A nonlocal problem arising in the study of magneto-elastic interactions

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Abstract

The energy of magneto-elastic materials is described by a nonconvex functional. Three terms of the total free energy are taken into account: the exchange energy, the elastic energy and the magneto-elastic energy usually adopted for cubic crystals. We focus our attention to a one dimensional penalty problem and study the gradient flow of the associated type Ginzburg-Landau functional. We prove the existence and uniqueness of a classical solution which tends asymptotically for subsequences to a stationary point of the energy functional.

1 Introduction

The paper deals with the analysis of the equation

$$\frac{d\mathbf{u}}{dt} = -\operatorname{grad} F(\mathbf{u}) \tag{1.1}$$

where $F(\mathbf{u})$ is a type Ginzburg-Landau functional, associated to the energy of a magneto-elastic material, which contains a nonlinear nonlocal term. The derivation of the energy functional $F(\mathbf{u})$ is detailed in the next section starting from a general 3D-model depending on the displacements and the magnetization and assuming some

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simplifications. In particular in one-dimensional case the energy functional can be expressed in terms of the only magnetization variable and the equation (1.1) reduces to the following one

$$\mathbf{u}_t = \mathbf{u}_{xx} - \varepsilon^{-1}(|\mathbf{u}|^2 - 1)\mathbf{u} + \mu\Lambda(\mathbf{u})[\Lambda(\mathbf{u})\cdot\mathbf{u} - \int_0^1 \Lambda(\mathbf{u})\cdot\mathbf{u}dx] \tag{1.2}$$

where $\mathbf{u} = (u_1, u_2)$ and $\Lambda(\mathbf{u}) = (u_2, u_1)$.

The parameter μ couples the elastic and magnetic processes and ε is the small positive parameter introduced to relax the constrain $|\mathbf{u}| = 1$.

To the equation (1.2) we assumed associated the boundary and initial conditions

$$\mathbf{u}_x(0,t) = \mathbf{u}_x(1,t) = 0, \qquad \mathbf{u}(\mathbf{x},0) = \mathbf{u}^0(\mathbf{x}) \tag{1.3}$$

Firstly we study the minimization problem related to the energy functional $F(\mathbf{u})$, then we provide the existence and uniqueness of the solution $\mathbf{u}$ to (1.2), (1.3) we show that $\lim_{t\to\infty}\mathbf{u}(t) = \mathbf{u}_\infty$ exists and the function $\mathbf{u}_\infty$ is a stationary point of the energy functional. The minimization problem presents some interesting features. For μ small enough the absolute minimum of the energy functional is zero and it is achieved only by one-modulus constants (trivial solutions). Moreover there exists a value μ^* such that for $\mu \geq \mu^*$ the global minimum is negative and it is achieved only by non trivial functions. The characterization of the critical value μ^* is the at the moment an open problem.

The rest of the paper is devoted to the analysis of the dynamical problem (1.2), (1.3) with the aim also to detect a procedure to reach nontrivial critical points of the energy functional.

2 The model

The behaviour of a magnetoelastic material is described by a system of differential equations in the two unknowns: the displacement vector and the magnetization vector. Let $\Omega \subset \mathbb{R}^3$ the volume of the magnetoelastic material and $\partial\Omega$ its boundary, the unknown magnetization vector $\mathbf{m}$ is a map from Ω to S^2 (the unit sphere of $\mathbb{R}^3$). The magnetization distribution is well described by a free energy functional which we assume composed of three terms namely the *exchange* energy $E_{\rm ex}$, the *elastic* energy $E_{\rm el}$ and the *elastic-magnetic* energy $E_{\rm em}$. Let $\mathbf{v}$ be the displacement vector, then the total free energy E for a deformable magnetoelastic material is given by

$$E(\mathbf{m},\mathbf{v}) \;=\; E_{\rm ex}(\mathbf{m}) \;+\; E_{\rm em}(\mathbf{m},\mathbf{v}) \;+\; E_{\rm el}(\mathbf{v})$$

We neglect here other contributions to the free energy due, for example, to anisotropy and demagnetization energy terms.

We refer to the books [1], [2]; moreover among the papers on this subject we quote [3], [4], [5], [6]. In the sequel we detail the three energetic terms and derive the governing differential equations. Some drastic hypotheses allows us to reach a reduced one dimensional problem and to carry out the variational analysis for the associated energy functional.

2.1 The general 3D model

Let x_i, $i = 1, 2, 3$ be the position of a point $\mathbf{x}$ of Ω and denote by

$$v_i = v_i(\mathbf{x}), \qquad i = 1, 2, 3$$

the components of the displacement vector $\mathbf{v}$ and by

$$\epsilon_{kl}(\mathbf{v}) = \frac{1}{2}(v_{k,l} + v_{l,k}), \qquad i, j = 1, 2, 3$$

the deformation tensor where, as a common praxis, $v_{k,l}$ stands for $\dfrac{\partial v_k}{\partial v_l}$.

