

# ASYMPTOTICS OF EIGENSTATES OF ELLIPTIC PROBLEMS WITH MIXED BOUNDARY DATA ON DOMAINS TENDING TO INFINITY

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ABSTRACT. We analyze the asymptotic behavior of eigenvalues and eigenfunctions of an elliptic operator with mixed boundary conditions on cylindrical domains when the length of the cylinder goes to infinity. We identify the correct limiting problem and show in particular, that in general the limiting behavior is very different from the one for the Dirichlet boundary conditions.

## 1. INTRODUCTION

Let  $\omega$  be a bounded open set in  $\mathbb{R}^{n-1}$ . For every  $\ell > 0$  set  $\Omega_\ell = (-\ell, \ell) \times \omega$  and write each  $x \in \Omega_\ell$  as  $x = (x_1, X_2)$  with  $X_2 = (x_2, \dots, x_n)$ . We assume that the matrices

$$A(X_2) = \begin{pmatrix} a_{11}(X_2) & A_{12}(X_2) \\ A_{12}^t(X_2) & A_{22}(X_2) \end{pmatrix}$$

are uniformly elliptic and uniformly bounded on  $\omega$  (precise assumptions will be made in Section 2). The limiting behavior, when  $\ell$  goes to infinity, of the eigenvalues and eigenfunctions of the elliptic operator  $-\operatorname{div}(A(X_2)\nabla u)$  on  $\Omega_\ell$  with zero Dirichlet boundary conditions, was studied by Chipot and Rougirel in [7]. We shall recall below one of their main results that was the principal motivation for the current paper. Let  $\mu^k$  and  $\sigma_\ell^k$  denote, respectively, the  $k$ th eigenvalues for the problems

$$(1.1) \quad \begin{cases} -\operatorname{div}(A_{22}(X_2)\nabla u) = \mu u & \text{in } \omega, \\ u = 0 & \text{on } \partial\omega, \end{cases}$$

and

$$(1.2) \quad \begin{cases} -\operatorname{div}(A(X_2)\nabla u) = \sigma u & \text{in } \Omega_\ell, \\ u = 0 & \text{on } \partial\Omega_\ell. \end{cases}$$

The following relation between problem (1.2) (for large  $\ell$ ) and problem (1.1) was established in [7].

**Theorem A** (Chipot-Rougirel).

$$(1.3) \quad \mu^1 \leq \sigma_\ell^1 \leq \mu^1 + \frac{C}{\ell^2},$$

where  $C$  is a constant independent of  $\ell$ .

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The main goal of the present article is to study the analogous problem for *mixed* boundary conditions, at least for  $k = 1$ . Let us write  $\partial\Omega_\ell = \Gamma_\ell \cup \gamma_\ell$  where

$$(1.4) \quad \Gamma_\ell = \{-\ell, \ell\} \times \omega \text{ and } \gamma_\ell = (-\ell, \ell) \times \partial\omega,$$

and denote by  $\lambda_\ell^k$  the  $k$ th eigenvalue for the mixed Neumann-Dirichlet problem

$$(1.5) \quad \begin{cases} -\operatorname{div}(A(X_2)\nabla u) = \sigma u & \text{in } \Omega_\ell, \\ u = 0 & \text{on } \gamma_\ell, \\ (A(X_2)\nabla u) \cdot \nu = 0 & \text{on } \Gamma_\ell. \end{cases}$$

One of our main results establishes that  $\lim_{\ell \rightarrow \infty} \lambda_\ell^1$  exists, but in general it is strictly smaller than  $\mu^1$ . This “gap phenomenon” is explained by the appearance of boundary effects near  $\Gamma_\ell$ . To gain better understanding of these effects we are led to consider first the limit  $\lim_{\ell \rightarrow 0} \lambda_\ell^1$ . Asymptotic behavior of elliptic problems set on domains shrinking to zero in some directions are generally known as “Dimension Reduction” problems and are addressed in [1, 3, 14] and in a setting particularly suitable for us, in [2]. Our work establishes a somewhat surprising connection between the theory of dimension reduction (i.e., “ $\ell \rightarrow 0$ ”) and the theory for “ $\ell \rightarrow \infty$ ”.

In order to have a more precise description of the boundary effects and to characterize the value of the limit  $\lim_{\ell \rightarrow \infty} \lambda_\ell^1$ , we introduce eigenvalue problems on the two semi-infinite cylinders  $\Omega_\infty^+ = (0, \infty) \times \partial\omega$  and  $\Omega_\infty^- = (-\infty, 0) \times \partial\omega$ , with mixed boundary conditions. Let  $\nu_\infty^\pm$  denote the first eigenvalue for the operator  $-\operatorname{div}(A(X_2)\nabla u)$  on  $\Omega_\infty^\pm$  with zero boundary condition on the lateral part of the boundary  $\partial\Omega_\infty^\pm$ . One might be tempted to expect that the equality  $\nu_\infty^+ = \nu_\infty^-$  always hold because of “symmetry considerations”. However, as we shall see in Section 6, this equality is false in general. Our main results are summarized in the next theorem, that combines the results of Theorem 4.2 and Theorem 5.2. We denote by  $W_1$  the positive normalized eigenfunction corresponding to  $\mu^1$ .

**Main Theorem.** We have  $\lim_{\ell \rightarrow \infty} \lambda_\ell^1 = \min(\nu_\infty^+, \nu_\infty^-)$ . If  $A_{12} \cdot \nabla W_1 \neq 0$  a.e. on  $\omega$ , then  $\lim_{\ell \rightarrow \infty} \lambda_\ell^1 < \mu^1$ . Otherwise,  $\lambda_\ell^1 = \mu^1, \forall \ell$ .

Many problems of the type “ $\ell \rightarrow \infty$ ” were studied in the past. Besides the eigenvalue problem already mentioned [7], these include elliptic and parabolic equations, variational inequalities and systems, see [5, 6, 8, 9, 10, 11, 12]. In all these problems it is found that the limit is characterized by the solution of the corresponding problem on the section  $\omega$ . We emphasize that the limiting behavior in our problem is very different.

The paper is organized as follows. In Section 2 we give the main definitions and notation needed in the subsequent sections. In Section 3 we illustrate the gap phenomenon in a simple model case where  $\omega = (-1, 1)$  and  $A$  is a  $2 \times 2$  matrix with constant coefficients, namely,  $A = A_\delta = \begin{pmatrix} 1 & \delta \\ \delta & 1 \end{pmatrix}$ . In Section 4 we prove the gap phenomenon for the general case. In Section 5 we prove that the limit  $\lim_{\ell \rightarrow \infty} \lambda_\ell^1$  exists, and identify its value using the eigenvalue problems on the semi-infinite cylinders  $\Omega_\infty^+$  and  $\Omega_\infty^-$ . In Section 6 we investigate further the problem on a semi-infinite cylinder and use it to give a more precise description of the first eigenfunction  $u_\ell$  for large  $\ell$ . In the last section, Section 7, we address briefly two natural related problems. First, we present a result on the asymptotics of the

second eigenvalue  $\lambda_\ell^2$  as  $\ell$  goes to infinity (under some symmetry assumption on the matrix  $A$ ). Second, we give a partial result for the more general case of a domain becoming large in several directions.

## 2. PRELIMINARIES

For each  $\ell > 0$  consider  $\Omega_\ell = (-\ell, \ell) \times \omega$  with  $\omega$  a bounded domain in  $\mathbb{R}^{n-1}$  as in the Introduction. The lateral part of  $\partial\Omega_\ell$  and the remaining part of the cylinder (i.e., the two ends) will be denoted by  $\gamma_\ell$  and  $\Gamma_\ell$ , respectively. Let us denote by  $H^1(\Omega_\ell)$  and  $H_0^1(\Omega_\ell)$  the usual spaces of functions defined by

$$H^1(\Omega_\ell) = \{v \in L^2(\Omega_\ell) \mid \partial_{x_i} v \in L^2(\Omega_\ell), i = 1, 2, \dots, n\},$$

and

$$H_0^1(\Omega_\ell) = \{v \in H^1(\Omega_\ell) \mid v = 0 \text{ on } \partial\Omega_\ell\},$$

or in a more precise way,  $H_0^1(\Omega_\ell)$  is the closure of  $C_c^\infty(\Omega_\ell)$  in  $H^1(\Omega_\ell)$ . The space  $H_0^1(\Omega_\ell)$  is equipped with the norm

$$(2.1) \quad \|\nabla v\|_{2, \Omega_\ell}^2 = \int_{\Omega_\ell} |\nabla v|^2.$$

A suitable space for our problem is

$$V(\Omega_\ell) = \{v \in H^1(\Omega_\ell) \mid v = 0 \text{ on } \gamma_\ell\},$$

where the boundary condition should be interpreted in the sense of traces. Thanks to the Poincaré inequality,  $V(\Omega_\ell)$  becomes an Hilbert space when equipped with the norm (2.1). For later use we define the sets

$$(2.2) \quad \Omega_\ell^+ = [0, \ell) \times \omega \text{ and } \Omega_\ell^- = (-\ell, 0) \times \omega,$$

We decompose  $\Gamma_\ell$  (see (1.4)) into two parts as  $\Gamma_\ell = \Gamma_\ell^+ \cup \Gamma_\ell^-$ , where

$$(2.3) \quad \Gamma_\ell^+ = \{\ell\} \times \omega \text{ and } \Gamma_\ell^- = \{-\ell\} \times \omega.$$

Similarly, for the lateral part of  $\partial\Omega_\ell$  we define,

$$(2.4) \quad \gamma_\ell^+ = (0, \ell) \times \partial\omega \text{ and } \gamma_\ell^- = (-\ell, 0) \times \partial\omega.$$

We shall be concerned with the operator  $-\operatorname{div}(A(X_2)\nabla u)$  where, for each  $X_2 \in \omega$ ,

$$A(X_2) = \begin{pmatrix} a_{11}(X_2) & A_{12}(X_2) \\ A_{12}^t(X_2) & A_{22}(X_2) \end{pmatrix}$$

is a symmetric  $n \times n$  matrix,  $a_{11} \in \mathbb{R}$ ,  $A_{12}$  is a  $1 \times (n-1)$  matrix and  $A_{22}$  is a  $(n-1) \times (n-1)$  matrix. The components of  $A(X_2)$  are assumed to be bounded measurable functions on  $\omega$  and we assume the following bound

$$(2.5) \quad \|A(X_2)\| \leq C_A, \text{ a.e. } X_2 \in \omega,$$

for the Euclidean operator norm. We also assume that  $A(X_2)$  is uniformly elliptic and denote by  $\lambda_A$  the largest positive number for which the following inequality holds,

$$(2.6) \quad A(X_2)\xi \cdot \xi \geq \lambda_A |\xi|^2, \forall \xi \in \mathbb{R}^n, \text{ a.e. } X_2 \in \omega.$$

The weak formulation of the eigenvalue problem (1.1) is to find  $u \in H_0^1(\omega) \setminus \{0\}$  and  $\mu \in \mathbb{R}$  such that

$$(2.7) \quad \int_{\omega} (A_{22} \nabla u) \cdot \nabla v \, dX_2 = \mu \int_{\omega} uv \, dX_2, \quad \forall v \in H_0^1(\omega).$$

Denote by  $\mu^1$  the first eigenvalue of the problem (2.7) with the corresponding normalized eigenfunction  $W_1$ , i.e.,  $\int_{\omega} |W_1|^2 = 1$ . It is well known that  $\mu^1$  has a variational characterization by the Rayleigh quotient:

$$(2.8) \quad \mu^1 = \inf \left\{ \int_{\omega} (A_{22}(X_2) \nabla u) \cdot \nabla u \mid u \in H_0^1(\omega) \text{ s.t. } \int_{\omega} u^2 = 1 \right\} \\ = \inf_{u \in H_0^1(\omega) \setminus \{0\}} \frac{\int_{\omega} (A_{22}(X_2) \nabla u) \cdot \nabla u}{\int_{\omega} u^2}.$$

Moreover,  $W_1$  is simple and has constant sign in  $\Omega$  (see [13]). The choice of positive sign leaves us with a unique  $W_1$ .

Similarly, the eigenvalue problem (1.5) has the following weak formulation: find  $u \in V(\Omega_{\ell}) \setminus \{0\}$  and a real number  $\lambda$  such that

$$(2.9) \quad \int_{\Omega_{\ell}} A \nabla u \cdot \nabla v \, dx = \lambda \int_{\Omega_{\ell}} uv \, dx, \quad \forall v \in V(\Omega_{\ell}).$$

It is well known, see [4], that the first eigenvalue  $\lambda_{\ell}^1$  for (2.9) is associated with a variational characterization,

$$(2.10) \quad \lambda_{\ell}^1 = \inf \left\{ \int_{\Omega_{\ell}} A \nabla u \cdot \nabla u : u \in V(\Omega_{\ell}), \int_{\Omega_{\ell}} u^2 = 1 \right\} = \inf_{u \in V(\Omega_{\ell}) \setminus \{0\}} \frac{\int_{\Omega_{\ell}} A(X_2) \nabla u \cdot \nabla u}{\int_{\Omega_{\ell}} u^2}.$$

It is also true, and can be proved in the same way as it is done for the corresponding Dirichlet problem, that  $\lambda_{\ell}^1$  is simple and the corresponding eigenfunction  $u_{\ell}$  has constant sign in  $\Omega_{\ell}$ , that we should fix as the positive sign in the sequel. For some of our results we shall need to impose a certain symmetry condition on  $\omega$  and  $A$ .

