A Comparison Principle for the p-Laplacian

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1 Introduction

The purpose of this note is to present a quite general comparison principle between sub and super solutions for singular equations involving the $p-$Laplacian. We are motivated by two such results in the case $p = 2$. The first is the following theorem of Agmon [2, Theorem 2.7] that we give in a slightly modified form.

Theorem 1.1 (Agmon). *Let $a(x) \in L^1_{loc}(\mathbb{R}^N \setminus \{0\})$, $R > 0$ and $u, v \in C(\overline{\Omega}_R) \cap H^1_{loc}(\Omega_R)$, with $\Omega_R := \{|x| > R\}$ and $v > 0$ on $\overline{\Omega}_R$, satisfy (in the weak sense),*

$$-\Delta v - a(x)v \geq 0 \geq -\Delta u - a(x)u \qquad in\ \Omega_R.$$

Suppose also that

$$v(x) \geq u(x) \qquad on\ \{|x| = R\},$$

and that for some $\alpha > 1$ we have,

$$\liminf_{K\to\infty} K^{-2} \int_{\{K<|x|<\alpha K\}} u^2 = 0. \tag{1.1}$$

Then, $v \geq u$ in Ω_R.

Adapting Agmon's method to a different setting, Marcus, Mizel and Pinchover proved in [5, Lemma 8] the following result (again we slightly modify the statement),

Theorem 1.2 (Marcus-Mizel-Pinchover). *Let Ω be a proper subdomain of $\mathbb{R}^N$ with compact boundary and $a(x) \in L^1_{loc}(\Omega)$. Set $\delta(x) = dist(x, \partial\Omega)$ and assume that $\beta > 0$ is such that $\Sigma_\beta := \{x \in \Omega : \delta(x) = \beta\} \neq \emptyset$. Suppose that $u, v \in C(\Omega_\beta \cup \Sigma_\beta) \cap H^1_{loc}(\Omega_\beta)$, with $\Omega_\beta := \{x \in \Omega : \delta(x) < \beta\}$ and $v > 0$ on $\Omega_\beta \cup \Sigma_\beta$, satisfy (in the weak sense),*

$$-\Delta v - a(x)v \geq 0 \geq -\Delta u - a(x)u \qquad in\ \Omega_\beta.$$

Suppose also that

$$v(x) \geq u(x) \quad on\ \Sigma_\beta,$$

and that for some $\alpha > 1$ we have,

$$\liminf_{r\to 0} r^{-2} \int_{\{x\in\Omega\,:\,r<\delta(x)<\alpha r\}} u^2 = 0. \tag{1.2}$$

Then, $v \geq u$ in Ω_β.

In [9, Proposition A.1] Shafrir proved a generalization of Theorem 1.2 for any $1 < p < \infty$. The purpose of this note is to present a comparison principle which unifies and generalizes all the above mentioned results for any $1 < p < \infty$. We basically follow the strategy of [9], but we make a new key observation about a choice of test functions which enables us to improve the known results even in the case $p = 2$. In particular, this improvement shows that in both (1.1) and (1.2) it is enough to require that the corresponding limits are *finite* (i.e. not necessarily zero) in order for the conclusions to hold. We also present some applications for uniqueness and Liouville type results.

We begin with some notations and definitions. In the sequel let Ω be a proper subdomain of $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a disjoint union of two compact sets Γ_0 and Γ_1. We assume that $\Gamma_1 \neq \emptyset$, but Γ_0 may be empty. When $\Gamma_0 \neq \emptyset$ we define the distance function $\delta_0(x) = \mathrm{dist}(x, \Gamma_0)$, $\forall x \in \mathbb{R}^N$. In case $\Gamma_0 = \emptyset$ it will be convenient to set $\delta_0 \equiv 0$. For $p \in (1,\infty)$ we denote by $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ the $p-$Laplacian of u. Given $a(x) \in L^1_{\mathrm{loc}}(\Omega)$ we shall say that a function $v \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ is a super solution for the equation,

$$-\Delta_p u - a(x)|u|^{p-2}u = 0, \qquad \text{in } \Omega, \tag{1.3}$$

if

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla\phi - a(x)|v|^{p-2}v\phi \geq 0, \qquad \text{for all } 0 \leq \phi \in C_c^\infty(\Omega). \tag{1.4}$$