Moreover we denote by

$$m_j = m_j(\mathbf{x}), \qquad j = 1, 2, 3$$

the component of the unit magnetization vector $\mathbf{m}$. In the sequel, where not specified, the Latin indices vary in the set {1,2,3} and the summation of the repeated indices is assumed. We define

$$E_{\text{ex}}(\mathbf{m}) = \frac{1}{2}\int_\Omega a_{ij} m_{k,i} m_{k,j} d\Omega \tag{2.1}$$

where (a_{ij}) is a symmetric positive definite matrix which is supposed diagonal for most materials with all diagonal elements equal to a positive number a. The magneto-elastic energy for cubic crystals is assumed, that implies

$$E_{\text{em}}(\mathbf{m}, \mathbf{v}) = \frac{1}{2}\int_\Omega \lambda_{ijkl} m_i m_j \epsilon_{kl}(\mathbf{v}) d\Omega \tag{2.2}$$

where $\lambda_{ijkl} = \lambda_1 \delta_{ijkl} + \lambda_2 \delta_{ij}\delta_{kl} + \lambda_3(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ with $\delta_{ijkl} = 1$ if $i = j = k = l$ and $\delta_{ijkl} = 0$ otherwise. Finally we introduce the elastic energy

$$E_{\text{el}}(\mathbf{v}) = \frac{1}{2}\int_\Omega \sigma_{klmn} \epsilon_{kl}(\mathbf{v}) \epsilon_{mn}(\mathbf{v}) d\Omega \tag{2.3}$$

where σ_{klmn} is the elasticity tensor satisfying the following symmetry property

$$\sigma_{klmn} = \sigma_{mnkl} = \sigma_{lkmn}$$

and moreover the inequality

$$\sum(\sigma_{klmn}\epsilon_{kl}\epsilon_{mn}) \geq \beta \sum |\epsilon_{kl}|^2$$

holds for some $\beta > 0$.

We consider the energy functional E is given by

$$E(\mathbf{m},\mathbf{v}) \;=\; E_{\rm ex}(\mathbf{m}) \;+\; E_{\rm em}(\mathbf{m},\mathbf{v}) \;+\; E_{\rm el}(\mathbf{v}) \tag{2.4}$$

We introduce two tensors $\mathcal{S} = \sigma_{ijkl}\epsilon_{ij}$ and $\mathcal{L} = \lambda_{ijkl}m_im_j$, moreover we denote $\mathbf{p}$ the vector $\mathbf{p} = \lambda_{ijkl}m_j\epsilon_{kl}m_i$
The system of differential equations associated to the functional (2.4) reads

$$\begin{cases} \operatorname{div}(\mathcal{S} + \frac{1}{2}\mathcal{L}) = 0 & \text{in } \Omega \\ \\ a\Delta\mathbf{m} - \mathbf{p} + (a|\nabla\mathbf{m}|^2 + \mathbf{p}\cdot\mathbf{m})\mathbf{m} = 0, & \text{in } \Omega \end{cases} \tag{2.5}$$

with boundary conditions

$$\mathbf{v} = 0, \qquad \frac{\partial \mathbf{m}}{\partial \nu} = 0 \qquad \text{on} \quad \partial\Omega \tag{2.6}$$

where ν is the outer unit normal at the boundary $\partial\Omega$.

An alternative form for describing the magnetoelastic interactions (2.5) is

$$\begin{cases} \operatorname{div}(\mathcal{S} + \frac{1}{2}\mathcal{L}) = 0 & \text{in } \Omega \\ \\ \mathbf{m}\times(a\Delta\mathbf{m} - \mathbf{p}) = 0, \quad |\mathbf{m}| = 1 & \text{in } \Omega \end{cases} \tag{2.7}$$

2.2 The proposed 1D problem

A simplified model can be obtain assuming that Ω is a subset of $\mathbb{R}$ and neglecting some components of the unknowns $\mathbf{v}$ and $\mathbf{m}$. More precisely we consider the single space variable x and assume $\Omega = (0,1)$, $\mathbf{v} = (0,w,0)$ and $\mathbf{m} = (m_1, m_2, 0,)$. Moreover we put $a = 1$, and the only elements of the tensors σ_{ijkl} and λ_{ijkl} which live on equal to 1 and λ respectively. The above system (2.5) reduces to

$$\begin{cases} w_{xx} + \frac{1}{2}\lambda\,(\Lambda(\mathbf{m})\cdot\mathbf{m})_x = 0 \\ \\ \mathbf{m}_{xx} - \lambda\Lambda(\mathbf{m})w_x + (|\mathbf{m}_x|^2 + \lambda\Lambda(\mathbf{m})\cdot\mathbf{m}w_x)\mathbf{m} = 0 \end{cases} \tag{2.8}$$

where Λ is a linear operator such that $\Lambda(\mathbf{m}) = (m_2, m_1)$. Formally solving the first equation with the boundary condition $(2.6)_1$ one gets to the following functional equation for the unknown $\mathbf{m}$,

$$\mathbf{m}_{xx} - \mu\Lambda(\mathbf{m})\,G(\mathbf{m}) + \big(|\mathbf{m}_x|^2 + \mu\Lambda(\mathbf{m})\cdot\mathbf{m}\,G(\mathbf{m})\big)\;\mathbf{m} = 0 \tag{2.9}$$

where $\mu = \lambda^2/2$ and

$$G(\mathbf{m}) = \Lambda(\mathbf{m})\cdot\mathbf{m} - \int_0^1 \Lambda(\mathbf{m})\cdot\mathbf{m}dx$$