**Definition 2.1.** *We shall say that property (S) holds if  $\omega$  is symmetric w.r.t. the origin (i.e.,  $-\omega = \omega$ ) and  $A(-X_2) = A(X_2)$ .*

From the uniqueness of  $u_{\ell}$  we deduce easily the following symmetry result.

**Proposition 2.1.** *If property (S) holds then  $u_{\ell}(x_1, X_2) = u_{\ell}(-x_1, -X_2)$ .*

*Proof.* Clearly  $v_{\ell}(x_1, X_2) := u_{\ell}(-x_1, -X_2)$  is a positive normalized eigenfunction for  $\lambda_{\ell}^1$ , so it must be equal to  $u_{\ell}$ .  $\square$

### 3. THE GAP PHENOMENON IN A MODEL PROBLEM

In this section we treat a two dimensional model problem in order to illustrate the main ideas behind the analysis of the general case in the next sections. Throughout this section  $\omega = (-1, 1)$ ,  $\Omega_{\ell} = (-\ell, \ell) \times (-1, 1)$ , and the matrix  $A$  is a constant matrix depending on the parameter  $\delta \in [0, 1)$ , namely,

$$(3.1) \quad A = A_{\delta} = \begin{pmatrix} 1 & \delta \\ \delta & 1 \end{pmatrix}.$$

Clearly  $A_\delta$  satisfies all the assumptions made on  $A$  in Section 2. Since the eigenvalues of  $A_\delta$  are  $1 \pm \delta$ ,  $\lambda_A = 1 - \delta$  (see (2.6)). In this section we shall denote a point in  $\mathbb{R}^2$  by  $x = (x_1, x_2)$ . The problem (2.7) has the following simple form

$$\begin{cases} -W_1'' = \mu^1 W_1 \text{ in } (-1, 1), \\ W_1(-1) = W_1(1) = 0. \end{cases}$$

where  $\mu^1$  denotes the first eigenvalue and  $W_1$  is the corresponding positive normalized eigenfunction. Therefore,  $\mu^1 = (\frac{\pi}{2})^2$  and  $W_1(t) = \cos(\frac{\pi}{2}t)$ .

**Proposition 3.1.** *For  $\delta = 0$  we have  $\lambda_\ell^1 = \mu^1$  for all  $\ell > 0$ . For  $\delta \in (0, 1)$  we have*

$$(3.2) \quad (1 - \delta^2)\mu^1 < \lambda_\ell^1 < \mu^1, \forall \ell > 0.$$

*Proof.* (i) Since  $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , the corresponding operator is just  $-\Delta$ , and the function  $v(x_1, x_2) = W_1(x_2)$  is clearly a positive eigenfunction in (1.5) with  $\sigma = \mu^1$ , for all  $\ell > 0$ . It follows that  $\lambda_\ell^1 = \mu^1$  as claimed.

(ii) Assume now that  $\delta \in (0, 1)$ . Using the function  $v(x_1, x_2) = W_1(x_2)$  in the Rayleigh quotient (2.10) yields the inequality

$$(3.3) \quad \lambda_\ell^1 \leq \mu^1.$$

We claim that the inequality in (3.3) is strict as stated in (3.2). Indeed, an equality would imply that the function  $v$  (as defined above) is a positive eigenfunction in (1.5) for  $\sigma = \lambda_\ell^1 = \mu^1$ , and in particular, it satisfies the Neumann boundary condition

$$0 = (A_\delta \nabla v) \cdot \nu = v_{x_1} + \delta v_{x_2} = \delta v_{x_2} \quad \text{on } \Gamma_\ell^+ = \{\ell\} \times (-1, 1).$$

But this clearly contradicts the fact that  $(W_1)'(x_2) \neq 0$  for  $x_2 \in (-1, 1) \setminus \{0\}$ . To prove the inequality of the left in (3.2) we first notice the elementary inequality

$$(3.4) \quad (A_\delta \xi) \cdot \xi \geq (1 - \delta^2)|\xi_2|^2, \quad \forall \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.$$

Indeed, (3.4) follows from the identity

$$(3.5) \quad (A_\delta \xi) \cdot \xi = \xi_1^2 + 2\delta\xi_1\xi_2 + \xi_2^2 = (1 - \delta^2)\xi_2^2 + (\xi_1 + \delta\xi_2)^2.$$

By (3.4) and (2.8) we get

$$(3.6) \quad \begin{aligned} \lambda_\ell^1 &= \int_{\Omega_\ell} (A_\delta \nabla u_\ell) \cdot \nabla u_\ell \geq (1 - \delta^2) \int_{\Omega_\ell} |\partial_{x_2} u_\ell|^2 \\ &\geq (1 - \delta^2)\mu^1 \int_{\Omega_\ell} |u_\ell|^2 = (1 - \delta^2)\mu^1. \end{aligned}$$

To conclude, we show that the inequality  $\lambda_\ell^1 \geq (1 - \delta^2)\mu^1$  is strict. Indeed, equality would imply equalities in all the inequalities in (3.6), implying in particular that  $u_\ell(x_1, x_2) = W_1(x_2)$  in  $\Omega_\ell$ . It would then follow that  $\lambda_\ell^1 = \mu^1$ . Contradiction.  $\square$

From now on we shall assume that  $\delta \in (0, 1)$  (the first part of Proposition 3.1 settles completely the case  $\delta = 0$ ). Our main result in this section establishes the following estimate about the behavior of  $\lambda_\ell^1$  as  $\ell$  goes to infinity.

**Theorem 3.1.**  $\limsup_{\ell \rightarrow \infty} \lambda_\ell^1 < \mu^1$ , for every  $\delta \in (0, 1)$ .

In the next section, when dealing with the general case, we shall actually see that the limit  $\lim_{\ell \rightarrow \infty} \lambda_\ell^1$  exists. As mentioned in the Introduction, an important ingredient in the proof of Theorem 3.1 is a study of the asymptotic behavior of  $\lambda_\ell^1$  as  $\ell \rightarrow 0$  (a dimension reduction problem).

**Theorem 3.2.** *We have  $\lim_{\ell \rightarrow 0} \lambda_\ell^1 = (1 - \delta^2)\mu^1$ .*

*Proof.* It suffices to consider  $\ell < 1$ . Fix any  $\alpha \in (0, 1)$  and let  $\rho_\ell$  be the piecewise-linear function defined by

$$\rho_\ell(t) = \begin{cases} \frac{t+1}{\ell^\alpha} & t \in [-1, -1 + \ell^\alpha), \\ 1 & t \in [-1 + \ell^\alpha, 1 - \ell^\alpha], \\ \frac{1-t}{\ell^\alpha} & t \in (1 - \ell^\alpha, 1]. \end{cases}$$

Consider the following test function

$$(3.7) \quad v_\ell(x_1, x_2) = W_1(x_2) - \delta x_1 W_1'(x_2) \rho_\ell(x_2).$$

Then clearly  $v_\ell \in V(\Omega_\ell)$  is a valid test function. From (2.10), we have

$$(3.8) \quad \lambda_\ell^1 \leq \frac{\int_{\Omega_\ell} A_\delta \nabla v_\ell \cdot \nabla v_\ell}{\int_{\Omega_\ell} v_\ell^2} = \frac{\int_{\Omega_\ell} |\partial_{x_1} v_\ell|^2 + \int_{\Omega_\ell} |\partial_{x_2} v_\ell|^2 + 2\delta \int_{\Omega_\ell} \partial_{x_1} v_\ell \partial_{x_2} v_\ell}{\int_{\Omega_\ell} v_\ell^2} \\ = \frac{I_1 + I_2 + I_3}{I}.$$

We consider each of the terms  $I_1, I_2, I_3$  and  $I$  separately. First,

$$(3.9) \quad I_1 = \delta^2 \int_{\Omega_\ell} \rho_\ell^2 |W_1'(x_2)|^2 dx = 2\ell\delta^2 \int_{-1}^1 \rho_\ell^2 |W_1'(x_2)|^2 dx_2.$$

Next, calculating for  $I_2$ ,

$$I_2 = \int_{\Omega_\ell} \left[ W_1'(x_2) - \delta x_1 \{ \rho_\ell W_1''(x_2) + W_1'(x_2) \rho_\ell'(x_2) \} \right]^2 \\ = \int_{\Omega_\ell} |W_1'|^2 - 2\delta \int_{\Omega_\ell} x_1 W_1'(x_2) \{ \rho_\ell W_1''(x_2) + W_1'(x_2) \rho_\ell'(x_2) \} \\ + \delta^2 \int_{\Omega_\ell} x_1^2 | \rho_\ell W_1''(x_2) + W_1'(x_2) \rho_\ell'(x_2) |^2.$$

The integral in the middle vanishes since  $\int_{-\ell}^\ell x_1 = 0$ . Hence, using  $|\rho_\ell'| \leq \frac{1}{\ell^\alpha}$  and (2.8) we get

$$(3.10) \quad I_2 = 2\ell\mu^1 + \frac{2\delta^2\ell^3}{3} \int_{-1}^1 | \rho_\ell W_1'' + W_1' \rho_\ell' |^2 \leq 2\ell\mu^1 + \frac{2\delta^2\ell^3}{3} (C_1 + C_2\ell^{-2\alpha}),$$

where  $C_1, C_2$  are two constants independent of  $\ell$ . Next, for  $I_3$  we find,

$$(3.11) \quad I_3 = 2\delta \int_{\Omega_\ell} -\delta W_1' \rho_\ell [W_1' - x_1 \delta \{ W_1' \rho_\ell' + \rho_\ell W_1'' \}] \\ = -4\ell\delta^2 \int_{-1}^1 \rho_\ell |W_1'|^2 + 2\delta^3 \int_{\Omega_\ell} x_1 W_1' \rho_\ell \{ W_1' \rho_\ell' + \rho_\ell W_1'' \} = -4\ell\delta^2 \int_{-1}^1 \rho_\ell |W_1'|^2.$$

Finally we compute the term  $I$ .

$$(3.12) \quad I = \int_{\Omega_\ell} (W_1 - \delta x_1 W_1' \rho_\ell)^2 = \int_{\Omega_\ell} W_1^2 + \delta^2 \int_{\Omega_\ell} x_1^2 \rho_\ell^2 |W_1'|^2$$

$$= 2\ell + \frac{2\ell^3\delta^2}{3} \int_{-1}^1 \rho_\ell^2 |W_1'|^2 \geq 2\ell.$$

Plugging (3.9)–(3.12) in (3.8) yields

$$(3.13) \quad \lambda_\ell^1 \leq \delta^2 \int_{-1}^1 \rho_\ell^2 |W_1'|^2 + \mu^1 - 2\delta^2 \int_{-1}^1 \rho_\ell |W_1'|^2 + \varepsilon(\ell),$$

where  $\varepsilon(\ell) \rightarrow 0$  as  $\ell \rightarrow 0$ . Since  $\rho_\ell \rightarrow 1$  pointwise, passing to the limit  $\ell \rightarrow 0$  and using dominated convergence for the RHS of (3.13) gives

$$(3.14) \quad \limsup_{\ell \rightarrow 0} \lambda_\ell^1 \leq (1 - \delta^2)\mu^1.$$

Combining (3.14) with (3.2) we obtain the result of the theorem.  $\square$

Now we turn to the proof of Theorem 3.1.