The notions of subsolution and solution are defined similarly by replacing "$\geq$" in (1.4) by "$\leq$" and "$=$" respectively. For $0 < a < b$ we denote,

$$\Omega_{(a,b)} = \{x \in \Omega \,:\, \delta_0(x) \in (a,b)\} \quad \text{ and } \Omega^{(a,b)} = \{x \in \Omega \,:\, |x| \in (a,b)\}.$$

Consider first $p \in (1,2]$ and a function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$. We shall say that u satisfies condition (C_0) if for some $0 < \sigma < 1$ there exist for each $n \geq 2$, n positive numbers $\{r_j^{(n)}\}_{j=1}^n$ such that,

$$r_{j+1}^{(n)} \leq \sigma r_j^{(n)}, \qquad j = 1,\dots,n-1,$$

with $\lim_{n\to\infty} r_1^{(n)} = 0$ and

$$\lim_{n\to\infty} \frac{1}{n^p} \sum_{j=1}^n \int_{\Omega_{(\sigma r_j^{(n)}, r_j^{(n)})}} \Big(\frac{|u|}{\delta_0}\Big)^p = 0. \tag{1.5}$$

For $p > 2$ we replace in the definition (1.5) by

$$\lim_{n\to\infty} \frac{1}{n^p} \sum_{j=1}^n \int_{\Omega_{(\sigma r_j^{(n)}, r_j^{(n)})}} \Big(\frac{|u|}{\delta_0}\Big)^p + \frac{1}{n^2} \sum_{j=1}^n \int_{\Omega_{(\sigma r_j^{(n)}, r_j^{(n)})}} |\nabla u|^{p-2}\Big(\frac{|u|}{\delta_0}\Big)^2 = 0. \tag{1.6}$$

Condition (C_0) is a growth condition for v near Γ_0 (it is trivially satisfied when $\Gamma_0 = \emptyset$). We now define analogously a growth condition at infinity (C_∞), in the case of unbounded Ω. As above, consider first the case $p \in (1,2]$ and a function $u \in W^{1,p}_{\text{loc}}(\Omega)$. We shall say that u satisfies condition (C_∞) if for some $0 < \tau < 1$ there exist for each $n \geq 2$, n positive numbers $\{\rho_j^{(n)}\}_{j=1}^n$ such that,

$$\rho_{j+1}^{(n)} \leq \tau \rho_j^{(n)}, \qquad j = 1, \ldots, n-1,$$

with $\lim_{n\to\infty} \rho_n^{(n)} = \infty$ and

$$\lim_{n\to\infty} \frac{1}{n^p} \sum_{j=1}^{n} \int_{\Omega^{(\tau\rho_j^{(n)},\rho_j^{(n)})}} \Big(\frac{|u|}{|x|}\Big)^p = 0. \tag{1.7}$$

For $p > 2$ we replace in the definition (1.7) by

$$\lim_{n\to\infty} \frac{1}{n^p} \sum_{j=1}^{n} \int_{\Omega^{(\tau\rho_j^{(n)},\rho_j^{(n)})}} \Big(\frac{|u|}{|x|}\Big)^p + \frac{1}{n^2} \sum_{j=1}^{n} \int_{\Omega^{(\tau\rho_j^{(n)},\rho_j^{(n)})}} |\nabla u|^{p-2} \Big(\frac{|u|}{|x|}\Big)^2 = 0. \tag{1.8}$$

Our main result is the following.

Theorem 1.3. *Consider Ω as above, $p \in (1,\infty)$ and $a(x) \in L^1_{loc}(\Omega)$. Suppose that $u, v \in C(\Omega \cup \Gamma_1) \cap W^{1,p}_{loc}(\Omega)$ are such that v is a super solution for (1.3), $v > 0$ on $\Omega \cup \Gamma_1$, u is a subsolution for (1.3) satisfying condition (C_0), if $\Gamma_0 \neq \emptyset$, and condition (C_∞), if Ω is unbounded. Assume also that*

$$v \geq u \quad \text{on } \Gamma_1. \tag{1.9}$$

Then,

$$v \geq u \qquad \text{in } \Omega. \tag{1.10}$$

2 Proof of the main result

For the proof we use two basic tools that we shall now recall. The first is a Picone identity for the p-Laplacian, which is due to Allegretto and Huang [3, Theorem 1.1]. We summarize it in the next lemma.