Our aim is to study the problem (2.9) with the boundary condition

$$\mathbf{m}_x(0) = \mathbf{m}_x(1) = 0 \tag{2.10}$$

We propose to relax the constraint $|\mathbf{m}| = 1$ and introduce a penalty version of the previous system (2.8), (2.6) depending on a small positive parameter ε that is

$$\begin{cases} \mathbf{m}^\varepsilon_{xx} - \lambda\Lambda(\mathbf{m}^\varepsilon) w_x - \varepsilon^{-1}(|\mathbf{m}^\varepsilon|^2 - 1)\mathbf{m}^\varepsilon = 0 \\ w^\varepsilon_{xx} + \frac{1}{2}\lambda\left(\Lambda(\mathbf{m}^\varepsilon)\cdot\mathbf{m}^\varepsilon\right)_x = 0 \end{cases} \tag{2.11}$$

We get to the following penalty nonlocal equation

$$\mathbf{m}^\varepsilon_{xx} - \varepsilon^{-1}(|\mathbf{m}^\varepsilon|^2 - 1)\mathbf{m}^\varepsilon + \mu\,\Lambda(\mathbf{m}^\varepsilon)[\Lambda(\mathbf{m}^\varepsilon)\cdot\mathbf{m}^\varepsilon - \int_0^1 \Lambda(\mathbf{m}^\varepsilon)\cdot\mathbf{m}^\varepsilon dx] = 0 \tag{2.12}$$

with boundary conditions

$$\mathbf{m}^\varepsilon_x(0) = \mathbf{m}^\varepsilon_x(1) = 0 \tag{2.13}$$

Here we focus our attention at the analysis of the equations (2.12)–(2.13).

3 The minimization problem

The equation (2.12) is the Euler-Lagrange equation of the energy functional

$$F(\mathbf{m}) = F_{\mu,\varepsilon}(\mathbf{m}) = \frac{1}{2}\int_0^1 |\mathbf{m}_x|^2\,dx + \frac{\varepsilon^{-1}}{4}\int_0^1 |\,|\mathbf{m}|^2 - 1|^2\,dx$$
$$- \frac{\mu}{4}\left[\int_0^1 (\Lambda(\mathbf{m})\cdot\mathbf{m})^2\,dx - (\int_0^1 \Lambda(\mathbf{m})\cdot\mathbf{m}\,dx)^2\right] \tag{3.1}$$

We consider the minimization problem

$$\mathcal{F}_{\mu,\varepsilon} = \inf_{\mathbf{m}\in\mathbf{H}^1(0,1)} F(\mathbf{m})\,. \tag{3.2}$$

Theorem 3.1 *For each μ and for each positive ε small enough, i.e., such that $\varepsilon^{-1}-\mu > 0$, the minimum of the functional $F(\mathbf{m})$ is achieved by a function $\mathbf{m}^\varepsilon \in \mathbf{H}^1(0,1)$. Furthermore, $\mathbf{m}^\varepsilon$ is a solution (2.12)–(2.13) and is therefore of class C^∞.*

Proof. First of all we observe that

$$\left(\int_0^1 \Lambda(\mathbf{m})\cdot\mathbf{m}\,dx\right)^2 \le \int_0^1 (\Lambda(\mathbf{m})\cdot\mathbf{m})^2 dx \le \int_0^1 |\mathbf{m}|^4\,dx =$$

$$= \int_0^1 (|\mathbf{m}|^2 - 1 + 1)^2\,dx \le (1 + \frac{1}{\delta}) + (1+\delta)\int_0^1 (|\mathbf{m}|^2 - 1)^2 dx$$

So we have:
(i) If $\varepsilon^{-1} - \mu > 0$ then for δ small enough $\varepsilon^{-1} - (1+\delta)\mu \geq 0$ and the functional $F(\mathbf{m})$ is bounded from below. Indeed,

$$F(\mathbf{m}) \geq \frac{1}{2}\int_0^1 |\mathbf{m}_x|^2 dx + \frac{\varepsilon^{-1} - (1+\delta)\mu}{4}\int_0^1 (|\mathbf{m}|^2 - 1)^2 dx - (1+\frac{1}{\delta})\frac{\mu}{4} \geq -\frac{\mu}{2}$$

(ii) The functional $F(\mathbf{m})$ is coercive, i.e.,

$$F(\mathbf{m}) \to +\infty, \qquad \text{as} \quad \|\mathbf{m}\|_{\mathbf{H}^1(0,1)} \to \infty\,.$$