*Proof of Theorem 3.1.* Let  $\ell_0$  and  $\eta$  be two positive constants whose values will be determined later. For  $\ell > \ell_0 + \eta$  define  $\phi_\ell$  by

$$\phi_\ell = \begin{cases} v_{\ell_0}(x_1 - \ell + \ell_0, x_2) & \text{on } (\ell - \ell_0, \ell) \times (-1, 1), \\ \frac{(x_1 - (\ell - \ell_0 - \eta))W_1(x_2)}{\eta} & \text{on } (\ell - \ell_0 - \eta, \ell - \ell_0) \times (-1, 1), \\ 0 & \text{on } \Omega_{\ell - \ell_0 - \eta}, \\ \frac{(-x_1 - (\ell - \ell_0 - \eta))W_1(x_2)}{\eta} & \text{on } (\ell_0 - \ell, -\ell + \ell_0 + \eta) \times (-1, 1), \\ v_{\ell_0}(x_1 + \ell - \ell_0, x_2) & \text{on } (-\ell, \ell_0 - \ell) \times (-1, 1), \end{cases}$$

where  $v_{\ell_0}$  is given by (3.7). We have

$$(3.15) \quad \begin{aligned} \int_{\Omega_\ell} \phi_\ell^2 &= \int_{\Omega_\ell \setminus \Omega_{\ell - \ell_0}} \phi_\ell^2 + \int_{\Omega_{\ell - \ell_0}} \phi_\ell^2 \\ &= \int_{\Omega_{\ell_0}} v_{\ell_0}^2 + 2 \left( \int_{\ell - \ell_0 - \eta}^{\ell - \ell_0} \frac{(x_1 - \ell + \ell_0 + \eta)^2}{\eta^2} dx_1 \right) \left( \int_{-1}^1 W_1^2 \right) \\ &= \int_{\Omega_{\ell_0}} v_{\ell_0}^2 + \frac{2}{3}\eta, \end{aligned}$$

where we used the fact that  $\phi_\ell$  is an even function in  $x_1$  on  $\Omega_\ell \setminus \Omega_{\ell - \ell_0}$ . Also,

$$(3.16) \quad \int_{\Omega_\ell} A_\delta \nabla \phi_\ell \cdot \nabla \phi_\ell = \int_{\Omega_{\ell_0}} A_\delta \nabla v_{\ell_0} \cdot \nabla v_{\ell_0} + \int_{\Omega_{\ell - \ell_0}} A_\delta \nabla \phi_\ell \cdot \nabla \phi_\ell.$$

Setting  $\mathcal{D} = \Omega_{\ell - \ell_0} \setminus \Omega_{\ell - \ell_0 - \eta}$  and using the fact that  $\phi_\ell$  is even in  $\mathcal{D}$  while  $\partial_{x_1} \phi_\ell$  is odd on  $\mathcal{D}$  we get

$$(3.17) \quad \begin{aligned} \int_{\Omega_{\ell - \ell_0}} A_\delta \nabla \phi_\ell \cdot \nabla \phi_\ell &= \frac{1}{\eta^2} \int_{\mathcal{D}} W_1^2 + 2\delta \int_{\mathcal{D}} \partial_{x_1} \phi_\ell \partial_{x_2} \phi_\ell \\ &\quad + \frac{2}{\eta^2} \int_{(\ell - \ell_0 - \eta, \ell - \ell_0) \times (-1, 1)} |W_1'|^2 (x_1 - \ell + \ell_0 + \eta)^2 \\ &= \frac{2}{\eta} \int_{-1}^1 W_1^2 + \frac{2\eta}{3} \int_{-1}^1 |W_1'|^2 = \frac{2}{\eta} + \frac{2\eta\mu^1}{3}. \end{aligned}$$

From (3.15)–(3.17) we obtain

$$(3.18) \quad \lambda_\ell^1 \leq \frac{\int_{\Omega_{\ell_0}} A_\delta \nabla v_{\ell_0} \cdot \nabla v_{\ell_0} + \frac{2}{\eta} + \frac{2\eta\mu^1}{3}}{\int_{\Omega_{\ell_0}} v_{\ell_0}^2 + \frac{2}{3}\eta}.$$

Noting that Theorem 3.2 implies that

$$\frac{\int_{\Omega_{\ell_0}} A_\delta \nabla v_{\ell_0} \cdot \nabla v_{\ell_0}}{\int_{\Omega_{\ell_0}} v_{\ell_0}^2} = (1 - \delta^2)\mu^1 + \varepsilon(\ell_0),$$

we obtain from (3.18) that

$$(3.19) \quad \begin{aligned} \lambda_\ell^1 - \mu^1 &\leq \frac{\{(1 - \delta^2)\mu^1 + \varepsilon(\ell_0)\} \int_{\Omega_{\ell_0}} v_{\ell_0}^2 + \frac{2}{\eta} + \frac{2\eta\mu^1}{3}}{\int_{\Omega_{\ell_0}} v_{\ell_0}^2 + \frac{2}{3}\eta} - \mu^1 \\ &= \frac{(\varepsilon(\ell_0) - \delta^2\mu^1) \int_{\Omega_{\ell_0}} v_{\ell_0}^2 + \frac{2}{\eta}}{\int_{\Omega_{\ell_0}} v_{\ell_0}^2 + \frac{2}{3}\eta}. \end{aligned}$$

Choosing  $\ell_0$  small enough such that  $\varepsilon(\ell_0) - \delta^2\mu^1 < 0$ , and then taking  $\eta$  sufficiently large, makes the RHS of (3.19) equal a negative number, say  $-\delta_0$ . Hence,  $\lambda_\ell^1 \leq \mu^1 - \delta_0$  for  $\ell > \ell_0 + \eta$ , and the result follows.  $\square$

#### 4. THE GAP PHENOMENON IN THE GENERAL CASE.

In this section we extend the results from Section 3 to a more general framework. We shall use the notation from Section 2 and study the limit  $\lim_{\ell \rightarrow \infty} \lambda_\ell^1$  for  $\lambda_\ell^1$  given by (2.10). As in Section 3 our strategy is to study first the limit as  $\ell$  goes to 0.

**Theorem 4.1.** *We have  $\lim_{\ell \rightarrow 0} \lambda_\ell^1 = \Lambda^1$  where*

$$(4.1) \quad \Lambda^1 = \inf \left\{ \int_{\omega} A_{22}(X_2) \nabla u \cdot \nabla u - \frac{|A_{12}(X_2) \cdot \nabla u|^2}{a_{11}(X_2)} : u \in H_0^1(\omega), \int_{\omega} u^2 = 1 \right\}.$$

*Proof.* The reason why we find  $\Lambda^1$  as the limiting value will be clarified by the following simple observation. Let  $B = \begin{pmatrix} b_{11} & B_{12} \\ B_{12}^t & B_{22} \end{pmatrix}$  be a positive definite  $n \times n$  matrix and represent any vector  $\mathbf{z}$  in  $\mathbb{R}^n$  as  $\mathbf{z} = (z_1, Z_2)$  with  $Z_2 \in \mathbb{R}^{n-1}$ . Then, elementary calculus shows that for any fixed  $Z_2 \in \mathbb{R}^{n-1}$  we have

$$(4.2) \quad \min_{z_1 \in \mathbb{R}} (B\mathbf{z}) \cdot \mathbf{z} = (B_{22}Z_2) \cdot Z_2 - \frac{|B_{12}Z_2|^2}{b_{11}}.$$

Furthermore, the minimum in (4.2) is attained for

$$(4.3) \quad z_1 = -\frac{B_{12}Z_2}{b_{11}}.$$

Applying (4.2) with  $B = A(X_2)$  we obtain, for any  $\ell > 0$ ,

$$(4.4) \quad \begin{aligned} \int_{\Omega_\ell} (A(X_2) \nabla u_\ell) \cdot \nabla u_\ell &\geq \int_{\Omega_\ell} (A_{22}(X_2) \nabla_{X_2} u_\ell) \cdot \nabla_{X_2} u_\ell - \frac{|A_{12}(X_2) \nabla_{X_2} u_\ell|^2}{a_{11}(X_2)} \\ &\geq \Lambda^1 \int_{\Omega_\ell} u_\ell^2. \end{aligned}$$

By (4.4) the lower-bound

$$(4.5) \quad \liminf_{\ell \rightarrow 0} \lambda_\ell^1 \geq \Lambda^1,$$



is clear. We note that from the above it follows in particular that

$$\Lambda^1 \geq \lambda_A \cdot \inf \left\{ \int_{\omega} |\nabla u|^2 : u \in H_0^1(\omega), \int_{\omega} u^2 = 1 \right\}.$$

(see (2.6)) and the infimum in (4.1) is actually a minimum, which is realized by a positive function  $w_1 \in H_0^1(\omega)$ .

In order to complete the proof of Theorem 4.1 we need to establish the upper-bound part. A natural generalization of the construction used in the proof of Theorem 3.2 would be to use

$$(4.6) \quad v_{\ell}(x) = w_1(X_2) - \frac{(A_{12}(X_2) \cdot \nabla w_1) x_1 \rho_{\ell}(X_2)}{a_{11}(X_2)},$$

where  $\rho_{\ell}$  is an appropriate cut-off function. However, since the coefficients of the matrix  $A(X_2)$  are only assumed to be  $L^{\infty}$ -functions, the function on the RHS of (4.6) does not necessarily belong to  $H^1$ . To overcome this difficulty, we use an approximation argument, motivated by [2, Ch. 14]. We apply standard mollification to define a family of functions  $\{G_{\varepsilon}\}_{\varepsilon>0} \subset C_c^{\infty}(\omega)$  satisfying

$$(4.7) \quad \lim_{\varepsilon \rightarrow 0} G_{\varepsilon}(X_2) = \frac{A_{12}(X_2) \cdot \nabla w_1}{a_{11}(X_2)} \quad \text{in } L^2(\omega) \text{ and a.e..}$$

We then define

$$(4.8) \quad v_{\ell}^{\varepsilon}(x_1, X_2) = w_1(X_2) - G_{\varepsilon}(X_2)x_1.$$

First notice that

$$(4.9) \quad \int_{\Omega_{\ell}} |v_{\ell}^{\varepsilon}|^2 = \int_{-\ell}^{\ell} \int_{\omega} w_1^2 - 2x_1 w_1 G_{\varepsilon} + (x_1 G_{\varepsilon})^2 \geq 2\ell \int_{\omega} w_1^2 = 2\ell,$$

since  $\int_{-\ell}^{\ell} x_1 dx_1 = 0$ . Now

$$\begin{aligned} \int_{\Omega_{\ell}} A \nabla v_{\ell}^{\varepsilon} \cdot \nabla v_{\ell}^{\varepsilon} &= \int_{\Omega_{\ell}} a_{11} (\partial_{x_1} v_{\ell}^{\varepsilon})^2 + 2(A_{12} \cdot \nabla_{X_2} v_{\ell}^{\varepsilon}) \partial_{x_1} v_{\ell}^{\varepsilon} + (A_{22} \nabla_{X_2} v_{\ell}^{\varepsilon}) \cdot \nabla_{X_2} v_{\ell}^{\varepsilon} \\ &= I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon). \end{aligned}$$

For the first integral we have

$$(4.10) \quad I_1(\varepsilon) = \int_{\Omega_{\ell}} a_{11} G_{\varepsilon}^2 = 2\ell \int_{\omega} a_{11} G_{\varepsilon}^2.$$

For the second integral,

$$(4.11) \quad I_2(\varepsilon) = 2 \int_{-\ell}^{\ell} \int_{\omega} A_{12} \cdot \left\{ \nabla w_1 - x_1 \nabla G_{\varepsilon}(X_2) \right\} \left\{ -G_{\varepsilon}(X_2) \right\}.$$

Since the integral of the term containing  $x_1$  vanishes, we get

$$(4.12) \quad I_2(\varepsilon) = -4\ell \int_{\omega} (A_{12} \cdot \nabla w_1) G_{\varepsilon}.$$

For the last integral we have (after dropping the term with the vanishing integral),

$$(4.13) \quad \begin{aligned} I_3(\varepsilon) &= \int_{-\ell}^{\ell} \int_{\omega} (A_{22} \nabla w_1) \cdot \nabla w_1 + x_1^2 (A_{22} \nabla G_{\varepsilon}) \cdot \nabla G_{\varepsilon} \\ &= 2\ell \left\{ \int_{\omega} (A_{22} \nabla w_1) \cdot \nabla w_1 + \frac{\ell^2}{3} \int_{\omega} (A_{22} \nabla G_{\varepsilon}) \cdot \nabla G_{\varepsilon} \right\}. \end{aligned}$$

By (4.9)–(4.13) we deduce that

$$(4.14) \quad \limsup_{\ell \rightarrow 0} \lambda_\ell^1 \leq \limsup_{\ell \rightarrow 0} \frac{\int_{\Omega_\ell} A \nabla v_\ell^\varepsilon \cdot \nabla v_\ell^\varepsilon}{\int_{\Omega_\ell} |v_\ell^\varepsilon|^2} \leq \int_\omega a_{11} (G_\varepsilon)^2 - 2 \int_\omega (A_{12} \cdot \nabla w_1) G_\varepsilon + \int_\omega (A_{22} \nabla w_1) \cdot \nabla w_1.$$

Passing to the limit  $\varepsilon \rightarrow 0$  in (4.14), using (4.7), gives

$$\limsup_{\ell \rightarrow 0} \lambda_\ell^1 \leq \int_\omega (A_{22} \nabla w_1) \cdot \nabla w_1 - \frac{|A_{12} \nabla w_1|^2}{a_{11}} = \Lambda^1,$$

which together with (4.5) yields the result.  $\square$

**Remark 4.1.** Replacing (4.8) by

$$(4.15) \quad \tilde{v}_\ell^\varepsilon(x_1, X_2) = W_1(X_2) - \tilde{G}_\varepsilon(X_2)x_1,$$

where  $\tilde{G}_\varepsilon$  is defined as in (4.7), but with  $w_1$  replacing  $W_1$ , and carrying out the same computation as in the last part of the proof of Theorem 4.1 yields

$$(4.16) \quad \inf_{\varepsilon > 0} \lim_{\ell \rightarrow 0} \frac{\int_{\Omega_\ell} (A \nabla \tilde{v}_\ell^\varepsilon) \cdot \nabla \tilde{v}_\ell^\varepsilon}{\int_{\Omega_\ell} |\tilde{v}_\ell^\varepsilon|^2} = \int_\omega (A_{22} \nabla W_1) \cdot \nabla W_1 - \frac{|A_{12} \nabla W_1|^2}{a_{11}}.$$

Our next theorem provides an analog of Theorem 3.1 to the general case.

