfaites:*

$$\Lambda_J(\boldsymbol{M}) \geq 0, \quad \forall J \subseteq I,\ J \neq \emptyset. \tag{4}$$

La preuve du Théorème 1 repose sur l'étude d'un problème dual, concernant la fonctionnelle

$$\Psi(\boldsymbol{\rho}) = \sum_{i=1}^{n}\int_{S^2}\rho_i \log \rho_i + \frac{1}{4\pi}\sum_{i,j=1}^{n} a_{i,j}\int_{S^2}\int_{S^2}\rho_i(x)\log|x-y|\rho_j(y)\,dx\,dy\,, \tag{5}$$

sur l'ensemble

$$\boldsymbol{\Gamma_M} = \{\boldsymbol{\rho} \in (\mathcal{L}\ln\mathcal{L}(S^2))^n \text{ avec } \rho_i \geq 0 \text{ et } \int_{S^2}\rho_i = M_i,\ \forall i\}. \tag{6}$$

On considère ensuite un problème lié à la résolution d' une inégalité d'entropie sur $\mathbb{R}^2$.

Théorème 2. *Soit A une matrice symétrique dont les termes sont positifs ou nuls. Alors, la fonctionnelle*

$$\overline{\Psi}(\tilde{\boldsymbol{\rho}}) = \sum_{i=1}^{n}\int_{\mathbb{R}^2}\tilde{\rho}_i \log \tilde{\rho}_i\,dx + \frac{1}{4\pi}\sum_{i,j=1}^{n} a_{i,j}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\tilde{\rho}_i(x)\log|x-y|\tilde{\rho}_j(y)\,dx\,dy, \tag{7}$$

est bornée inférieurement sur l'ensemble $\widetilde{\mathbf{\Gamma}}_{\boldsymbol{M}}$ *(défini comme dans* (6) *en remplaçant* S^2 *par* $\mathbb{R}^2$*) si est seulement si l'on a,*

$$\Lambda_I(\boldsymbol{M}) = 0 \ \text{et}\ \Lambda_J(\boldsymbol{M}) \geq 0, \quad \forall J \subsetneq I,\ J \neq \emptyset. \tag{8}$$

On conclut avec une version de l'inégalité de Moser-Trudinger sur un domaine borné de $\mathbb{R}^2$. Le théorème suivant améliore un résultat de [3].

Théorème 3. *Soit* A *une matrice définie positive avec des éléments positifs ou nuls. Alors,*

$$\frac{1}{2}\sum_{i,j=1}^{n}\int_{\Omega} a_{i,j}\nabla u_i \nabla u_j - \sum_{i=1}^{n} M_i \log\Big(\int_{\Omega}\exp(\sum_{j=1}^{n} a_{i,j}u_j)\Big) \geq -C, \quad \forall \boldsymbol{u} \in (H_0^1(\Omega))^n,$$

si et seulement si les conditions (4) *sont satisfaites.*

The following Moser-Trudinger inequality on the sphere S^2 (see [8]),

$$\int_{S^2}(\frac{1}{2}|\nabla u|^2 + 2u)\,d\omega - 8\pi\log\Big(\int_{S^2} e^u \frac{d\omega}{4\pi}\Big) \geq -C, \quad \forall u \in H^1(S^2), \tag{1}$$

plays an important role in problems of prescribing Gauss curvature on the sphere, like Nirenberg's problem and Yamabe's problem, see Aubin [1], Chang-Yang [6] and the references therein. A sharp version of (1), which is due to Onofri [9], states that (1) is valid with the optimal $C = 0$ on the right hand side. An alternative derivation of Onofri's inequality by Beckner [2], see also Carlen and Loss [4], which allows an extension of the inequality to S^N, $N \geq 2$, is of particular interest to us. This approach uses a duality argument and an entropy inequality, a logarithmic HLS (Hardy-Littlewood-Sobolev) inequality which is derived from Lieb's sharp form of the HLS inequality ([7]). The main objective of the present paper is to establish an extension of (1) to systems. We shall also prove an analogous inequality for systems defined on a bounded domain in $\mathbb{R}^2$ (see Theorem 3 below) and an entropy inequality on all of $\mathbb{R}^2$ (see Theorem 2).

In order to state our main results we shall need some notations. Let $A = (a_{i,j})$ be a $n \times n$ symmetric matrix with constant nonnegative coefficients. For some of the results we shall also assume that A is positive definite. For a matrix A as above we seek conditions on the vector $\boldsymbol{M} \in (\mathbb{R}_+)^n$ which will ensure that the following functional,

$$F^{\boldsymbol{M}}(\boldsymbol{u}) = \frac{1}{2}\sum_{i,j=1}^{n} a_{i,j}\int_{S^2}\Big(\nabla u_i\nabla u_j + (M_i/2\pi)u_j\Big) - \sum_{i=1}^{n} M_i\log\Big(\int_{S^2}\exp(\sum_{j=1}^{n} a_{i,j}u_j)\Big) \tag{2}$$

is bounded from below on the space $(H^1(S^2))^n$. Note that in the *scalar* case $n = 1$, it follows easily from (1) that a necessary and sufficient condition for the boundedness from below of F^M over $H^1(S^2)$ is $M \leq 8\pi$. The analogue of this condition to the system case turns out to be a set of inequalities involving the quadratic polynomials

$$\Lambda_J(\boldsymbol{M}) = 8\pi \sum_{i\in J} M_i - \sum_{i,j\in J} a_{i,j} M_i M_j, \quad \emptyset \neq J \subset I \equiv \{1,\ldots,n\}\,. \tag{3}$$

The detailed proofs of the theorems below are given in [10].