This follows easily from the inequality $(|\mathbf{m}|^2 - 1)^2 \geq |\mathbf{m}|^2 - 5/4$.
(iii) The functional is weakly lower semicontinuous, that is: if $\{\mathbf{m}_n\}$ is a sequence of functions in $\mathbf{H}^1(0,1)$ such that $\mathbf{m}_n \rightharpoonup \mathbf{m}$ weakly in $\mathbf{H}^1(0,1)$, then

$$\liminf_{n\to\infty} F(\mathbf{m}_n) \geq F(\mathbf{m})\,.$$

Indeed, for such a weakly convergent sequence we have

$$\int_0^1 |\mathbf{m}_x|^2\, dx \leq \liminf_{n\to\infty} \int_0^1 |(\mathbf{m}_n)_x|^2\, dx\,,$$

$|\mathbf{m}_n|^2 \to |\mathbf{m}|^2$ and $\Lambda(\mathbf{m}_n)\cdot\mathbf{m}_n \to \Lambda(\mathbf{m})\cdot\mathbf{m}$ strongly in $L^2(0,1)$.

Since the functional (3.1) is C^1, it follows that the stationary points of F are solutions to the Euler-Lagrange equations (2.12)–(2.13), and it is easily verified that any solution to this one-dimensional problem is of class C^∞. □

Remark 3.2 *The result is sharp since for $\varepsilon > \frac{1}{\mu}$ then F is unbounded from below. Indeed suppose that $1 - \frac{1}{\mu\varepsilon} > 0$. Consider f the following function $f = (\delta - x)^+$. One has*

$$\Big(\int_0^1 f^2\Big)^2 \Big/ \int_0^1 f^4 = \Big(\int_0^\delta (\delta - x)^2\Big)^2 \Big/ \int_0^\delta (\delta - x)^4 = \frac{\delta^6}{9}\Big/\frac{\delta^5}{5} = \frac{5}{9}\delta < 1 - \frac{1}{\mu\varepsilon}$$

for δ small enough. So we choose δ small enough such that

$$\Big(\int_0^1 f^2\Big)^2 < \Big(1 - \frac{1}{\mu\varepsilon}\Big)\int_0^1 f^4.$$

Next we consider $m = \alpha f(x)(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Then we have

$$\begin{aligned} F(\mathbf{m}) &= \frac{1}{2}\alpha^2\int_0^1 f'(x)^2 + \frac{1}{4\varepsilon}\int_0^1 (\alpha^2 f(x)^2 - 1)^2 - \frac{\mu}{4}\int_0^1 \alpha^4 f^4 + \frac{\mu}{4}\Big(\int_0^1 \alpha^2 f^2\Big)^2 \\ &= \frac{1}{2}\alpha^2\int_0^1 f'(x)^2 + \frac{\alpha^4}{4}\mu\Big\{\frac{1}{\varepsilon\mu}\int_0^1 \Big(f^2 - \frac{1}{\alpha^2}\Big)^2 - \int_0^1 f^4 + \Big(\int_0^1 f^2\Big)^2\Big\}. \end{aligned}$$

For α large enough one has

$$\frac{1}{\varepsilon\mu}\int_0^1 (f^2-\frac{1}{\alpha^2})^2 - \int_0^1 f^4 + (\int_0^1 f^2)^2$$

close to $(\frac{1}{\varepsilon\mu}-1)\int_0^1 f^4 + (\int_0^1 f^2)^2 < 0$ *and thus* $F(\mathbf{m}) \to -\infty$ *when* $\alpha \to +\infty$.

The functional $F(\mathbf{m})$ has some obvious symmetry properties. We have clearly $F(\mathcal{S}_i(\mathbf{m})) = F(\mathbf{m})$ for each $\mathcal{S}_i$ in the group

$$\mathcal{G} = \{\mathcal{S}_0, \dots, \mathcal{S}_7\} \tag{3.3}$$

generated by the rotation by $\pi/2$ and the complex conjugation.

Lemma 3.3 *Let* $\mathbf{m}$ *be a solution of the problem* (2.12))–(2.13) *which verifies* $F(\mathbf{m}) \le 0$. *Then, for each* ε *such that* $\varepsilon^{-1} \ge 2\mu$ *the following a-priori estimate holds,*

$$|\mathbf{m}|^2 \;\le\; K \;:=\; \frac{\varepsilon^{-1} + \mu\sqrt{\frac{\varepsilon^{-1}}{\varepsilon^{-1}+\mu}}}{\varepsilon^{-1}-\mu}. \tag{3.4}$$

PROOF. By the assumption on $\mathbf{m}$ we have

$$\frac{\varepsilon^{-1}}{4}\int_0^1 |\,|\mathbf{m}|^2-1|^2\,dx + \frac{\mu}{4}\left[(\int_0^1 \Lambda(\mathbf{m})\cdot\mathbf{m}\,dx)^2 - \int_0^1 (\Lambda(\mathbf{m})\cdot\mathbf{m})^2\,dx\right] \le 0\,.$$