Lemma 2.1. *Let G be a domain in $\mathbb{R}^N$. For $u, v \in C(G) \cap W^{1,p}_{loc}(G)$ such that $u \geq 0$ and $v > 0$ in G, denote*

$$\begin{aligned} L(u,v) &= |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}\nabla u |\nabla v|^{p-2}\nabla v\,, \\ R(u,v) &= |\nabla u|^p - \nabla\Big(\frac{u^p}{v^{p-1}}\Big)|\nabla v|^{p-2}\nabla v\,. \end{aligned} \tag{2.1}$$

Then $L(u,v) = R(u,v)$ a.e. in G. Moreover, $L(u,v) \geq 0$ a.e. in G, and $L(u,v) = 0$ a.e in G if and only if $u = kv$ for some constant k.

The second tool is given by the following simple inequalities from [9, Lemma A.4].

Lemma 2.2. (i) *If $p \geq 2$ then*

$$|z_1 + z_2|^p - |z_1|^p - p|z_1|^{p-2} z_1 \cdot z_2 \leq \frac{p(p-1)}{2}(|z_1| + |z_2|)^{p-2}|z_2|^2, \quad \forall z_1, z_2 \in \mathbb{R}^N. \tag{2.2}$$

(ii) *If $p \leq 2$ then there exists a constant $\gamma_p > 0$ such that*

$$|z_1 + z_2|^p - |z_1|^p - p|z_1|^{p-2} z_1 \cdot z_2 \leq \gamma_p |z_2|^p, \quad \forall z_1, z_2 \in \mathbb{R}^N. \tag{2.3}$$

Proof of Theorem 1.3. Assume by negation that (1.10) does not hold. Then, the open set $\Omega^+ = \{x \in \Omega : u(x) > v(x)\}$ is nonempty. Put

$$K := \sup\{\log(u(x)) - \log(v(x)) : x \in \Omega, \, u(x) > v(x)\} \in (0, \infty], \tag{2.4}$$

and fix a positive constant b such that $5b < K$. Let $\theta \in C^1(\mathbb{R})$ be a non-decreasing function such that

$$\theta(t) = 0 \text{ for } t \leq 2b, \quad \theta(t) = 1 \text{ for } t \geq 5b \quad \text{and } \theta'(t) > 0 \text{ for } 3b \leq t \leq 4b. \tag{2.5}$$

Let

$$\xi = \chi_{\{u>v\}} \, \theta\big(\log(u/v)\big) \quad \text{in } \Omega. \tag{2.6}$$

If $\Gamma_0 \neq \emptyset$ then for any $n \geq 2$ we define,

$$f_n(r) = \begin{cases} 1 & \text{for } r \geq r_1^{(n)}, \\ (1 - j/n) + \frac{\log(r/r_j^{(n)}) - \log\sigma}{n|\log\sigma|} & \text{for } \sigma r_j^{(n)} \leq r \leq r_j^{(n)}, \quad 1 \leq j \leq n, \\ 1 - j/n & \text{for } r_{j+1}^{(n)} \leq r \leq \sigma r_j^{(n)}, \quad 1 \leq j \leq n-1, \\ 0 & \text{for } r \leq \sigma r_n^{(n)}, \end{cases} \tag{2.7}$$

where $\{r_j^{(n)}\}_{j=1}^n$ and σ are given by condition (C_0). If $\Gamma_0 = \emptyset$, simply set $f_n \equiv 1$. Similarly, if Ω is unbounded, using $\{\rho_j^{(n)}\}_{j=1}^n$ and τ given by condition (C_∞), we define,

$$g_n(r) = \begin{cases} 0 & \text{for } \rho \geq \rho_1^{(n)}, \\ j/n - \frac{\log(r/\rho_j^{(n)}) - \log\tau}{n|\log\tau|} & \text{for } \tau\rho_j^{(n)} \leq r \leq \rho_j^{(n)}, \quad 1 \leq j \leq n, \\ j/n & \text{for } \rho_{j+1}^{(n)} \leq r \leq \tau\rho_j^{(n)}, \quad 1 \leq j \leq n-1, \\ 1 & \text{for } r \leq \tau\rho_n^{(n)}. \end{cases} \tag{2.8}$$

In case Ω is bounded, simply set $g_n \equiv 1$. Clearly, both f_n and g_n are Lipschitz functions on $\mathbb{R}$. There exists n_0 such that:

(i) $\partial\Omega \subset B_{\rho_n^{(n)}}(0)$ for $n \geq n_0$.