Our first main result provides a complete answer to the question posed above.

Theorem 1. *Assume the matrix A is positive definite with nonnegative coefficients. Then the functional* (2) *is bounded below on $(H^1(S^2))^n$ if and only if the following conditions hold,*

$$\Lambda_J(\boldsymbol{M}) \geq 0, \quad \forall J \subseteq I,\ J \neq \emptyset. \tag{4}$$

We recall that the polynomial Λ_I was introduced by Chanillo and Kiessling [5] in their study of entire solutions of Liouville systems in $\mathbb{R}^2$. A set of conditions similar to (4), but with strict inequalities, was used in [3] for the study of a related variational problem on bounded domains in $\mathbb{R}^2$ and the associated minimizers, see also below. Two special cases of Theorem 1 were proved by Wang [11] (for functionals defined on general closed surfaces, i.e. not necessarily the sphere). The first of these is when the inequalities in (4) are strict (i.e. this is the *subcritical case*). The second case studied by Wang is when the matrix A is stochastic and the vector of masses $\boldsymbol{M}$ satisfies $M_i = 8\pi$, $\forall i$ (which implies also the case $M_i \leq 8\pi$, $\forall i$). In fact, this is a particular example of what we shall call the *conformal case*, i.e. when we have,

$$\sum_{j=1}^{n} a_{i,j} M_j = 8\pi, \quad i = 1,\ldots,n. \tag{5}$$

Note that (5) implies that $\Lambda_I(\boldsymbol{M}) = 0$ and $\Lambda_J(\boldsymbol{M}) \geq 0$ for any $J \subseteq I$, but this is of course just a special case of the general *critical case*, i.e. when some equalities are allowed in (4). The particular feature of the conformal case is that it induces full invariance of the functional with respect to the conformal group of the sphere S^2 (just as in the case $M = 8\pi$ for $n = 1$). The result of Theorem 1 in the *conformal case*, even in a sharp form, can be easily deduced from the scalar case, see [10] for details.

Our proof of Theorem 1 uses a duality principle, similarly to the approach used by Beckner [2] and Carlen and Loss [4] in the scalar case ($n = 1$) in their proof of Onofri's inequality (and its generalization to higher dimensions). We are thus led to investigate the question of boundedness from below of the functional

$$\Psi(\boldsymbol{\rho}) = \sum_{i=1}^{n} \int_{S^2} \rho_i \log \rho_i + \frac{1}{4\pi} \sum_{i,j=1}^{n} a_{i,j} \int_{S^2} \int_{S^2} \rho_i(x) \log |x-y| \rho_j(y)\, dx\, dy\,, \tag{6}$$

(where $|x-y|$ stands for the Euclidean distance in $\mathbb{R}^3$) over the class

$$\boldsymbol{\Gamma_M} = \{\boldsymbol{\rho} \in (\mathcal{L} \ln \mathcal{L}(S^2))^n \text{ with } \rho_i \geq 0 \text{ and } \int_{S^2} \rho_i = M_i,\ \forall i\}. \tag{7}$$

The Proof of Theorem 1 is obtained as a corollary of the characterization of the vectors $\boldsymbol{M}$ for which the functional $\Psi(\boldsymbol{\rho})$ is bounded below on $\boldsymbol{\Gamma_M}$. This later question makes sense and is resolved for any symmetric matrix A with nonnegative elements (i.e. without the assumption of positive definiteness of A).

Our second main result is about entropy inequalities for systems in $\mathbb{R}^2$. It involves the functional

$$\overline{\Psi}(\tilde{\boldsymbol{\rho}}) = \sum_{i=1}^{n} \int_{\mathbb{R}^2} \tilde{\rho}_i \log \tilde{\rho}_i\, dx + \frac{1}{4\pi} \sum_{i,j=1}^{n} a_{i,j} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \tilde{\rho}_i(x) \log |x-y| \tilde{\rho}_j(y)\, dx\, dy, \tag{8}$$

where $|x-y|$ stands, this time, for the Euclidean distance in $\mathbb{R}^2$. The functional $\overline{\Psi}$ is defined on the set $\widetilde{\boldsymbol{\Gamma}}_{\boldsymbol{M}}$ which is defined analogously to (7), with $\mathbb{R}^2$ replacing S^2, for a given symmetric A with nonnegative elements. It is interesting to compare the functional (8) to the functional (6) when transformed to a functional on $\mathbb{R}^2$ using stereographic projection. The result is,

$$\begin{aligned} \widetilde{\Psi}(\tilde{\boldsymbol{\rho}}) = \sum_{i=1}^{n} \int_{\mathbb{R}^2} \tilde{\rho}_i \log \tilde{\rho}_i\, dx + \frac{1}{4\pi} \sum_{i,j=1}^{n} a_{i,j} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \tilde{\rho}_i(x) \log |x-y| \tilde{\rho}_j(y)\, dx\, dy \\ + \sum_{i=1}^{n} \nu_i \int_{\mathbb{R}^2} \tilde{\rho}_i \log(1+|x|^2)\, dx\,, \end{aligned} \tag{9}$$

with

$$\nu_i = 2 - \frac{1}{4\pi} \sum_{j=1}^{n} a_{i,j} M_j, \quad i = 1, \ldots, n. \tag{10}$$