Combining it with the inequality $\int_0^1 |\mathbf{m}|^4\,dx \le (1+\delta)\int_0^1 (|\mathbf{m}|^2-1)^2\,dx + (1+\frac{1}{\delta})$, yields

$$\frac{\varepsilon^{-1}}{4}\int_0^1 |\,|\mathbf{m}|^2-1|^2\,dx + \frac{\mu}{4}(\int_0^1 \Lambda(\mathbf{m})\cdot\mathbf{m}\,dx)^2 - (1+\delta)\frac{\mu}{4}\int_0^1 (|\mathbf{m}|^2-1)^2\,dx \le \frac{\mu}{2}.$$

Therefore, for $\varepsilon^{-1} > \mu$ for any δ such that $\varepsilon^{-1}-(1+\delta)\mu \ge 0$ i.e. $\frac{1}{\delta} \le \frac{\mu}{\varepsilon^{-1}-\mu}$ we have

$$(\int_0^1 \Lambda(\mathbf{m})\cdot\mathbf{m}\,dx)^2 \le (1+\frac{1}{\delta}). \tag{3.5}$$

Now we multiply the Euler equation (2.12) by $\mathbf{m}$ and write the equation for $|\mathbf{m}|^2$:

$$-\frac{1}{2}\frac{d^2}{dx^2}|\mathbf{m}|^2 + |\mathbf{m}_x|^2 + \varepsilon^{-1}(|\mathbf{m}|^2-1)|\mathbf{m}|^2 - \mu(\Lambda(\mathbf{m})\cdot\mathbf{m})^2 + \mu\Lambda(\mathbf{m})\cdot\mathbf{m}\int_0^1 \Lambda(\mathbf{m})\cdot\mathbf{m}\,dx = 0\,.$$

Using (3.5) we obtain

$$-\frac{1}{2}\frac{d^2}{dx^2}|\mathbf{m}|^2 + \varepsilon^{-1}(|\mathbf{m}|^2-1)|\mathbf{m}|^2 - \mu|\mathbf{m}|^4 - \mu\sqrt{1+\frac{1}{\delta}}|\mathbf{m}|^2 \le 0\,,$$

that is

$$-\frac{1}{2}\frac{d^2}{dx^2}|\mathbf{m}|^2 + (\varepsilon^{-1}-\mu)\,|\mathbf{m}|^2\,(|\mathbf{m}|^2 - \frac{\varepsilon^{-1}+\mu\sqrt{1+\frac{1}{\delta}}}{\varepsilon^{-1}-\mu})\ \le 0\,,$$

or equivalently, choosing $\frac{1}{\delta} = \frac{\mu}{\varepsilon^{-1}-\mu}$ setting $K = (\varepsilon^{-1}+\mu\sqrt{\frac{\varepsilon^{-1}}{\varepsilon^{-1}-\mu}})/(\varepsilon^{-1}-\mu)$,

$$-\frac{1}{2}\frac{d^2}{dx^2}(|\mathbf{m}|^2 - K) + (\varepsilon^{-1}-\mu)\,|\mathbf{m}|^2\,(|\mathbf{m}|^2 - K)\ \le 0\,.$$

By the maximum principle, applied to the function $h = |\mathbf{m}|^2 - K$, we get that $h \le 0$, i.e., $|\mathbf{m}|^2 \le K$. $\square$

Let us denote by λ_2 the first nontrivial eigenvalue for the Neumann problem:

$$\begin{cases} -f_{xx} = \lambda f \quad \text{in } (0,1)\,, \\ \ \ f_x(0) = f_x(1) = 0\,. \end{cases} \tag{3.6}$$

It is well known that $\lambda_2 = \pi^2$ and that it yields the optimal constant in the following Poincaré inequality,

$$\int_0^1 |g_x|^2\,dx \ge \lambda_2 \int_0^1 \big(g(x) - \int_0^1 g(t)\,dt\big)^2\,dx\,, \quad \forall g \in H^1(0,1)\,. \tag{3.7}$$

Next, we analyze the minimization problem (3.2) restricted to S^1-valued maps.

Proposition 3.4 *Put*

$$I(\mu) = \inf_{\mathbf{m}\in H^1((0,1);S^1)} F(\mathbf{m})\,. \tag{3.8}$$

Then:

(i) *For $\mu \le \lambda_2/2$ we have $I(\mu) = 0$ and the minimum is attained only by constant functions, $\mathbf{m} \equiv \boldsymbol{\alpha} \in S^1$.*
(ii) *For $\mu > \lambda_2/2$ we have $I(\mu) < 0$ and the minimum is attained by $\mathbf{m}^0 = e^{i\phi^0}$ where ϕ^0 is a nontrivial solution of the problem*

$$\begin{cases} -\phi^0_{xx} = \mu\big(\sin 2\phi^0 - \int_0^1 \sin 2\phi^0\,dt\big)\cos 2\phi^0 \qquad \textit{in } (0,1)\,, \\ \ \ \phi^0_x(0) = \phi^0_x(1) = 0\,. \end{cases} \tag{3.9}$$