**Theorem 4.2.** We have

$$(4.17) \quad \limsup_{\ell \rightarrow \infty} \lambda_\ell^1 < \mu^1,$$

provided the following condition holds,

$$(4.18) \quad A_{12} \cdot \nabla W_1 \neq 0 \text{ a.e. on } \omega.$$

In case (4.18) does not hold we have  $\lambda_\ell^1 = \mu^1$  for all  $\ell > 0$ .

**Remark 4.2.** It is easy to construct examples where condition (4.18) doesn't hold.

Take for example for  $\omega$  the unit disc in  $\mathbb{R}^2$ . For  $A_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , the eigenfunction  $W_1$  is radially symmetric. We use polar coordinates on  $\omega$  and represent each  $X_2$  as  $X_2 = r(\cos \theta, \sin \theta)$ . Taking  $a_{11} = 1$  and  $A_{12}(X_2) = t(-\sin \theta, \cos \theta)$  for  $|t|$  small enough (in order for the uniform ellipticity condition (2.6) to hold for the 3 by 3 matrix  $A$ ) yields an example for which (4.18) doesn't hold.

*Proof.* (i) Assume first that (4.18) holds. Then,

$$(4.19) \quad \Lambda_1 < \mu^1.$$

Indeed, this follows from

$$\Lambda_1 \leq \int_\omega A_{22}(X_2) \nabla W_1 \cdot \nabla W_1 - \frac{|A_{12}(X_2) \nabla W_1|^2}{a_{11}(X_2)} < \int_\omega A_{22}(X_2) \nabla W_1 \cdot \nabla W_1 = \mu^1.$$

By the proof of Theorem 4.1 there exist positive values of  $\ell_0$  and  $\varepsilon_0$  such that  $\tilde{v}_{\ell_0}^{\varepsilon_0}$  defined by (4.15) satisfies

$$(4.20) \quad \int_{\Omega_{\ell_0}} A \nabla \tilde{v}_{\ell_0}^{\varepsilon_0} \cdot \nabla \tilde{v}_{\ell_0}^{\varepsilon_0} < \mu^1 \int_{\Omega_{\ell_0}} |\tilde{v}_{\ell_0}^{\varepsilon_0}|^2.$$

Notice that  $\tilde{v}_{\ell_0}^{\varepsilon_0}(0, X_2) = W_1(X_2)$ . Let  $\eta > 0$  be a parameter whose value will be determined later. For  $\ell > \ell_0 + \eta$  define  $\phi_\ell$  as follows,

$$(4.21) \quad \phi_\ell = \begin{cases} \tilde{v}_{\ell_0}^{\varepsilon_0}(x_1 - \ell + \ell_0, X_2) & \text{on } (\ell - \ell_0, \ell) \times \omega, \\ \frac{(x_1 - (\ell - \ell_0 - \eta))W_1(X_2)}{\eta} & \text{on } (\ell - \ell_0 - \eta, \ell - \ell_0) \times \omega, \\ 0 & \text{on } \Omega_{\ell - \ell_0 - \eta}, \\ \frac{(-x_1 - (\ell - \ell_0 - \eta))W_1(X_2)}{\eta} & \text{on } (\ell_0 - \ell, -(\ell - \ell_0 - \eta)) \times \omega, \\ \tilde{v}_{\ell_0}^{\varepsilon_0}(x_1 + \ell - \ell_0, X_2) & \text{on } (-\ell, \ell_0 - \ell) \times \omega. \end{cases}$$

Since

$$\int_{\Omega_\ell \setminus \Omega_{\ell - \ell_0}} \phi_\ell^2 = \int_{\Omega_{\ell_0}} |\tilde{v}_{\ell_0}^{\varepsilon_0}|^2,$$

and

$$\int_{\Omega_{\ell - \ell_0}} \phi_\ell^2 = 2 \left( \int_{\ell - \ell_0 - \eta}^{\ell - \ell_0} \frac{(x_1 - \ell + \ell_0 + \eta)^2}{\eta^2} dx_1 \right) \left( \int_\omega W_1^2 dX_2 \right) = \frac{2}{3}\eta,$$

we have

$$(4.22) \quad \int_{\Omega_\ell} \phi_\ell^2 = \int_{\Omega_{\ell_0}} |\tilde{v}_{\ell_0}^{\varepsilon_0}|^2 + \frac{2}{3}\eta.$$

Similarly

$$(4.23) \quad \int_{\Omega_\ell} A \nabla \phi_\ell \cdot \nabla \phi_\ell = \int_{\Omega_{\ell_0}} A \nabla \tilde{v}_{\ell_0}^{\varepsilon_0} \cdot \nabla \tilde{v}_{\ell_0}^{\varepsilon_0} + \int_{\Omega_{\ell - \ell_0}} A \nabla \phi_\ell \cdot \nabla \phi_\ell.$$

Setting  $D = \Omega_{\ell - \ell_0} \setminus \Omega_{\ell - \ell_0 - \eta}$  and  $D^+ = (\ell - \ell_0 - \eta, \ell - \ell_0) \times \omega$ , the last integral above can be written as

$$\begin{aligned} \int_{\Omega_{\ell - \ell_0}} A \nabla \phi_\ell \cdot \nabla \phi_\ell &= \frac{1}{\eta^2} \int_D a_{11} W_1^2 + 2 \int_D (A_{12} \cdot \nabla_{X_2} \phi_\ell) \partial_{x_1} \phi_\ell \\ &\quad + \frac{2}{\eta^2} \int_{D^+} (x_1 - \ell + \ell_0 + \eta)^2 A_{22} \nabla W_1 \cdot \nabla W_1. \end{aligned}$$

The second integral vanishes since its integrand is an odd function of  $x_1$  on  $D$ . Therefore,

$$(4.24) \quad \int_{\Omega_{\ell - \ell_0}} A \nabla \phi_\ell \cdot \nabla \phi_\ell = \frac{2}{\eta} \int_\omega a_{11} W_1^2 + \frac{2\eta}{3} \int_\omega A_{22} \nabla W_1 \cdot \nabla W_1 = \frac{2}{\eta} \int_\omega a_{11} W_1^2 + \frac{2\eta\mu^1}{3}.$$

Combining (4.22), (4.23) and (4.24) we obtain

$$\lambda_\ell^1 \leq \frac{\int_{\Omega_\ell} A \nabla \phi_\ell \cdot \nabla \phi_\ell}{\int_{\Omega_\ell} \phi_\ell^2} \leq \frac{\int_{\Omega_{\ell_0}} A \nabla \tilde{v}_{\ell_0}^{\varepsilon_0} \cdot \nabla \tilde{v}_{\ell_0}^{\varepsilon_0} + \frac{2}{\eta} \int_\omega a_{11} W_1^2 + \frac{2}{3}\eta\mu^1}{\int_{\Omega_{\ell_0}} |\tilde{v}_{\ell_0}^{\varepsilon_0}|^2 + \frac{2}{3}\eta}.$$

Therefore,

$$(4.25) \quad \lambda_\ell^1 - \mu^1 \leq \frac{\int_{\Omega_{\ell_0}} A \nabla \tilde{v}_{\ell_0}^{\varepsilon_0} \cdot \nabla \tilde{v}_{\ell_0}^{\varepsilon_0} - \mu^1 \int_{\Omega_{\ell_0}} |\tilde{v}_{\ell_0}^{\varepsilon_0}|^2 + \frac{2}{\eta} \int_\omega a_{11} W_1^2}{\int_{\Omega_{\ell_0}} |\tilde{v}_{\ell_0}^{\varepsilon_0}|^2 + \frac{2}{3}\eta}.$$

By (4.20) it is clear that we can fix a large enough value for  $\eta$  such that the RHS of (4.25) is negative, and the result for case (i) follows.

(ii) By (4.4) we have  $\Lambda_1 \leq \lambda_\ell^1$  for all  $\ell > 0$ . On the other hand, using  $u(x) = W_1(X_2)$  as a test function in (2.10) gives  $\lambda_\ell^1 \leq \mu^1$ . Thus we have,

$$(4.26) \quad \Lambda_1 \leq \lambda_\ell^1 \leq \mu^1, \quad \forall \ell > 0.$$

In view of (4.26), the result for the case where (4.18) doesn't hold would follow once we show that in this case  $\Lambda_1 = \mu^1$ . The Euler-Lagrange equation for an eigenfunction  $v$  of the quadratic form in (4.1), with eigenvalue  $\lambda$  is

$$(4.27) \quad \begin{cases} -\operatorname{div}(A_{22}\nabla v) + \operatorname{div}((A_{12}\cdot\nabla v)A_{12}^t/a_{11}) = \lambda v & \text{in } \omega, \\ v = 0 & \text{on } \partial\omega. \end{cases}$$

Of course  $v = w_1$  satisfies (4.27) with  $\lambda = \Lambda_1$ . But since we assume that (4.18) doesn't hold,  $v = W_1$  is also a solution of (4.27) with  $\lambda = \mu^1$ . However, only the first eigenvalue of the problem (4.27) can have a positive eigenfunction, so we must have  $\Lambda_1 = \mu^1$  as claimed.  $\square$

## 5. CHARACTERIZATION OF THE LIMIT $\lim_{\ell \rightarrow \infty} \lambda_\ell^1$

In this section we obtain more precise results on the asymptotic behavior of the eigenfunctions  $\{u_\ell\}$  and the eigenvalues  $\{\lambda_\ell^1\}$  as  $\ell$  goes to infinity. We shall see that when (4.18) holds, the eigenfunctions decay to zero in the bulk of the cylinder and concentration occurs near the bases of the cylinder. We denote by  $[x]$  the integer part of  $x$ .

**Theorem 5.1.** *Assume (4.18) holds. Then, there exist  $\alpha \in (0, 1)$  and a positive constant  $c$  such that for  $\ell > \ell_0$  we have, for every  $0 < r \leq \ell - 1$ ,*

$$(5.1) \quad \int_{\Omega_r} u_\ell^2 \leq \alpha^{[\ell-r]},$$

and

$$(5.2) \quad \int_{\Omega_r} |\nabla u_\ell|^2 \leq c\alpha^{[\ell-r]}.$$

*Proof.* Let  $\ell$  and  $\ell'$  satisfy  $0 < \ell' \leq \ell - 1$ . Define  $\rho_{\ell'} = \rho_{\ell'}(x_1)$  by

$$(5.3) \quad \rho_{\ell'}(x_1) = \begin{cases} 1 & |x_1| \leq \ell', \\ \ell' + 1 - |x_1| & |x_1| \in (\ell', \ell' + 1), \\ 0 & |x_1| \geq \ell' + 1. \end{cases}$$

Using  $v = \rho_{\ell'}^2 u_\ell \in V(\Omega_\ell)$  in (2.9), we get

$$\int_{\Omega_\ell} (A\nabla u_\ell) \cdot \nabla(\rho_{\ell'}^2 u_\ell) = \lambda_\ell^1 \int_{\Omega_\ell} \rho_{\ell'}^2 u_\ell^2,$$

i.e.,

$$(5.4) \quad \int_{\Omega_\ell} (A\nabla(\rho_{\ell'} u_\ell)) \cdot \nabla(\rho_{\ell'} u_\ell) - \int_{\Omega_\ell} u_\ell^2 (A\nabla \rho_{\ell'}) \cdot \nabla \rho_{\ell'} = \lambda_\ell^1 \int_{\Omega_\ell} \rho_{\ell'}^2 u_\ell^2.$$

Since  $\rho_{\ell'} u_\ell \in H_0^1(\Omega_\ell)$ , by the Rayleigh quotient characterization of  $\sigma_\ell^1$  (see (1.2)) we have

$$(5.5) \quad \sigma_\ell^1 \int_{\Omega_\ell} u_\ell^2 \rho_{\ell'}^2 \leq \int_{\Omega_\ell} A\nabla(\rho_{\ell'} u_\ell) \cdot \nabla(\rho_{\ell'} u_\ell).$$

Combining (5.4)–(5.5) with (2.5) we get

$$(5.6) \quad \begin{aligned} (\sigma_\ell^1 - \lambda_\ell^1) \int_{\Omega_\ell} u_\ell^2 \rho_{\ell'}^2 &\leq \int_{\Omega_\ell} u_\ell^2 (A \nabla \rho_{\ell'}) \cdot \nabla \rho_{\ell'} = \int_{\Omega_{\ell'+1} \setminus \Omega_{\ell'}} u_\ell^2 (A \nabla \rho_{\ell'}) \cdot \nabla \rho_{\ell'} \\ &\leq C_A \int_{\Omega_{\ell'+1} \setminus \Omega_{\ell'}} u_\ell^2. \end{aligned}$$

By (1.3) and (4.17) there exists  $\beta > 0$  such that for  $\ell > \ell_0$  we have  $\sigma_\ell^1 - \lambda_\ell^1 \geq \beta$ . Therefore, from (5.6) we deduce that

$$(C_A + \beta) \int_{\Omega_{\ell'}} u_\ell^2 \leq C_A \int_{\Omega_{\ell'+1}} u_\ell^2.$$

This leads to

$$(5.7) \quad \int_{\Omega_{\ell'}} u_\ell^2 \leq \alpha \int_{\Omega_{\ell'+1}} u_\ell^2,$$

with  $\alpha = \frac{C_A}{C_A + \beta} < 1$ . Applying (5.7) successively for  $\ell' = r, r+1, \dots, r + [\ell - r] - 1$  yields

$$(5.8) \quad \int_{\Omega_r} u_\ell^2 \leq \alpha^{[\ell - r]} \int_{\Omega_\ell} u_\ell^2 = \alpha^{[\ell - r]}.$$

To prove (5.2), we fix  $r \in (0, \ell - 2)$  and then use (5.4), with  $\ell' = r$ , combined with (2.6) and (3.3), to obtain

$$(5.9) \quad \begin{aligned} \lambda_A \int_{\Omega_r} |\nabla u_\ell|^2 &\leq \int_{\Omega_\ell} A \nabla(\rho_r u_\ell) \cdot \nabla(\rho_r u_\ell) \\ &= \int_{\Omega_\ell} u_\ell^2 (A \nabla \rho_r) \cdot \nabla \rho_r + \lambda_\ell^1 \int_{\Omega_\ell} \rho_r^2 u_\ell^2 \leq (C_A + \mu^1) \int_{\Omega_{r+1}} u_\ell^2. \end{aligned}$$

Finally, (5.2) follows from (5.8)–(5.9) for  $r \leq \ell - 2$ . Choosing a step size of  $\frac{1}{2}$  in the first part of the proof would allow  $r \leq \ell - 1$ .  $\square$

The decay of the eigenfunction in the bulk immediately implies concentration near the two ends of the cylinder.