It then follows that the two minimization problems are equivalent only in the conformal case (5) (i.e. $\nu_i = 0$). A criterion for the boundedness from below of $\overline{\Psi}$ is given by the next theorem.

Theorem 2. *The functional* $\overline{\Psi}$ *is bounded from below on* $\widetilde{\mathbf{\Gamma}}_{\boldsymbol{M}}$ *if and only if*

$$\Lambda_I(\boldsymbol{M}) = 0 \text{ and } \Lambda_J(\boldsymbol{M}) \geq 0, \quad \forall J \subsetneq I,\ J \neq \emptyset. \tag{11}$$

Finally we turn to a version of Moser-Trudinger inequality for systems on a bounded domain $\Omega \subset \mathbb{R}^2$. By Moser's inequality [8]

$$\frac{1}{2}\int_\Omega |\nabla u|^2 - 8\pi \log\Big(\int_\Omega e^u\Big) \geq -C, \quad \forall u \in H_0^1(\Omega). \tag{12}$$

The extension to systems should be an inequality of the form,

$$\frac{1}{2}\sum_{i,j=1}^n \int_\Omega a_{i,j}\nabla u_i \nabla u_j - \sum_{i=1}^n M_i \log\Big(\int_\Omega \exp(\sum_{j=1}^n a_{i,j}u_j)\Big) \geq -C, \quad \forall \boldsymbol{u} \in (H_0^1(\Omega))^n, \tag{13}$$

where A is a positive definite matrix with nonnegative elements. In [3] it was shown that (13) holds for $\boldsymbol{M}$ satisfying

$$\Lambda_J(\boldsymbol{M}) > 0, \quad \forall J \subseteq I,\ J \neq \emptyset. \tag{14}$$

The question whether the same result remains valid under the weaker assumptions (4) was left open. Our last theorem provides a positive answer to that question by establishing the optimality of (4).

Theorem 3. *Let* A *be a positive definite matrix with nonnegative elements. Then* (13) *holds if and only if* (4) *is satisfied.*

The basic ideas of the proofs are:

1. Reduction to radially symmetric functions. This is done by symmetrization, either on the plane (in Theorem 2 and Theorem 3) or on the sphere (in Theorem 1).

2. Using the result of [3] we know that a minimizer exists on a finite disc $\{|x| < R\}$ under the assumption of strict inequalities (14). We then proceed to show that the lower bound realized by this minimizer is independent of R and $\Lambda_J(\boldsymbol{M})$, $J \subseteq I$ as long as $\Lambda_J(\boldsymbol{M}) \geq 0$, $J \subseteq I$.

Acknowledgment. This research was supported by M. and M. L. Bank Mathematics research fund. The research of I.S. was also supported by the fund for promotion of research at the Technion.

References

[1] T. Aubin, *Meilleures constantes dans le théorème d'inclusion de Sobolev et un théorème de Fredholm non linéaire pour la transformation conforme de la courbure scalaire*, J. Funct. Anal. **32** (1979), 148–174.

[2] W. Beckner, *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality*, Ann. of Math. **128** (1993), 213–242.

[3] M. Chipot, I. Shafrir and G. Wolansky, *On the solutions of Liouville systems*, J. Diff. Eq. **140** (1997), 59–105.

[4] E. Carlen and M. Loss, *Competing symmetries, the logarithmic HLS inequality and Onofri's inequality on S^n*, Geom. Func. Anal. **1** (1992), 90–104.

[5] S. Chanillo and M. K.-H. Kiessling, *Conformally invariant systems of nonlinear pde of Liouville type*, Geom. Func. Anal. **5** (1995), 924-947.

[6] S.-Y.A. Chang and P. Yang, *Prescribing Gaussian curvature on S^2*, Acta Math. **159** (1987), 215–259.

[7] E. H. Lieb, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. of Math. **118** (1983), 349–374.

[8] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1971), 1077-1092.

[9] E. Onofri, *On the positivity of the effective action in a theory of random surfaces*, Comm. Math. Phys. **86** (1982), 321–326.

[10] I. Shafrir and G. Wolansky, *Moser-Trudinger and logarithmic HLS inequalities for systems*, preprint.

[11] G. Wang, *Moser-Trudinger inequalities and Liouville systems*, C. R. Acad. Sci. Paris Sér. I Math. **328** (1999), 895–900.

Technion - Israel Institute of Technology, 32000 Haifa, ISRAEL

email: shafrir@math.technion.ac.il

gershonw@math.technion.ac.il