Proof. Each $\mathbf{m} \in H^1((0,1);S^1)$ can be written as $\mathbf{m} = e^{i\phi}$ for some $\phi \in H^1((0,1);\mathbb{R})$. For such $\mathbf{m}$ we may rewrite the energy in (3.1) as

$$F(\mathbf{m}) = \frac{1}{2}\int_0^1 |\phi_x|^2\,dx - \frac{\mu}{4}\int_0^1 \Big(\sin 2\phi - \int_0^1 \sin 2\phi\,dt\Big)^2\,dx\,. \tag{3.10}$$

The function $f = \sin 2\phi$ satisfies $f_x = 2(\cos 2\phi)\phi_x$, so that

$$|\phi_x| = \frac{|f_x|}{2|\cos 2\phi|} \geq \frac{|f_x|}{2}\,. \tag{3.11}$$

Write the r.h.s. of (3.10) as a sum of two integrals to obtain

$$F(\mathbf{m}) = \int_0^1 \Big(\frac{1}{2}\phi_x^2 - \frac{1}{8}f_x^2\Big)\,dx + \int_0^1 \Big(\frac{1}{8}f_x^2 - \frac{\mu}{4}\big(\sin 2\phi - \int_0^1 \sin 2\phi\,dt\big)^2\Big)\,dx := I_1 + I_2\,. \tag{3.12}$$

Clearly, for $\mu < \lambda_2/2$ and any $f \not\equiv$ const we have by (3.11) and (3.7) that $I_1 > 0$ and $I_2 > 0$. For $\mu = \lambda_2/2$ and $f \not\equiv$ const we have still $I_1 > 0$ while I_2 is nonnegative. This yields assertion (i) of the proposition.

Assume next that $\mu > \lambda_2/2$. From the optimality of λ_2 in (3.7) follows the existence of $\tilde{f} \in H^1((0,1);\mathbb{R})$ with

$$\int_0^1 \big(\frac{1}{8}|\tilde{f}_x|^2 - \frac{\mu}{4}\tilde{f}^2\big)\,dx = -c < 0 \quad \textit{and} \quad \int_0^1 \tilde{f}\,dx = 0\,.$$

For $t > 0$ small enough set $\psi^{(t)} = \frac{1}{2}\arcsin(t\tilde{f})$ and then $\mathbf{m}^{(t)} = e^{i\psi^{(t)}}$. Using (3.12) we get

$$F(\mathbf{m}^{(t)}) = -ct^2 + O(t^4) < 0\,, \qquad \text{for } t \text{ small enough.}$$

This yields $I(\mu) < 0$, and the existence of a minimizer, $\mathbf{m}^0 = e^{i\phi^0}$ with ϕ^0 a nontrivial solution of (3.9) is obvious. This completes the proof of assertion (ii). □

A more precise description of the minimizers in the case $\mu > \lambda_2/2 = \pi^2/2$ is given by the next proposition.

Proposition 3.5 *In the case $\mu > \lambda_2/2$ the minimizer $\mathbf{m}^0 = e^{i\phi^0}$ is unique modulo the operation of the symmetry group $\mathcal{G}$ (see (3.3)), namely, up to performing the operations:*

$$\phi^0 \leftarrow \phi^0 + k\pi/2 \quad \textit{or} \quad \phi^0 \leftarrow -\phi^0 + k\pi/2,\ k \in \mathbb{Z}\,. \tag{3.13}$$

Such a unique representative of the minimizers can be chosen which is a strictly increasing function on $[0,1]$ that satisfies

$$\phi_0(x) = -\phi_0(1-x) \quad x \in [0,1]\,. \tag{3.14}$$

Proof. Setting

$$a = \int_0^1 \sin 2\phi^0\,dx\,,$$

we can rewrite (3.9) as

$$\begin{cases} -\phi^0_{xx} = \mu\big(\sin 2\phi^0 - a\big)\cos 2\phi^0 & \text{in } (0,1)\,, \\ \phi^0_x(0) = \phi^0_x(1) = 0\,. \end{cases} \tag{3.15}$$

The rest of the proof is divided to several steps.

Step 1: ϕ^0 is strictly monotone.

Replacing ϕ^0 by its increasing rearrangement $(\phi^0)^*$ will decrease the first term on the r.h.s. of (3.10) (strictly, if ϕ^0 is not a monotone function), without changing the second term on the r.h.s. of (3.10). Since we may replace ϕ^0 by $-\phi^0$ we can assume in the sequel that $\phi^0_x \geq 0$ in $[0,1]$. We next claim that actually we have:

$$\phi^0_x > 0 \quad \text{ on } (0,1)\,. \tag{3.16}$$

Indeed, the function $\psi = \phi^0_x$ satisfies

$$\begin{cases} -\psi_{xx} = 2\mu\big(\cos 4\phi^0 + a\sin 2\phi^0\big)\psi \quad \text{ in } (0,1)\,, \\ \quad \psi \geq 0 \text{ in } (0,1),\ \psi(0) = \psi(1) = 0\,. \end{cases} \tag{3.17}$$

Since $\psi \not\equiv 0$ we deduce (3.16) from the maximum principle.

Step 2: $|\sin 2\phi^0| < 1$ in $(0,1)$ and $\sin 2\phi^0$ is strictly monotone increasing on $[0,1]$.