**Corollary 5.1.** *If (4.18) holds then for every  $r \in (0, \ell - 1]$  we have*

$$(5.10) \quad \int_{\Omega_\ell \setminus \Omega_r} u_\ell^2 \geq 1 - \alpha^{[\ell - r]} \quad \text{and} \quad \int_{\Omega_\ell \setminus \Omega_r} A \nabla u_\ell \cdot \nabla u_\ell \geq \lambda_\ell^1 - c_1 \alpha^{[\ell - r]}.$$

To have a more precise description of the asymptotic behavior of  $\lambda_\ell^1$  we introduce two variational problems on semi-infinite cylinders. Set

$$\Omega_\infty^+ = (0, \infty) \times \omega \quad \text{and} \quad \Omega_\infty^- = (-\infty, 0) \times \omega,$$

and denote the corresponding lateral parts of the boundary by

$$\gamma_\infty^+ = (0, \infty) \times \partial\omega \quad \text{and} \quad \gamma_\infty^- = (-\infty, 0) \times \partial\omega.$$

Define the spaces

$$V(\Omega_\infty^\pm) := \{u \in H^1(\Omega_\infty^\pm) : u = 0 \text{ on } \gamma_\infty^\pm\},$$

and set

$$(5.11) \quad \nu_\infty^\pm = \inf_{0 \neq u \in V(\Omega_\infty^\pm)} \frac{\int_{\Omega_\infty^\pm} A \nabla u \cdot \nabla u}{\int_{\Omega_\infty^\pm} u^2}.$$

**Remark 5.1.** In case property (S) holds (see Definition 2.1) we clearly have  $\nu_\infty^+ = \nu_\infty^-$  as we can use the transformation  $v(x_1, X_2) \mapsto v(-x_1, -X_2)$  to pass from a function in  $V(\Omega_\infty^+)$  to a function in  $V(\Omega_\infty^-)$  (and vice versa) that has the same Rayleigh quotient. In general we can only assert that  $\nu_\infty^- = \tilde{\nu}_\infty^+$  where  $\tilde{\nu}_\infty^+$  is defined as in (5.11), but with  $A$  being replaced by  $\tilde{A}$ , given by

$$\tilde{A}(X_2) = \begin{pmatrix} a_{11}(X_2) & -A_{12}(X_2) \\ -A_{12}^t(X_2) & A_{22}(X_2) \end{pmatrix}.$$

This is easily seen by applying the transformation  $v(x_1, X_2) \mapsto v(-x_1, X_2)$ .

The next lemma gives the possible range of values for  $\nu_\infty^\pm$ .

**Lemma 5.1.** We have

$$(5.12) \quad 0 < \nu_\infty^\pm \leq \mu^1.$$

*Proof.* By Remark 5.1 it is enough to consider  $\nu_\infty^+$ . The fact that  $\nu_\infty^+ > 0$  follows from the Poincaré inequality. In order to show that  $\nu_\infty^+ \leq \mu^1$  we set for each  $\varepsilon > 0$ ,

$$v_\varepsilon(x) = e^{-\varepsilon x_1} W_1(X_2).$$

Clearly  $v_\varepsilon \in V(\Omega_\infty^+)$  and a direct computation gives

$$(5.13) \quad \begin{aligned} \int_{\Omega_\infty^+} A \nabla v_\varepsilon \cdot \nabla v_\varepsilon &= \int_{\Omega_\infty^+} e^{-2\varepsilon x_1} \left( a_{11} \varepsilon^2 W_1^2 - 2\varepsilon (A_{12} \cdot \nabla W_1) W_1 + A_{22} \nabla W_1 \cdot \nabla W_1 \right) \\ &= \left( \int_0^\infty e^{-2\varepsilon x_1} \right) \left( \mu^1 + \varepsilon^2 \int_\omega a_{11} W_1^2 - 2\varepsilon \int_\omega (A_{12} \cdot \nabla W_1) W_1 \right), \end{aligned}$$

and

$$(5.14) \quad \int_{\Omega_\infty^+} v_\varepsilon^2 = \int_0^\infty e^{-2\varepsilon x_1} \left( = \frac{1}{2\varepsilon} \right).$$

By (5.13)–(5.14) we obtain

$$\frac{\int_{\Omega_\infty^+} A \nabla v_\varepsilon \cdot \nabla v_\varepsilon}{\int_{\Omega_\infty^+} v_\varepsilon^2} = \mu^1 - 2\varepsilon \int_\omega (A_{12} \cdot \nabla W_1) W_1 + \varepsilon^2 \int_\omega a_{11} W_1^2,$$

so by sending  $\varepsilon$  to 0 we deduce that  $\nu_\infty^+ \leq \mu^1$ .  $\square$

It is easy to identify  $\nu_\infty^\pm$  with the limits, as  $\ell \rightarrow \infty$ , of certain minimization problems on  $\Omega_\ell^\pm$ . This is the content of the next lemma (see (2.3) and (2.4) for the definitions of  $\gamma_\ell^\pm$  and  $\Gamma_\ell^\pm$ ).

**Lemma 5.2.** We have  $\nu_\infty^\pm = \lim_{\ell \rightarrow \infty} \tilde{\lambda}_\ell^{1,\pm}$ , where

$$(5.15) \quad \tilde{\lambda}_\ell^{1,\pm} = \inf \left\{ \int_{\Omega_\ell^\pm} A \nabla u \cdot \nabla u : u \in H^1(\Omega_\ell^\pm), \int_{\Omega_\ell^\pm} u^2 = 1, u = 0 \text{ on } \gamma_\ell^\pm \cup \Gamma_\ell^\pm \right\}.$$

**Remark 5.2.** It is a standard fact that the infimum in (5.15) is actually attained. The unique positive normalized minimizers will be denoted by  $\tilde{u}_\ell^\pm$ .

*Proof.* We present the proof for  $\tilde{\lambda}_\ell^{1,+}$  as the proof for  $\tilde{\lambda}_\ell^{1,-}$  is completely analogous. Note first that the limit  $\lim_{\ell \rightarrow \infty} \tilde{\lambda}_\ell^{1,+}$  exists since the function  $\ell \mapsto \tilde{\lambda}_\ell^{1,+}$  is non increasing. Indeed, if  $\ell_1 < \ell_2$  then any admissible function in (5.15) for  $\tilde{\lambda}_{\ell_1}^{1,+}$  can be extended to an admissible function for  $\tilde{\lambda}_{\ell_2}^{1,+}$  by setting it to zero on  $\Omega_{\ell_2}^+ \setminus \Omega_{\ell_1}^+$ .

A similar argument shows that  $\tilde{\lambda}_\ell^{1,+} \geq \nu_\infty^+$ , for any  $\ell > 0$ . On the other hand, the density of the space

(5.16)  
 $V_s(\Omega_\infty^+) = \{u \in C^\infty(\Omega_\infty^+) \cap V(\Omega_\infty^+) : \exists M = M(u) > 0 \text{ s.t. } u = 0 \text{ on } (M, \infty) \times \omega\}$ ,  
 in  $V(\Omega_\infty^+)$  implies that for each  $u \in V(\Omega_\infty^+) \setminus \{0\}$  and any  $\varepsilon > 0$  we can find an  $\ell_\varepsilon$  and  $v_\varepsilon \in V_s(\Omega_\infty^+)$  with  $\text{supp}(v_\varepsilon) \subset \Omega_{\ell_\varepsilon}^+$  such that

$$\left| \frac{\int_{\Omega_\infty^+} (A\nabla v_\varepsilon) \cdot \nabla v_\varepsilon}{\int_{\Omega_\infty^+} v_\varepsilon^2} - \frac{\int_{\Omega_\infty^+} (A\nabla u) \cdot \nabla u}{\int_{\Omega_\infty^+} u^2} \right| \leq \varepsilon,$$

and (5.15) follows (for  $\tilde{\lambda}_\ell^{1,+}$ ).  $\square$

Our next result complements the result of Theorem 4.2 in two ways: by showing that the limit  $\lim_{\ell \rightarrow \infty} \lambda_\ell^1$  exists and by identifying its value.

**Theorem 5.2.** *We have*

$$(5.17) \quad \lim_{\ell \rightarrow \infty} \lambda_\ell^1 = \min(\nu_\infty^+, \nu_\infty^-).$$

*Proof.* (i) We shall first show that

$$(5.18) \quad \limsup_{\ell \rightarrow \infty} \lambda_\ell^1 \leq \min(\nu_\infty^+, \nu_\infty^-).$$

We may assume w.l.o.g. that  $\nu_\infty^+ = \min(\nu_\infty^+, \nu_\infty^-)$ . Given  $\varepsilon > 0$  we may find by Lemma 5.2 an  $\ell_\varepsilon > 1/\varepsilon$  such that  $\tilde{\lambda}_{\ell_\varepsilon}^{1,+} \leq \nu_\infty^+ + \varepsilon$ . Since  $\lambda_{\ell/2}^1 \leq \tilde{\lambda}_\ell^{1,+}$  by the definitions (2.10) and (5.15), we easily deduce (5.18).

(ii) We now treat the case where (4.18) holds. Let  $u_\ell$  denote the positive normalized minimizer in (2.10). Define  $v_\ell(x) = \rho(x_1)u_\ell(x)$  where  $\rho$  is given by

$$(5.19) \quad \rho(x_1) = \begin{cases} 0 & x_1 \leq -1, \\ 1 + x_1 & x_1 \in (-1, 0), \\ 1 & x_1 \geq 0. \end{cases}$$

By (2.5) and (5.19) we have

$$(5.20) \quad \int_{(-1, \ell) \times \omega} (A\nabla v_\ell) \cdot \nabla v_\ell \leq \int_{\Omega_\ell^+} (A\nabla u_\ell) \cdot \nabla u_\ell + C_A \int_{(-1, 0) \times \omega} |\nabla v_\ell|^2.$$

Define  $w_{\ell+1}(x_1, X_2) = v_\ell(x_1 + \ell, X_2)$  on  $\Omega_{\ell+1}^-$  and notice that it is an admissible function for the infimum defining  $\tilde{\lambda}_{\ell+1}^{1,-}$  (see (5.15)). By (5.20) and (5.1)–(5.2) we obtain, for some positive constant  $C$ ,

$$(5.21) \quad \int_{\Omega_{\ell+1}^-} (A\nabla w_{\ell+1}) \cdot \nabla w_{\ell+1} \leq \int_{\Omega_\ell^+} (A\nabla u_\ell) \cdot \nabla u_\ell + C\alpha^\ell.$$

Denote

$$N_\ell^\pm = \int_{\Omega_\ell^\pm} (A\nabla u_\ell) \cdot \nabla u_\ell \quad \text{and} \quad D_\ell^\pm = \int_{\Omega_\ell^\pm} |u_\ell|^2,$$

so that in particular we have

$$(5.22) \quad N_\ell^+ + N_\ell^- = \lambda_\ell^1 \quad \text{and} \quad D_\ell^+ + D_\ell^- = 1.$$

By (5.21) and an analogous construction on  $\Omega_{\ell+1}^+$  we have

$$(5.23) \quad \tilde{\lambda}_{\ell+1}^{1,-} \leq \frac{N_\ell^+ + C\alpha^\ell}{D_\ell^+} \quad \text{and} \quad \tilde{\lambda}_{\ell+1}^{1,+} \leq \frac{N_\ell^- + C\alpha^\ell}{D_\ell^-}.$$

From (5.23) and (5.22) it follows that

$$(5.24) \quad \min\{\tilde{\lambda}_{\ell+1}^{1,-}, \tilde{\lambda}_{\ell+1}^{1,+}\} \leq D_\ell^+ \tilde{\lambda}_{\ell+1}^{1,-} + D_\ell^- \tilde{\lambda}_{\ell+1}^{1,+} \leq \lambda_\ell^1 + C\alpha^\ell.$$

Passing to the limit  $\ell \rightarrow \infty$  in (5.24) and using Lemma 5.2 yields

$$(5.25) \quad \min(\nu_\infty^+, \nu_\infty^-) \leq \liminf_{\ell \rightarrow \infty} \lambda_\ell^1,$$

which combined with (5.18) clearly implies (5.17) (when (4.18) holds).