(ii) If $\Gamma_0 \neq \emptyset$ then $\mathrm{dist}(\Gamma_1, \Gamma_0) > r_1^{(n)}$ for $n \geq n_0$.

For every $n > n_0$ we define a Lipschitz function ψ_n on $\mathbb{R}^N$ by

$$\psi_n(x) = f_n(\delta_0(x)) \cdot g_n(|x|). \tag{2.9}$$

Using (1.9) and our assumptions on τ and ψ_n we get that the nonnegative function

$$w := \Big(\frac{\psi_n^p u^p}{v^{p-1}}\Big)\,\xi$$

has a compact support in Ω^+. Therefore, $w \in W_0^{1,p}(\Omega)$ and we can use it as a test function in the inequality satisfied by v to get

$$\int\limits_{\Omega^+} \xi\,|\nabla v|^{p-2}\nabla v \cdot \nabla\Big(\frac{\psi_n^p u^p}{v^{p-1}}\Big) \geq \int\limits_{\Omega^+} a\psi_n^p u^p \xi - \int\limits_{\Omega^+} \psi_n^p u^p |\nabla \log v|^{p-2}\,\nabla \log v \cdot \nabla\xi\,. \tag{2.10}$$

By (2.10) we infer that

$$\begin{aligned} 0 \leq \int_{\Omega^+} L(\psi_n u, v)\,\xi \;&= \int_{\Omega^+} R(\psi_n u, v)\,\xi \\ &= \int_{\Omega^+} |\nabla(\psi_n u)|^p \xi \,-\int_{\Omega^+} \xi\,|\nabla v|^{p-2}\nabla v \cdot \nabla\Big(\frac{\psi_n^p u^p}{v^{p-1}}\Big) \\ &\leq \int_{\Omega^+} |\nabla(\psi_n u)|^p \xi \,-\int_{\Omega^+} a\psi_n^p u^p \xi \\ &\quad + \int_{\Omega^+} \psi_n^p u^p |\nabla \log v|^{p-2}\,\nabla \log v \cdot \nabla\xi\,, \end{aligned} \tag{2.11}$$

where $L(u,v)$ and $R(u,v)$ are defined in (2.1). Similarly, the nonnegative function

$$z(x) := \psi_n^p |u|\,\xi = \psi_n^p u \xi$$

has compact support in Ω^+ and therefore $z \in W_0^{1,p}(\Omega)$. Testing the inequality satisfied by u against z yields

$$\begin{aligned} 0 &\geq \int_{\Omega^+} (-\operatorname{div}(|\nabla u|^{p-2}\nabla u) - a|u|^{p-2}u)\psi_n^p u\xi \\ &= \int_{\Omega^+} \xi|\nabla u|^{p-2}\nabla u \cdot \nabla(\psi_n^p u) + \int_{\Omega^+} \psi_n^p u\,|\nabla u|^{p-2}\nabla u \cdot \nabla\xi \,-\int_{\Omega^+} a\psi_n^p u^p \xi\,. \end{aligned} \tag{2.12}$$

Subtracting (2.12) from (2.11) we finally obtain that

$$\begin{aligned} \int_{\Omega^+} \psi_n^p u^p \left[|\nabla \log u|^{p-2}\,\nabla \log u - |\nabla \log v|^{p-2}\,\nabla \log v\right] \cdot \nabla\xi \;\leq \\ \int_{\Omega^+} \left[|\nabla(\psi_n u)|^p - |\nabla u|^{p-2}\nabla u \cdot \nabla(\psi_n^p u)\right]\xi\,. \end{aligned} \tag{2.13}$$

Next we note that

$$\begin{aligned} &|\nabla(\psi_n u)|^p - |\nabla u|^{p-2}\nabla u \cdot \nabla(\psi_n^p u) \\ &\quad = |\psi_n \nabla u + u\nabla\psi_n|^p - p|\psi_n \nabla u|^{p-2}(\psi_n \nabla u)\cdot(u\nabla\psi_n) - |\psi_n \nabla u|^p. \end{aligned}$$