Looking for contradiction, assume for example that $\sin 2\phi^0(x_0) = 1$ for some $x_0 \in (0,1)$. By (3.13) we may assume that $\phi^0(x_0) = \pi/4$. Set $\tilde{\phi}(x) = \pi/2 - \phi^0(2x_0 - x)$. It is easy to verify that $\tilde{\phi}$ satisfies the equation in (3.15), and also $\tilde{\phi}(x_0) = \phi^0(x_0)$, $\tilde{\phi}_x(x_0) = \phi^0_x(x_0)$. By the uniqueness theory for ODE we deduce that $\tilde{\phi} = \phi^0$, i.e., $\phi^0(x) = \pi/2 - \phi^0(2x_0 - x)$. For the boundary conditions in (3.15) to hold, the only possibility is that $x_0 = 1/2$. We thus conclude that

$$\phi^0(x) = \pi/2 - \phi^0(1-x), \quad x \in (0,1)\,. \tag{3.18}$$

The relation (3.18) implies that

$$a = \int_0^1 \sin 2\phi^0\,dx = 2\int_0^{1/2} \sin 2\phi^0\,dx = \fint_0^{1/2} \sin 2\phi^0\,dx\,.$$

Defining the following functional on $H^1(0,1/2)$,

$$F_{1/2}(e^{i\phi}) = \frac{1}{2}\int_0^{1/2} |\phi_x|^2\,dx - \frac{\mu}{4}\int_0^{1/2}\Big(\sin 2\phi - \fint_0^{1/2} \sin 2\phi\,dt\Big)^2\,dx\,,$$

we conclude that

$$F(e^{i\phi^0}) = 2F_{1/2}(e^{i\phi^0})\,. \tag{3.19}$$

Set, analogously to (3.8),

$$I_{1/2}(\mu) = \inf_{\mathbf{m}\in H^1((0,1/2);S^1)} F_{1/2}(\mathbf{m})\,. \tag{3.20}$$

The minimum in (3.20) is achieved by some function $\phi^1 \in H^1(0,1/2)$ by the same proof as that of Proposition 3.4. Since $\phi^0_x(1/2) > 0$, the restriction of ϕ^0 to $(0,1/2)$ is not a minimizer and therefore,

$$F_{1/2}(e^{i\phi^1}) < F_{1/2}(e^{i\phi^0})\,. \tag{3.21}$$

We can extend ϕ^1 to a function $\tilde{\phi}^1 \in H^1(0,1)$ by setting

$$\tilde{\phi}^1(x) = \phi^1(1-x) \quad \text{for } x \in [1/2, 1)\,.$$

Combining it with (3.21) and (3.19) we deduce that $F(e^{i\tilde{\phi}^1}) < F(e^{i\phi^0})$. This contradiction completes the proof of the assertion $|\sin 2\phi^0| < 1$ in $(0,1)$.

In view of the above and Step 1 we conclude that the function $\sin 2\phi^0$ is strictly increasing on $[0,1]$. By adding an integer multiple of $\pi/4$, see (3.13), we may assume that the image of the interval $(0,1)$ by ϕ^0 is contained in $(-\pi/4, \pi/4)$. The uniqueness for that representative of the phase of the minimizer will be established in the sequel.

Step 3: $a = 0$.

Multiplying the equation in (3.15) and integrating yields

$$(\phi^0_x)^2 = c^2 - \frac{\mu}{2}(\sin 2\phi^0 - a)^2 \quad \text{on } [0,1]\,, \tag{3.22}$$

for some positive constant c. Write the roots of the polynomial $p(t) = c^2 - (\mu/2)(t-a)^2$ as $a-b$ and $a+b$ for some $b > 0$, i.e., $p(t) = (\mu/2)(a+b-t)(t-a+b)$. By Steps 1 and 2, (3.22), and the boundary condition in (3.15) it follows that

$$\sin 2\phi^0(0) = a - b \quad \text{and} \quad \sin 2\phi^0(1) = a + b\,. \tag{3.23}$$

Assume by negation that $a \neq 0$. Then, from (3.23) it follows, in particular, that $-1 < \sin 2\phi^0(0) < \sin 2\phi^0(1) < 1$. Next, we exploit the following two integral conditions. First,

$$\begin{aligned} 1 = \int_0^1 dx = \int_{\frac{1}{2}\sin^{-1}(a-b)}^{\frac{1}{2}\sin^{-1}(a+b)} \frac{d\phi}{p^{\frac{1}{2}}(\sin 2\phi)} &= \int_{a-b}^{a+b} \frac{dt}{\sqrt{2\mu(a+b-t)(t-a+b)(1-t^2)}} \\ &= \int_{-b}^{b} \frac{ds}{\sqrt{2\mu(b-s)(b+s)(1-(a+s)^2)}}\,. \end{aligned} \tag{3.24}$$