(iii) Finally, we turn to the case where (4.18) doesn't hold. In this case we know already from Theorem 4.2 that  $\lambda_\ell^1 = \mu^1$  for all  $\ell$ . The proof of (5.17) will be clearly completed if we show that  $\nu_\infty^+ = \nu_\infty^- = \mu^1$ . We shall only show that  $\nu_\infty^+ = \mu^1$  as the argument for  $\nu_\infty^-$  is identical. By Lemma 5.1 we have  $\nu_\infty^+ \leq \mu^1$ . The reverse inequality is a special case of Theorem 6.1 (ii), see below.  $\square$

The argument of the above proof can be used to derive an additional information that will be useful in the next section.

**Proposition 5.1.** *If  $\nu_\infty^+ < \nu_\infty^-$  then  $\lim_{\ell \rightarrow \infty} \int_{\Omega_\ell^+} |\nabla u_\ell|^2 + |u_\ell|^2 = 0$ .*

*Proof.* We use the same notation as in the proof of Theorem 5.2. Passing to the limit  $\ell \rightarrow \infty$  in (5.24), using Lemma 5.2 and (5.17) yields

$$\left( \limsup_{\ell \rightarrow \infty} D_\ell^+ \right) \nu_\infty^- + \left( 1 - \limsup_{\ell \rightarrow \infty} D_\ell^+ \right) \nu_\infty^+ \leq \lim_{\ell \rightarrow \infty} \lambda_\ell^1 = \nu_\infty^+,$$

so necessarily  $\limsup_{\ell \rightarrow \infty} D_\ell^+ = 0$ . Next, by (5.23) we have for  $\ell$  large,

$$(5.26) \quad \frac{N_\ell^+}{D_\ell^-} + \tilde{\lambda}_{\ell+1}^{1,+} - C\alpha^\ell \leq \frac{N_\ell^+ + N_\ell^-}{D_\ell^-} \leq \frac{N_\ell^+ + N_\ell^-}{D_\ell^+ + D_\ell^-} = \lambda_\ell^1.$$

Since in our case,  $\lim_{\ell \rightarrow \infty} \lambda_\ell^1 = \lim_{\ell \rightarrow \infty} \tilde{\lambda}_{\ell+1}^{1,+} = \nu_\infty^+$ , and we know already that  $\lim_{\ell \rightarrow \infty} D_\ell^- = 1$ , we deduce from (5.26) that  $\lim_{\ell \rightarrow \infty} N_\ell^+ = 0$ .  $\square$

## 6. THE PROBLEM ON A SEMI-INFINITE CYLINDER

In this section we further investigate the minimization problem (5.11). By Remark 5.1 it is enough to consider  $\nu_\infty^+$ . There are two main questions we are interested in. First, we want to identify the conditions under which the infimum in (5.11) is attained. Second, we would like to know when the inequality  $\nu_\infty^+ < \mu^1$  hold. The next proposition shows that the two questions are closely related to each other.

**Proposition 6.1.** *If*

$$(6.1) \quad \nu_\infty^+ < \mu^1,$$

*then  $\nu_\infty^+$  is attained. The minimizer  $\tilde{u}^+$  is unique up to multiplication by a constant, has constant sign and satisfies*

$$(6.2) \quad \begin{cases} -\operatorname{div}(A(X_2)\nabla\tilde{u}^+) = \nu_\infty^+ \tilde{u}^+ & \text{in } \Omega_\infty^+, \\ \tilde{u}^+ = 0 & \text{on } \gamma_\infty^+, \\ (A(X_2)\nabla\tilde{u}^+).\nu = 0 & \text{on } \{0\} \times \omega. \end{cases}$$

*Proof.* The existence of a minimizer will be achieved by taking the limit  $\ell \rightarrow \infty$  of the minimizers  $\{\tilde{u}_\ell^+\}$  in (5.15) (see Remark 5.2). Since  $\{\tilde{u}_\ell^+\}$  is bounded in  $H^1(\Omega_\infty^+)$ , a subsequence  $\{\tilde{u}_{\ell_k}^+\}$  converges weakly to some limit  $\tilde{u}^+ \in H^1(\Omega_\infty^+)$ . Take



any  $\varphi \in V_s(\Omega_\infty^+)$ . Since  $\nu_\infty^+ = \lim_{k \rightarrow \infty} \tilde{\lambda}_{\ell_k}^{1,+}$  by Lemma 5.2, we can pass to the limit in the following equality, that holds for  $\ell_k > M(\varphi)$  (see (5.16)),

$$\int_{\Omega_\infty^+} A \nabla \tilde{u}_{\ell_k}^+ \cdot \nabla \varphi = \tilde{\lambda}_{\ell_k}^{1,+} \int_{\Omega_\infty^+} \tilde{u}_{\ell_k}^+ \varphi,$$

and obtain that

$$(6.3) \quad \int_{\Omega_\infty^+} A \nabla \tilde{u}^+ \cdot \nabla \varphi = \nu_\infty^+ \int_{\Omega_\infty^+} \tilde{u}^+ \varphi.$$

Since (6.3) is valid for any  $\varphi \in V_s(\Omega_\infty^+)$ , and by density also for any  $\varphi \in V(\Omega_\infty^+)$ , we obtain that  $\tilde{u}^+$  is a solution of (6.2). To conclude that it is a minimizer realizing  $\nu_\infty^+$  in (5.11) we only need to prove that it is *nontrivial*, i.e., that  $\tilde{u}^+ \not\equiv 0$ . Actually, we are going to show that  $\int_{\Omega_\infty^+} (\tilde{u}^+)^2 = 1$  and  $\tilde{u}^+ > 0$ . For that matter we will prove decay estimates for  $\tilde{u}_\ell^+$  for large  $x_1$ , that imply concentration near  $x_1 = 0$ , using the same technique as the one used in the proof of Theorem 5.1.

Let  $\ell$  and  $\ell'$  satisfy  $0 < \ell' \leq \ell - 1$ . Define  $\tilde{\rho}_{\ell'} = \tilde{\rho}_{\ell'}(x_1)$  by

$$\tilde{\rho}_{\ell'}(x_1) = \begin{cases} 0 & x_1 \leq \ell', \\ x_1 - \ell' & x_1 \in (\ell', \ell' + 1), \\ 1 & x_1 \geq \ell' + 1. \end{cases}$$

By the Euler-Lagrange equation satisfied by  $\tilde{u}_\ell^+$  we have

$$\int_{\Omega_\ell^+} (A \nabla \tilde{u}_\ell^+) \cdot \nabla (\tilde{\rho}_{\ell'}^2 \tilde{u}_\ell^+) = \tilde{\lambda}_\ell^{1,+} \int_{\Omega_\ell^+} \tilde{\rho}_{\ell'}^2 |\tilde{u}_\ell^+|^2.$$

Repeating the argument used to derive (5.6) we obtain

$$(6.4) \quad \begin{aligned} (\sigma_{\ell/2}^1 - \tilde{\lambda}_\ell^{1,+}) \int_{\Omega_\ell^+ \setminus \Omega_{\ell'+1}} |\tilde{u}_\ell^+|^2 &\leq (\sigma_{\ell/2}^1 - \tilde{\lambda}_\ell^{1,+}) \int_{\Omega_\ell^+} |\tilde{u}_\ell^+|^2 \tilde{\rho}_{\ell'}^2 \leq \int_{\Omega_\ell^+} |\tilde{u}_\ell^+|^2 (A \nabla \tilde{\rho}_{\ell'}) \cdot \nabla \tilde{\rho}_{\ell'} \\ &= \int_{\Omega_{\ell'+1}^+ \setminus \Omega_{\ell'}} |\tilde{u}_\ell^+|^2 (A \nabla \tilde{\rho}_{\ell'}) \cdot \nabla \tilde{\rho}_{\ell'} \leq C_A \int_{\Omega_{\ell'+1}^+ \setminus \Omega_{\ell'}} |\tilde{u}_\ell^+|^2. \end{aligned}$$

Using (1.3) together with (6.1) and Lemma 5.2 we deduce that there exist  $\tilde{\ell}_0 > 0$  and  $\tilde{\beta} > 0$  such that for  $\ell > \tilde{\ell}_0$  we have  $\sigma_{\ell/2}^1 - \tilde{\lambda}_\ell^{1,+} \geq \tilde{\beta}$ . Therefore, we deduce from (6.4) that

$$(6.5) \quad \int_{\Omega_\ell^+ \setminus \Omega_{\ell'+1}} |\tilde{u}_\ell^+|^2 \leq \tilde{\alpha} \int_{\Omega_\ell^+ \setminus \Omega_{\ell'}} |\tilde{u}_\ell^+|^2 \quad \text{with} \quad \tilde{\alpha} := \frac{C_A}{\tilde{\beta} + C_A}.$$

Fix any  $r > 1$ . Applying (6.5) successively for  $\ell' = r - 1, r - 2, \dots, r - [r]$  yields

$$\int_{\Omega_\ell^+ \setminus \Omega_r} |\tilde{u}_\ell^+|^2 \leq \tilde{\alpha}^{[r]} \int_{\Omega_\ell^+} |\tilde{u}_\ell^+|^2 = \tilde{\alpha}^{[r]}, \quad \forall \ell > r.$$

In other words,

$$(6.6) \quad \int_{\Omega_r^+} |\tilde{u}_\ell^+|^2 \geq 1 - \tilde{\alpha}^{[r]}.$$

Since  $\tilde{u}_{\ell_k} \rightarrow \tilde{u}^+$  strongly in  $L^2(\Omega_r^+)$ , we deduce from (6.6) that

$$(6.7) \quad \int_{\Omega_r^+} (\tilde{u}^+)^2 \geq 1 - \tilde{\alpha}^{[r]}.$$

This already implies that  $\tilde{u}^+$  is a nontrivial nonnegative solution to (6.2) and therefore, a minimizer in (5.11). Applying (6.7) with arbitrary large  $r$ , we get that  $\int_{\Omega_\infty^\pm} (\tilde{u}^+)^2 = 1$ . The uniqueness of the minimizer follows by a standard argument, using the fact that any minimizer must have a constant sign.  $\square$

Open Problem: Is it true that (6.1) is also a necessary condition for the existence of a minimizer realizing  $\nu_\infty^+$ ? In Theorem 6.1 below we will show nonexistence of a minimizer when  $\nu_\infty^+ = \mu^1$ , but under the additional condition (6.9).

The next result provides a sufficient condition for (6.1) to hold and another one for it to fail.