Therefore from Lemma 2.2 we infer that

$$|\nabla(\psi_n u)|^p - |\nabla u|^{p-2}\nabla u \cdot \nabla(\psi_n^p u) \le \begin{cases} \gamma_p u^p |\nabla \psi_n|^p & \text{if } p \in (1,2], \\ \\ C\big(u^p|\nabla\psi_n|^p + (u|\nabla\psi_n|)^2|\nabla u|^{p-2}\big) & \text{if } p > 2. \end{cases} \tag{2.14}$$

Assume first that $p \in (1,2]$. Using (2.13) and (2.14) we infer,

$$\begin{aligned} &\int_{\Omega^+} \psi_n^p u^p \left[|\nabla \log u|^{p-2}\,\nabla \log u - |\nabla \log v|^{p-2}\,\nabla \log v\right] \cdot \nabla\xi \\ &\quad \le C\sum_{j=1}^{n}\int_{\Omega_{(\sigma r_j^{(n)}, r_j^{(n)})}} u^p|\nabla(f_n(\delta_0))|^p + C\sum_{j=1}^{n}\int_{\Omega^{(\tau\rho_j^{(n)}, \rho_j^{(n)})}} u^p|\nabla(g_n(|x|))|^p \\ &\quad \le \frac{C}{n^p}\sum_{j=1}^{n}\int_{\Omega_{(\sigma r_j^{(n)}, r_j^{(n)})}} \Big(\frac{|u|}{\delta_0}\Big)^p + \frac{C}{n^p}\sum_{j=1}^{n}\int_{\Omega^{(\tau\rho_j^{(n)}, \rho_j^{(n)})}} \Big(\frac{|u|}{|x|}\Big)^p \to 0, \text{ as } n\to\infty, \end{aligned}$$

where we used (1.5) and (1.7). The computation in the case $p > 2$ is almost identical; we need only to use (1.6) and (1.8) instead of (1.5) and (1.7). Hence in all cases we obtain that

$$\limsup_{n\to\infty}\int_{\Omega^+} \psi_n^p u^p \left[|\nabla \log u|^{p-2}\,\nabla \log u - |\nabla \log v|^{p-2}\,\nabla \log v\right] \cdot \nabla\xi \le 0. \tag{2.15}$$

Since $\psi_n(x) \to 1$ a.e. in Ω we get from (2.15) and Fatou Lemma that

$$\int_{\Omega^+} u^p \left[|\nabla \log u|^{p-2}\,\nabla \log u - |\nabla \log v|^{p-2}\,\nabla \log v\right] \cdot \nabla\xi \le 0\,. \tag{2.16}$$

Note that in Ω^+

$$\nabla\xi = \theta'\big(\log(u/v)\big)\,(\nabla \log u - \nabla \log v),$$

and that

$$\big(|z_1|^{p-2}z_1 - |z_2|^{p-2}z_2\big)\cdot(z_1 - z_2) \ge 0, \quad \forall z_1, z_2 \in \mathbb{R}^N,$$

with equality if and only if $z_1 = z_2$ (by the strict convexity of the function $f(z) = |z|^p$). Therefore equality holds in (2.16) and consequently, from (2.5) we deduce that $u \equiv c_i v$ on each component E_i of the set $\{x \in \Omega;\ \log(u(x)/v(x)) \in (3b, 4b)\}$, for some positive constant c_i. But this is clearly a contradiction since the image of the function $\log(u/v)$ contains the interval $(3b, 4b)$ by (1.9) and (2.4). $\square$

3 Some applications of the comparison principle

This section is devoted to some consequences of Theorem 1.3. We first show how it can be used to deduce improvements of Theorem 1.2, Theorem 1.1 and [9, Proposition A.1]. We start with a simple lemma.

Lemma 3.1. *Let Ω with boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ be as in the previous section and let $u \in W^{1,p}_{loc}(\Omega)$. If for some $\sigma \in (0,1)$,*

$$\begin{aligned} &\liminf_{r\to 0} \int_{\Omega_{(\sigma r, r)}} \Big(\frac{|u|}{\delta_0}\Big)^p < \infty, \qquad \text{if } p \in (1,2], \\ &\liminf_{r\to 0} \int_{\Omega_{(\sigma r, r)}} \Big(\frac{|u|}{\delta_0}\Big)^p + |\nabla u|^{p-2}\Big(\frac{|u|}{\delta_0}\Big)^2 < \infty, \qquad \text{if } p \in (2,\infty), \end{aligned} \tag{3.1}$$