**Theorem 6.1.** (i) *Assume that (4.18) is satisfied. If the following condition holds,*

$$(6.8) \quad \int_{\omega} (A_{12} \cdot \nabla W_1) W_1 \geq 0,$$

*then (6.1) holds.*

(ii) *If*

$$(6.9) \quad A_{12} \cdot \nabla W_1 \leq 0 \text{ a.e. in } \omega$$

*then  $\nu_\infty^+ = \mu^1$ . Moreover, in this case there is no minimizer realizing  $\nu_\infty^+$ .*

*Proof.* (i) Assume that (6.8) is satisfied. A similar computation to the one done in the proof of Theorem 4.1 (see also Remark 4.1) shows that  $\{\tilde{v}_\ell^\varepsilon\}$  given by (4.15), satisfy not only (4.16), but also

$$\inf_{\varepsilon > 0} \lim_{\ell \rightarrow 0} \frac{\int_{\Omega_\ell^-} (A \nabla \tilde{v}_\ell^\varepsilon) \cdot \nabla \tilde{v}_\ell^\varepsilon}{\int_{\Omega_\ell^-} |\tilde{v}_\ell^\varepsilon|^2} = \int_{\omega} (A_{22} \nabla W_1) \cdot \nabla W_1 - \frac{|A_{12} \nabla W_1|^2}{a_{11}}.$$

Indeed, we only need to note that the term corresponding to the second term on the RHS of (4.11) is of the order  $O(\ell^2)$ . Hence, we can fix values of  $\ell_1$  and  $\varepsilon_1$  such that the following analog of (4.20) holds,

$$(6.10) \quad -\gamma_1 := \int_{\Omega_{\ell_1}^-} (A \nabla \tilde{v}_{\ell_1}^{\varepsilon_1}) \cdot \nabla \tilde{v}_{\ell_1}^{\varepsilon_1} - \mu^1 \int_{\Omega_{\ell_1}^-} |\tilde{v}_{\ell_1}^{\varepsilon_1}|^2 < 0.$$

For each  $\alpha > 0$  we define a test function in  $V_\infty(\Omega_\infty^+)$  by

$$z_\alpha(x_1, X_2) = \begin{cases} \tilde{v}_{\ell_1}^{\varepsilon_1}(x_1 - \ell_1, X_2) & x_1 \in [0, \ell_1), \\ W_1(X_2) e^{-\alpha(x_1 - \ell_1)} & x_1 \in [\ell_1, \infty). \end{cases}$$

Above we used the fact that  $\tilde{v}_{\ell_1}^{\varepsilon_1}(0, X_2) = W_1(X_2)$ . We have,

$$\int_{\Omega_\infty^\pm} |z_\alpha|^2 = \int_{\Omega_{\ell_1}^-} |\tilde{v}_{\ell_1}^{\varepsilon_1}|^2 + \left( \int_0^\infty e^{-2\alpha x_1} \right) \int_{\omega} W_1^2 = \int_{\Omega_{\ell_1}^-} |\tilde{v}_{\ell_1}^{\varepsilon_1}|^2 + \frac{1}{2\alpha},$$

and

$$\begin{aligned} \int_{\Omega_\infty^\pm} (A \nabla z_\alpha) \cdot \nabla z_\alpha &= \int_{\Omega_{\ell_1}^-} (A \nabla \tilde{v}_{\ell_1}^{\varepsilon_1}) \cdot \nabla \tilde{v}_{\ell_1}^{\varepsilon_1} \\ &\quad + \frac{1}{2\alpha} \left( \alpha^2 \int_{\omega} a_{11} W_1^2 - 2\alpha \int_{\omega} (A_{12} \cdot \nabla W_1) W_1 + \int_{\omega} (A_{22} \nabla W_1) \cdot \nabla W_1 \right) \end{aligned}$$

Therefore, using (6.10) we get

$$(6.11) \quad \nu_\infty^+ - \mu^1 \leq \frac{\int_{\Omega_\infty^+} A \nabla z_\alpha \cdot \nabla z_\alpha}{\int_{\Omega_\infty^+} |z_\alpha|^2} - \mu^1 < \frac{\frac{\alpha}{2} \int_\omega a_{11} W_1^2 - \int_\omega (A_{12} \cdot \nabla W_1) W_1 - \gamma_1}{\int_{\Omega_\ell^-} |v_{\ell_1}^{\varepsilon_1}|^2 + \frac{1}{2\alpha}}.$$

Since  $\gamma_1 > 0$  and  $\int_\omega (A_{12} \cdot \nabla W_1) W_1 \geq 0$  by (6.8), it is clear that we can choose  $\alpha$  small enough to ensure that the RHS of (6.11) is negative, completing the proof of (6.1).

(ii) We notice that not only  $V_s(\Omega_\mp^\infty)$  is dense in  $V(\Omega_\mp^\infty)$  (see (5.16)), but its subspace

$$V_s^0(\Omega_\infty^+) = \left\{ u \in V_s(\Omega_\infty^+) : \exists \delta = \delta(u) > 0 \text{ s.t. } u(x) = 0 \text{ for } \text{dist}(x, \gamma_\infty^+) \leq \delta \right\},$$

is dense as well. By elliptic regularity and the strong maximum principle we know that  $W_1$  is continuous and positive in  $\omega$  (see [15, Chapter 8]). We shall use the following version of Picone identity,

$$(6.12) \quad (A \nabla u) \cdot \nabla u - (A \nabla v) \cdot \nabla \left( \frac{u^2}{v} \right) = A \left( \nabla u - \frac{u}{v} \nabla v \right) \cdot \left( \nabla u - \frac{u}{v} \nabla v \right) \geq 0.$$

Using (6.12) with any  $u \in V_s^0(\Omega_\infty^+)$  and  $v = W_1$ , integrating and applying the generalized Green formula yields

$$(6.13) \quad \begin{aligned} 0 &\leq \int_{\Omega_\infty^+} A \left( \nabla u - \frac{u}{W_1} \nabla W_1 \right) \cdot \left( \nabla u - \frac{u}{W_1} \nabla W_1 \right) = \int_{\Omega_\infty^+} (A \nabla u) \cdot \nabla u - (A \nabla W_1) \cdot \nabla \left( \frac{u^2}{W_1} \right) \\ &= \int_{\Omega_\infty^+} (A \nabla u) \cdot \nabla u + \int_{\Omega_\infty^+} \text{div}(A \nabla W_1) \left( \frac{u^2}{W_1} \right) - \int_{\{0\} \times \omega} (A \nabla W_1 \cdot \nu) \left( \frac{u^2}{W_1} \right) \\ &= \int_{\Omega_\infty^+} (A \nabla u) \cdot \nabla u - \mu^1 u^2 + \int_\omega (A_{12} \cdot \nabla W_1) \frac{u^2(0, X_2)}{W_1(X_2)}. \end{aligned}$$

By (6.13) and (6.9) we deduce that

$$(6.14) \quad 0 \leq \int_{\Omega_\infty^+} A \left( \nabla u - \frac{u}{W_1} \nabla W_1 \right) \cdot \left( \nabla u - \frac{u}{W_1} \nabla W_1 \right) \leq \int_{\Omega_\infty^+} A \nabla u \cdot \nabla u - \mu^1 u^2.$$

By the density of  $V_s^0(\Omega_\infty^+)$  in  $V(\Omega_\infty^+)$  it follows that (6.14) holds for every  $u \in V(\Omega_\infty^+)$ , i.e.,  $\nu_\infty^+ \geq \mu^1$ . Finally, applying (5.12) we get that  $\nu_\infty^+ = \mu^1$ . To conclude, assume by negation that  $\nu_\infty^+$  is realized by a minimizer  $u$ . Then, by (6.14) we get that  $\nabla \left( \frac{u}{W_1} \right) = 0$  a.e., implying that  $u = cW_1$  for some constant  $c \neq 0$ . But this is clearly a contradiction since  $W_1 \notin V(\Omega_\infty^+)$ .  $\square$

**Remark 6.1.** *An immediate consequence of Theorem 6.1 and Remark 5.1 is that if (4.18) holds and  $\int_\omega (A_{12} \cdot \nabla W_1) W_1 = 0$ , then we have both  $\nu_\infty^+ < \mu^1$  and  $\nu_\infty^- < \mu^1$ . A special case is when property (S) holds. Another direct consequence is that whenever (4.18) holds we have  $\min(\nu_\infty^+, \nu_\infty^-) < \mu^1$ . However, this fact follows already from our previous results, by combining Theorem 4.2 and Theorem 5.2.*

Our last result provides a description of the asymptotic profile of the eigenfunctions  $\{u_\ell\}$  near the ends of the cylinder. We denote by  $\tilde{u}^\pm$  the unique positive renormalized minimizer for  $\nu_\infty^\pm$ , when it exists. For each  $\ell > 0$  we define:

$$(6.15) \quad \begin{aligned} \tilde{v}_\ell^+(x_1, X_2) &= u_\ell(x_1 - \ell, X_2) \quad \text{on } \Omega_\ell^+, \\ \tilde{v}_\ell^-(x_1, X_2) &= u_\ell(x_1 + \ell, X_2) \quad \text{on } \Omega_\ell^-. \end{aligned}$$

The next theorem describes two possible scenarios that may occur: concentration near one of the ends of the cylinder, or concentration near both ends.

**Theorem 6.2.** (i) If  $\nu_\infty^+ < \nu_\infty^-$  then, for every  $r > 0$ ,

$$(6.16) \quad \tilde{v}_\ell^+ \rightarrow \tilde{u}^+ \text{ in } H^1(\Omega_r^+) \text{ and } \tilde{v}_\ell^- \rightarrow 0 \text{ in } H^1(\Omega_r^-).$$

(ii) If both (6.2) and property (S) hold then we have  $\tilde{v}_\ell^+(x_1, X_2) = \tilde{v}_\ell^-(-x_1, -X_2)$  and for every  $r > 0$ ,

$$(6.17) \quad \tilde{v}_\ell^+ \rightarrow \tilde{u}^+ \text{ in } H^1(\Omega_r^+) \text{ and } \tilde{v}_\ell^- \rightarrow \tilde{u}^- \text{ in } H^1(\Omega_r^-).$$

*Proof.* (i) The convergence of  $\{\tilde{v}_\ell^-\}$  to 0 in  $H^1(\Omega_r^-)$  for all  $r > 0$  is clear from Proposition 5.1, so we only need to prove the result for  $\{\tilde{v}_\ell^+\}$ . Since  $\{\tilde{v}_\ell^+\}$  is bounded in  $H^1(\Omega_\ell^+)$ , given any sequence  $\ell_k \rightarrow \infty$ , we can apply a diagonal argument to  $\{\tilde{v}_{\ell_k}^+\}$  to extract a subsequence, still denoted by  $\{\ell_k\}$ , such that  $\tilde{v}_{\ell_k}^+$  converges weakly in  $H^1(\Omega_r^+)$  and strongly in  $L^2(\Omega_r^+)$  to some function  $v^+ \in H^1(\Omega_\infty^+)$ , for every  $r > 0$ . By (5.1) and Proposition 5.1 we have

$$(6.18) \quad \int_{\Omega_r^+} |\tilde{v}_\ell^+|^2 = \int_{\Omega_\ell^- \setminus \Omega_{\ell-r}^-} |u_\ell|^2 = 1 - \int_{\Omega_{\ell-r}^-} |u_\ell|^2 - \int_{\Omega_\ell^+} |u_\ell|^2 \geq 1 - \alpha^{[r]} + o(1),$$

where  $o(1)$  stands for a quantity that tends to 0 when  $\ell \rightarrow \infty$ . Passing to the limit in (6.18) with  $\ell = \ell_k$ , yields,

$$(6.19) \quad \int_{\Omega_r^+} |v^+|^2 \geq 1 - \alpha^{[r]},$$

and since  $r$  is arbitrary, we get that  $\int_{\Omega_\infty^+} |v^+|^2 = 1$ . In addition, we clearly have

$$(6.20) \quad \nu_\infty^+ = \lim_{\ell \rightarrow \infty} \lambda_\ell^1 \geq \lim_{\ell \rightarrow \infty} \int_{\Omega_\ell} (A \nabla u_\ell) \cdot \nabla u_\ell \\ \geq \limsup_{k \rightarrow \infty} \int_{\Omega_r^+} (A \nabla \tilde{v}_{\ell_k}^+) \cdot \nabla \tilde{v}_{\ell_k}^+ \geq \int_{\Omega_r^+} (A \nabla v^+) \cdot \nabla v^+.$$

From (6.19)–(6.20) we deduce that  $\int_{\Omega_\infty^+} (A \nabla v^+) \cdot \nabla v^+ = \nu_\infty^+$ , i.e.,  $v^+$  is a nonnegative normalized minimizer, realizing  $\nu_\infty^+$  in (5.11). Therefore, it must coincide with  $\tilde{u}^+$ . Finally, defining on  $(0, \infty)$  the function

$$f(r) = \limsup_{k \rightarrow \infty} \int_{\Omega_r^+} (A \nabla \tilde{v}_{\ell_k}^+) \cdot \nabla \tilde{v}_{\ell_k}^+ - \int_{\Omega_r^+} (A \nabla \tilde{u}^+) \cdot \nabla \tilde{u}^+,$$

we see that on the one hand it is a nonnegative and nondecreasing function, while on the other hand  $\lim_{r \rightarrow \infty} f(r) = 0$ . Hence  $f(r) \equiv 0$ , implying the strong convergence  $\tilde{v}_{\ell_k}^+ \rightarrow \tilde{u}^+$  in  $H^1(\Omega_r^+)$  for all  $r > 0$ . The uniqueness of the possible limit implies the the same convergence holds for the whole family  $\{\tilde{v}_\ell^+\}$ .

(ii) In this case we have the symmetry relation  $u_\ell(x_1, X_2) = u_\ell(-x_1, -X_2)$  by Proposition 2.1, and the same argument as in (i) gives the result.  $\square$

**Remark 6.2.** *Theorem 6.2 provides a description of the profile of  $u_\ell$  near the ends of the cylinder. As pointed to us by Y. Pinchover, a description of the profile of  $u_\ell$  in the bulk can be given using the characterization of positive solutions in an infinite cylinder, given in [17]. Indeed, setting  $v_\ell(x) = u_\ell(x)/u_\ell(0)$ , and employing Harnack's inequality and the boundary Harnack principle (see [16, Theorems 1.2 and 1.3]) we obtain a subsequence  $\{v_{\ell_k}\}$  that converges uniformly on each cylinder*

$\Omega_r$ ,  $r > 0$ , to a limit  $v$ . The function  $v$  is a positive solution on the infinite cylinder  $\Omega_\infty = (-\infty, \infty) \times \omega$  of  $-\operatorname{div}(A(X_2)\nabla v) = \lambda_\infty v$  satisfying  $v = 0$  on  $\partial\Omega_\infty = (-\infty, \infty) \times \partial\omega$ , where  $\lambda_\infty = \lim_{\ell \rightarrow \infty} \lambda_\ell^1 = \min(\nu_\infty^+, \nu_\infty^-)$  (by Theorem 5.2). From [17, Theorem 5.1] (that handles a much more general situation) it follows that such  $v$  is a linear combination of one or two exponential solutions of the form  $v_\alpha(x) = \Phi_\alpha(X_2)e^{\alpha x_1}$ . In particular, when property (S) holds it follows that  $v$  takes the form

$$v(x) = g(X_2)e^{\alpha x_1} + g(-X_2)e^{-\alpha x_1}$$

for some  $\alpha > 0$ , if (4.18) holds, and  $v(x) = cW_1(X_2)$  if (4.18) doesn't hold.

## 7. SOME ADDITIONAL RESULTS

So far we only studied the asymptotic behavior of the first eigenvalue  $\lambda_\ell^1$  and the corresponding eigenfunction  $u_\ell$ . The analogous behavior of the other eigenvalues  $\lambda_\ell^2, \lambda_\ell^3$ , etc., is also of interest. For the case of Dirichlet boundary condition this was done in [7]. For our case of mixed boundary conditions we have the following partial result for  $\lambda_\ell^2$ .

**Theorem 7.1.** *If property (S) holds then*

$$\lim_{\ell \rightarrow \infty} \lambda_\ell^2 = \lim_{\ell \rightarrow \infty} \lambda_\ell^1.$$

*Proof.* Define  $h_\ell^-$  and  $h_\ell^+$  on  $\Omega_\ell$  by

$$h_\ell^-(x) = \begin{cases} \tilde{u}_\ell^+(x_1 + \ell, X_2) & \text{on } \Omega_\ell^-, \\ 0 & \text{on } \Omega_\ell^+ \end{cases}$$

and

$$h_\ell^+ = \begin{cases} \tilde{u}_\ell^-(x_1 - \ell, X_2) & \text{on } \Omega_\ell^+, \\ 0 & \text{on } \Omega_\ell^-, \end{cases}$$

where  $\tilde{u}_\ell^-, \tilde{u}_\ell^+$  are defined in Remark 5.2. Set  $\mathcal{H}_\ell = \alpha_\ell h_\ell^- + \beta_\ell h_\ell^+$ , where  $\alpha_\ell, \beta_\ell$  are chosen such that

$$\int_{\Omega_\ell} u_\ell \mathcal{H}_\ell = 0 \quad \text{and} \quad \alpha_\ell^2 + \beta_\ell^2 > 0.$$

Such a choice is possible since we have to satisfy one linear equation in two unknowns. From the Rayleigh quotient characterization of  $\lambda_\ell^2$  we get, since the functions  $h_\ell^+$  and  $h_\ell^-$  have disjoint supports,

$$(7.1) \quad \lambda_\ell^2 = \min \left\{ \frac{\int_{\Omega_\ell} (A\nabla u) \cdot \nabla u}{\int_{\Omega_\ell} u^2} \mid 0 \neq u \in V(\Omega_\ell), \int_{\Omega_\ell} uu_\ell = 0 \right\} \leq \frac{\int_{\Omega_\ell} A\nabla \mathcal{H}_\ell \cdot \nabla \mathcal{H}_\ell}{\int_{\Omega_\ell} \mathcal{H}_\ell^2} \\ = \frac{\alpha_\ell^2 \int_{\Omega_\ell^-} (A\nabla h_\ell^-) \cdot \nabla h_\ell^- + \beta_\ell^2 \int_{\Omega_\ell^+} (A\nabla h_\ell^+) \cdot \nabla h_\ell^+}{\alpha_\ell^2 \int_{\Omega_\ell^-} (h_\ell^-)^2 + \beta_\ell^2 \int_{\Omega_\ell^+} (h_\ell^+)^2} = \frac{\alpha_\ell^2 \tilde{\lambda}_\ell^{1,+} + \beta_\ell^2 \tilde{\lambda}_\ell^{1,-}}{\alpha_\ell^2 + \beta_\ell^2}.$$

But the symmetry property (S) implies, by the same proof as that of Proposition 2.1, that  $\tilde{u}_\ell^+(x_1, X_2) = \tilde{u}_\ell^-(-x_1, -X_2)$  and  $\tilde{\lambda}_\ell^{1,+} = \tilde{\lambda}_\ell^{1,-}$ . Therefore, (7.1) implies that the RHS of (7.1) equals  $\tilde{\lambda}_\ell^{1,+}$  and we obtain that

$$\lambda_\ell^1 < \lambda_\ell^2 \leq \tilde{\lambda}_\ell^{1,+} = \tilde{\lambda}_\ell^{1,-}.$$

The theorem then follows from Lemma 5.2 and Theorem 5.2.  $\square$

In the previous sections we considered the case of a cylinder which goes to infinity in one direction. We now consider the more general case of a domain that tends to infinity in several directions. In the rest of the paper we set

$$\Omega_\ell = (-\ell, \ell)^p \times \omega,$$

where  $1 \leq p < n$  and  $\omega$  is a bounded subset of  $\mathbb{R}^{n-p}$ . The points in  $\Omega_\ell$  are denoted by

$$X = (X_1, X_2) \text{ with } X_1 = (x_1, \dots, x_p) \text{ and } X_2 = (x_{p+1}, \dots, x_n).$$

Let  $A(X_2)$  be a  $n \times n$  symmetric, positive definite matrix, uniformly elliptic and uniformly bounded on  $\omega$ , as in the previous sections. Now we consider the following decomposition to sub-matrices:

$$A(X_2) = \begin{pmatrix} A_{11}(X_2) & A_{12}(X_2) \\ A_{12}^t(X_2) & A_{22}(X_2) \end{pmatrix}$$

where  $A_{11}, A_{12}$  and  $A_{22}$  are  $p \times p, p \times (n-p)$  and  $(n-p) \times (n-p)$  matrices, respectively. We still denote by  $\mu^1$  and  $W_1$  the first eigenvalue and the corresponding eigenfunction for the problem (1.1). Let  $C_i$  denote the  $i$ -th row of the matrix  $A_{12}$ , and denote by  $B_i$  the  $(n-p+1) \times (n-p+1)$  matrix

$$B_i(X_2) = \begin{pmatrix} a_{ii}(X_2) & C_i(X_2) \\ C_i^t(X_2) & A_{22}(X_2) \end{pmatrix},$$

for  $1 \leq i \leq p$ . Since the matrix  $B_i$  can be viewed as a representation of the restriction of the operator associated with  $A$  to the subspace of  $\mathbb{R}^n$  consisting of the vectors  $v = (v_1, \dots, v_n)$  satisfying  $v_j = 0$  for all  $j$  such that  $i \neq j \leq p$ , we conclude that the matrices  $B_i(X_2)$  are also uniformly elliptic for  $X_2 \in \omega$ .

The following eigenvalue problem is the generalization of (1.5) to our setting:

$$(7.2) \quad \begin{cases} -\operatorname{div}(A(X_2)\nabla u) = \sigma u & \text{in } \Omega_\ell, \\ u = 0 & \text{on } (-\ell, \ell)^p \times \partial\omega, \\ (A(X_2)\nabla u) \cdot \nu = 0 & \text{on } \partial(-\ell, \ell)^p \times \omega. \end{cases}$$

As before we denote by  $\lambda_\ell^1$  the first eigenvalue and by  $u_\ell$  the corresponding normalized positive eigenfunction. We have the following generalization of Theorem 4.2.

**Theorem 7.2.** *We have*

$$\limsup_{\ell \rightarrow \infty} \lambda_\ell^1 < \mu^1,$$

provided the following condition holds,

$$(7.3) \quad A_{12} \cdot \nabla W_1 \neq \mathbf{0} \text{ a.e. on } \omega,$$

where  $\mathbf{0}$  denotes the zero element in  $\mathbb{R}^p$ . In case (7.3) does not hold we have  $\lambda_\ell^1 = \mu^1$  for all  $\ell > 0$ .

*Proof.* Assume first that (7.3) doesn't hold. Then there exists  $i \in \{1, \dots, p\}$  for which  $(A_{12}\nabla W_1)_i$  is not identically zero (a.e.) on  $\omega$ . It follows that the hypotheses of Theorem 4.2 (for the case where (4.18) holds) are satisfied for the eigenvalue problem associated with the operator  $-\operatorname{div}(B_i(X_2)\nabla v)$  on the domain  $\tilde{\Omega}_\ell = (-\ell, \ell) \times \omega$  in  $\mathbb{R}^{n-p+1}$ . Hence, there exist functions  $\phi_\ell(x_1, X_2) \in V(\tilde{\Omega}_\ell)$ ,  $\ell > 0$ , such that

$$(7.4) \quad \limsup_{\ell \rightarrow \infty} \frac{\int_{\tilde{\Omega}_\ell} (B_i(X_2)\nabla \phi_\ell) \cdot \nabla \phi_\ell}{\int_{\tilde{\Omega}_\ell} \phi_\ell^2} < \mu^1.$$

Define on  $\Omega_\ell$ ,  $v_\ell(X_1, X_2) := \phi_\ell(x_i, X_2)$ . Noting that

$$\int_{\Omega_\ell} (A\nabla v_\ell) \cdot \nabla v_\ell = (2\ell)^{p-1} \int_{\tilde{\Omega}_\ell} (B_i \nabla \phi_\ell) \cdot \nabla \phi_\ell \quad \text{and} \quad \int_{\Omega_\ell} v_\ell^2 = (2\ell)^{p-1} \int_{\tilde{\Omega}_\ell} \phi_\ell^2,$$

we get from (7.4) that

$$\limsup_{\ell \rightarrow \infty} \lambda_\ell^1 \leq \limsup_{\ell \rightarrow \infty} \frac{\int_{\Omega_\ell} A\nabla v_\ell \cdot \nabla v_\ell}{\int_{\Omega_\ell} v_\ell^2} < \mu^1.$$

Assume now that (7.3) does hold. Next we apply a simple generalization of an argument from Theorem 4.1. Let  $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^t & B_{22} \end{pmatrix}$  be a positive definite  $n \times n$  matrix, where  $B_{11}$  and  $B_{22}$  are  $p \times p$  and  $(n-p) \times (n-p)$  matrices, respectively. Represent any vector  $\mathbf{z}$  in  $\mathbb{R}^n$  as  $\mathbf{z} = (Z_1, Z_2)$  with  $Z_1 \in \mathbb{R}^p$  and  $Z_2 \in \mathbb{R}^{n-p}$ . Then, by a similar computation to the one leading to (4.2)–(4.3) we get that for any fixed  $Z_2 \in \mathbb{R}^{n-p}$  we have

$$(7.5) \quad \min_{Z_1 \in \mathbb{R}^p} (B\mathbf{z}) \cdot \mathbf{z} = (B_{22}Z_2) \cdot Z_2 - (B_{11}^{-1}B_{12}Z_2) \cdot B_{12}Z_2,$$

and the minimum in (7.5) is attained for

$$Z_1 = -B_{11}^{-1}(B_{12}Z_2).$$

Applying (7.5) with  $B = A(X_2)$  we obtain, for any  $\ell > 0$ ,

$$(7.6) \quad \begin{aligned} \int_{\Omega_\ell} (A(X_2)\nabla u_\ell) \cdot \nabla u_\ell &\geq \int_{\Omega_\ell} (A_{22}\nabla_{X_2}u_\ell) \cdot \nabla_{X_2}u_\ell - (A_{11}^{-1}A_{12}\nabla_{X_2}u_\ell) \cdot A_{12}\nabla_{X_2}u_\ell \\ &\geq \Lambda^1 \int_{\Omega_\ell} u_\ell^2, \end{aligned}$$

where  $\Lambda^1$  is defined, generalizing (4.1), by

$$(7.7) \quad \Lambda^1 = \inf \left\{ \int_{\omega} A_{22}\nabla u \cdot \nabla u - (A_{11}^{-1}A_{12}\nabla u) \cdot A_{12}\nabla u : u \in H_0^1(\omega), \int_{\omega} u^2 = 1 \right\}.$$

The infimum in (7.7) is attained by a unique positive function, denoted again by  $w_1$ , that satisfies

$$(7.8) \quad \begin{cases} -\operatorname{div}(A_{22}\nabla w_1) + \operatorname{div}(A_{12}^t A_{11}^{-1} A_{12}\nabla w_1) = \Lambda^1 w_1 & \text{in } \omega, \\ w_1 = 0 & \text{on } \partial\omega. \end{cases}$$

But if (7.3) holds, then  $W_1$  is also a positive eigenfunction in (7.8), with eigenvalue  $\mu^1$ . As in the proof of Theorem 4.2 we conclude that  $\Lambda^1 = \mu^1$  and the result follows from (7.6) (since clearly  $\lambda_\ell^1 \leq \mu^1$ ).  $\square$

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