then u satisfies condition (C_0). Similarly, if for some $\tau \in (0,1)$,

$$\begin{aligned} &\liminf_{\rho\to \infty} \int_{\Omega(\tau\rho, \rho)} \Big(\frac{|u|}{|x|}\Big)^p < \infty, \qquad \text{if } p \in (1,2], \\ &\liminf_{\rho\to \infty} \int_{\Omega(\tau\rho, \rho)} \Big(\frac{|u|}{|x|}\Big)^p + |\nabla u|^{p-2}\Big(\frac{|u|}{|x|}\Big)^2 < \infty, \qquad \text{if } p \in (2,\infty), \end{aligned}$$

then u satisfies condition (C_∞).

Proof. It is enough to prove the first assertion for $p \in (1,2]$; all the other cases are similar. By assumption we can find a sequence $\{r_n\}_{n=1}^{\infty}$ such that $r_n \searrow 0$, $r_{n+1} \le \sigma r_n$, $\forall n$ and

$$a_n := \int_{\Omega_{(\sigma r_n, r_n)}} \Big(\frac{|u|}{\delta_0}\Big)^p \le C, \ \forall n.$$

Clearly $\lim_{n\to\infty} \frac{1}{n^p} \sum_{j=1}^{n} a_{n+j} = 0$, so setting $r_j^{(n)} = r_{n+j}$ for all $n \ge 1$ and $1 \le j \le n$ we see that (1.5) is satisfied. □

An immediate consequence is the following improvement of Theorem 1.1.

Corollary 3.1. *Let $a(x) \in L^1_{loc}(\mathbb{R}^N \setminus \{0\})$, $R > 0$, $\Omega_R := \{|x| > R\}$ and let $u, v \in C(\overline{\Omega}_R) \cap W^{1,p}_{loc}(\Omega_R)$ be respectively a sub and super solutions for (1.3) on $\Omega = \Omega_R$. Assume that $v > 0$ on $\overline{\Omega}_R$, u satisfies condition (C_∞), and that*

$$v(x) \ge u(x) \qquad \text{on } \{|x| = R\}.$$

Then, $v \ge u$ in Ω_R. In particular, the result holds for u which satisfies for some $\tau \in (0,1)$,

$$\begin{aligned} &\liminf_{\rho\to \infty} \int_{\{\tau\rho < |x| < \rho\}} \Big(\frac{|u|}{|x|}\Big)^p < \infty, \qquad \text{if } p \in (1,2], \\ &\liminf_{\rho\to \infty} \int_{\{\tau\rho < |x| < \rho\}} \Big(\frac{|u|}{|x|}\Big)^p + |\nabla u|^{p-2}\Big(\frac{|u|}{|x|}\Big)^2 < \infty, \qquad \text{if } p \in (2,\infty). \end{aligned} \tag{3.2}$$

Proof. The assertions follow easily from Theorem 1.3 and Lemma 3.1 applied with $\Omega = \Omega_R$, $\Gamma_1 = \{|x| = R\}$ and $\Gamma_0 = \emptyset$. □

In the next proposition we present some common situations in which condition (C_0) is satisfied.

Proposition 3.1. *The function u satisfies condition (C_0) if either of the following holds:*

(i) Γ_0 is a compact Lipschitz hypersurface and for some $0 < r < dist(\Gamma_0, \Gamma_1)$ we have: $u \in W^{1,p}(\Omega_r)$ with $\Omega_r := \{x \in \Omega : \delta_0(x) < r\}$ and for some $\Psi \in C^\infty(\Omega)$ such that $\Psi \equiv 0$ on $\Omega \setminus \Omega_{r/2}$ and $\Psi \equiv 1$ on $\Omega_{r/3}$, we have $\Psi u \in W_0^{1,p}(\Omega_r)$.
(ii) $\Gamma_0 = \{x_0\}$ and for some $0 < r < dist(x_0, \Gamma_1)$, and $\Psi \in C_c^\infty(B_r(x_0))$ such that $\Psi \equiv 1$ on $B_{r/2}(x_0)$, we have $\Psi u \in W_0^{1,p}(B_r(x_0) \setminus \{x_0\})$.

Proof. (i) Since Hardy inequality holds for domains with Lipschitz boundary (see [7]) we have, for some constant $c > 0$,

$$\int_{\Omega_r} |\nabla(\Psi u)|^p \geq c \int_{\Omega_r} (|\Psi u|/\delta_0)^p. \tag{3.3}$$

It follows from (3.3) that u/δ_0 is an L^p function in a neighborhood of Γ_0. In case $p \in (1,2]$ this clearly implies (3.1) (in this case the limit in (3.1) is actually 0). In case $p > 2$, we have by Hölder inequality,

$$\int_{\Omega_{r/3}} |\nabla u|^{p-2}(|u|/\delta_0)^2 \leq \Big(\int_{\Omega_{r/3}} |\nabla u|^p\Big)^{\frac{p-2}{p}} \cdot \Big(\int_{\Omega_{r/3}} (|u|/\delta_0)^p\Big)^{\frac{2}{p}},$$

and (3.1) follows in this case too. By Lemma 3.1 we deduce that condition (C_0) holds.

(ii) Without loss of generality we can assume that $x_0 = 0$, hence $\delta_0(x) = |x|$. Put $w = \Psi u$ and $w_1 = u(1 - \Psi)$ on $B_r(0)$. Assume first that $p \neq N$. Since $w \in W_0^{1,p}(B_r(0))$ we get from Hardy inequality (see [7]),

$$\int_{B_r(0)} |\nabla w|^p \geq |1 - N/p|^p \int_{B_r(0)} (|w|/|x|)^p, \tag{3.4}$$

so that $w/|x| \in L^p(B_r(0))$. Since clearly $w_1/|x| \in L^p(B_r(0))$ it follows that $u/|x| \in L^p(B_r(0))$. But we have also $\nabla u \in L^p(B_r(0))$. We deduce thus as above that (3.1) holds, hence condition (C_0) is satisfied. In case $p = N$ we argue as above, but with (3.4) replaced by the following improved Hardy inequality due to Adimurthi, Chaudhuri and Ramaswamy [1, Theorem 1.1],

$$\int_{B_r(0)} |\nabla w|^p \geq C \int_{B_r(0)} (|w|/|x|)^p \, (\log R/|x|)^{-p}, \tag{3.5}$$

with $R \geq re^{2/p}$. It follows then that,

$$\frac{u}{|x| \log(R/|x|)} \in L^p(B_r(0)). \tag{3.6}$$

In order to show that condition (C_0) is satisfied we take $\sigma = 1/2$ and set,

$$r_j^{(n)} = (1/2)^{n+j-1}, \quad j = 1, \ldots, n,$$

for $n > -\frac{\log r}{\log 2}$. Since we always assume that $p > 1$ we have $p = N \geq 2$. By Young inequality,

$$\Big(\frac{|u|}{n|x|}\Big)^2 |\nabla u|^{p-2} \leq (2/p) \, \frac{|u|^p}{n^p |x|^p} + \big(1 - 2/p\big)|\nabla u|^p.$$

Therefore,

$$\frac{1}{n^p}\sum_{j=0}^{n-1}\int_{\{2^{-(n+j+1)}<|x|<2^{-(n+j)}\}}(|u|/|x|)^p+\frac{1}{n^2}\sum_{j=0}^{n-1}\int_{\{2^{-(n+j+1)}<|x|<2^{-(n+j)}\}}|\nabla u|^{p-2}(|u|/|x|)^2 \le$$
$$\frac{C}{n^p}\int_{\{2^{-2n}<|x|<2^{-n}\}}(|u|/|x|)^p+C\int_{\{2^{-2n}<|x|<2^{-n}\}}|\nabla u|^p := I_1^{(n)}+I_2^{(n)}.$$

Since $\nabla u \in L^p(B_r(0))$ it is clear that $\lim_{n\to\infty} I_2^{(n)} = 0$. Further,

$$I_1^{(n)} \le \frac{C}{n^p}(\log R + 2n\log 2)^p \int_{\{2^{-2n}<|x|<2^{-n}\}}\Big(\frac{|u|}{|x|\log(R/|x|)}\Big)^p \to 0 \text{ (by (3.6))}.$$

It follows that condition (C_0) is satisfied. □

Remark 3.1. *A comparison lemma related to (ii) of Proposition 3.1 was proved by Sreenadh [10].*

Next we prove a uniqueness result for global solutions.

Theorem 3.1. *Let $a(x) \in L^1_{loc}(\mathbb{R}^N \setminus \{0\})$ and suppose that $u \in C(\mathbb{R}^N \setminus \{0\}) \cap W^{1,p}_{loc}(\mathbb{R}^N \setminus \{0\})$ is a positive solution of the equation (1.3) in $\Omega = \mathbb{R}^N \setminus \{0\}$ which satisfies condition (3.2), and for some $\sigma \in (0,1)$,*

$$\begin{aligned}\liminf_{r\to 0}\int_{\{\sigma r<|x|<r\}}(|u|/|x|)^p < \infty, \qquad &\text{if } p\in(1,2],\\ \liminf_{r\to 0}\int_{\{\sigma r<|x|<r\}}(|u|/|x|)^p+|\nabla u|^{p-2}(|u|/|x|)^2 < \infty, \qquad &\text{if } p\in(2,\infty),\end{aligned} \tag{3.7}$$

or more generally, assume u satisfies conditions (C_∞) and (C_0) with $\Gamma_0 = \{0\}$. Let $v \in C(\mathbb{R}^N \setminus \{0\}) \cap W^{1,p}_{loc}(\mathbb{R}^N \setminus \{0\})$ be another positive solution to (1.3). Then, $v = cu$ for some constant $c > 0$.

(3.8)

Proof. Consider any $x_1 \in \mathbb{R}^N \setminus \{0\}$ and set $\alpha := v(x_1)/u(x_1)$. Put $\Omega_1 = \mathbb{R}^N \setminus \{0, x_1\}$ and apply Theorem 1.3 for $\Omega = \Omega_1$ and $\Gamma_1 = \{x_1\}$ to deduce that $v(x)/u(x) \ge \alpha_1 = v(x_1)/u(x_1)$ for every $x \ne 0$. Since x_1 and x are arbitrary, we deduce that $u/v \equiv c$. □

Remark 3.2. *Theorem 3.1 generalizes Theorem 1.3 of [8] which treated the special case $a(x) = |1 - N/p|^p|x|^{-p}$ in which a positive solution to (1.3) is given explicitly by $u(x) = |x|^{1-N/p}$.*

We close with a uniqueness result in bounded domains.

Theorem 3.2. *Let Ω be a bounded domain with boundary Γ_0. Let $a(x) \in L^1_{loc}(\Omega)$ and suppose that $u \in C(\Omega) \cap W^{1,p}_{loc}(\Omega)$ is a positive solution of (1.3) which satisfies (3.1), or more generally, condition (C_0). Let $v \in C(\Omega) \cap W^{1,p}_{loc}(\Omega)$ be another positive solution to (1.3). Then, $v = cu$ for some constant $c > 0$.*

Proof. Simply repeat the argument of Theorem 3.1, this time applying Theorem 1.3 to the domain $\Omega \setminus \{x_1\}$, whose boundary is the disjoint union of $\Gamma_0 = \partial\Omega$ and $\Gamma_1 = \{x_1\}$, for any $x_1 \in \Omega$. □

Remark 3.3. *Let Ω be a bounded domain of class C^2 and let $\eta \in C(\overline{\Omega})$ satisfy $\eta > 0$ in Ω and $\eta = 0$ on $\partial\Omega$. In [6, Theroem 1.2] it was shown that for a certain critical value λ^*, there exists a positive solution u_* to the equation,*

$$-\Delta_p u_* = \big(\lambda^* \eta + (1 - 1/p)^p\big) \frac{|u_*|^{p-2} u_*}{\delta^p}, \tag{3.9}$$

which satisfies the growth condition,

$$C^{-1} \delta^{1-1/p} \le u_* \le C \delta^{1-1/p}, \text{ in } \Omega, \quad \text{for some } C > 0, \tag{3.10}$$

where we denoted $\delta(x) = dist(x, \partial\Omega)$. It is easy to show that (3.10) *implies* (3.1) *(with $\delta_0 = \delta$). Indeed, in case $p \in (1, 2]$ this is immediate, while for $p > 2$ we need only to remark that by [9, Lemma A.3],* (3.10) *implies that $|\nabla u_*| \le C \delta^{-1/p}$ in Ω, and the conclusion follows as well. Therefore we infer from Theorem 3.2 that u_* is actually the* unique *positive solution to* (3.9)*, up to a multiplicative factor.*

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