Similarly,

$$\begin{aligned} a = \int_0^1 \sin 2\phi^0(x) dx = \int_{\frac{1}{2}\sin^{-1}(a-b)}^{\frac{1}{2}\sin^{-1}(a+b)} &\frac{\sin 2\phi\, d\phi}{p^{\frac{1}{2}}(\sin 2\phi)} \\ &= \int_{-b}^{b} \frac{(s+a)\, ds}{\sqrt{2\mu(b-s)(b+s)(1-(a+s)^2)}}\,. \end{aligned} \tag{3.25}$$

From (3.24) and (3.25) we deduce that

$$\begin{aligned} 0 &= \int_{-b}^{b} \frac{s\, ds}{\sqrt{2\mu(b-s)(b+s)(1-(a+s)^2)}} \\ &= \int_0^b \frac{s}{\sqrt{2\mu(b-s)(b+s)}} \Big(\frac{1}{\sqrt{(1-(a+s)^2)}} - \frac{1}{\sqrt{(1-(a-s)^2)}} \Big)\, ds\,. \end{aligned} \tag{3.26}$$

But it is clear that the r.h.s. of (3.26) is strictly positive for $a > 0$ and strictly negative for $a < 0$, so in either case we are led to a contradiction.

Step 4: Conclusion.

Going back to (3.22) we can now write

$$(\phi_x^0)^2 = c^2 - \frac{\mu}{2}\sin^2 2\phi^0 = \frac{\mu}{2}(b - \sin 2\phi^0)(b + \sin 2\phi^0) \quad \text{on } [0,1]\,,$$

with $b = c\sqrt{2/\mu}$. The equation (3.24) now reads

$$\sqrt{2\mu} = \int_{-b}^{b} \frac{ds}{\sqrt{(b-s)(b+s)(1-s^2)}} = \int_{-\pi/2}^{\pi/2} \frac{d\theta}{\sqrt{1 - b^2\sin^2\theta}}\,. \tag{3.27}$$

Since we assume that $\mu > \frac{\pi^2}{2}$, it follows that there is a *unique* $b > 0$ for which (3.27) holds.

Next, there is a unique point $x_0 \in (0,1)$ where $0 = \phi^0(x_0) = \sin 2\phi^0(x_0)$. At that point, $\phi_x^0(x_0) = b\sqrt{\mu/2}$. The function $\tilde{\phi}(x) = -\phi^0(2x_0 - x)$ solves the equation

$$-\tilde{\phi}_{xx} = \mu \sin 2\tilde{\phi}\cos 2\tilde{\phi} \quad \text{in } (0,1)\,, \tag{3.28}$$

with the initial conditions

$$\tilde{\phi}(x_0) = \phi^0(x_0) = 0 \quad \text{and} \quad \tilde{\phi}_x(x_0) = \phi_x^0(x_0) = \sqrt{\frac{\mu}{2}}\,b\,. \tag{3.29}$$

Since there is a unique solution to (3.28)–(3.29), it follows that $\phi^0 = \tilde{\phi}$. Since $\tilde{\phi}_x(2x_0) = 0$ we must have $x_0 = 1/2$ and the symmetry property (3.14) holds. The uniqueness assertion of the proposition follows from the uniqueness for the initial problem (3.28)–(3.29) for $x_0 = 1/2$. $\square$

Next we present a convergence result that will be used in our main theorem.

Proposition 3.6 *For each $\mu > 0$, any sequence of minimizers $\{\mathbf{m}_{\varepsilon_n}\}$, with $\varepsilon_n \to 0$, has a subsequence which converges in $H^1(0,1)$ and in $C[0,1]$ to $\mathbf{m}^0 \in C^\infty([0,1]; S^1)$ which is a minimizer for $I(\mu)$.*

PROOF. Note that $F(\mathbf{m}^\varepsilon) \le F(\boldsymbol{\alpha}) = 0$ for any constant $\boldsymbol{\alpha} \in S^1$. Using (3.4) we conclude that for $\varepsilon < \frac{1}{2\mu}$ we have

$$\int_0^1 |\mathbf{m}_x^\varepsilon|^2\,dx \le C \quad \text{and} \quad \int_0^1 (1 - |\mathbf{m}^\varepsilon|^2)^2 \le C$$

for some constant C (which is independent of ε). Since $H^1(0,1)$ is compactly embedded in $C[0,1]$, we can extract a subsequence, still denoted by $\{\mathbf{m}_{\varepsilon_n}\}$, such that converges weakly in $H^1(0,1)$ and strongly in $C[0,1]$ to a limit $\mathbf{m}^0 \in H^1((0,1); S^1)$. Since for each ε, and each $\mathbf{m} \in H^1((0,1); S^1)$, $F(\mathbf{m}^\varepsilon) \le F(\mathbf{m})$, we get that

$$\limsup_{\varepsilon_n \to 0} F(\mathbf{m}^{\varepsilon_n}) \le F(\mathbf{m})\,. \tag{3.30}$$

On the other hand, the weak lower-semicontinuity of the L^2-norm of the gradient, combined with the uniform convergence of $\{\mathbf{m}^{\varepsilon_n}\}$ towards $\mathbf{m}^0$, yields

$$F(\mathbf{m}^0) \le \liminf_{\varepsilon_n \to 0} F(\mathbf{m}^{\varepsilon_n})\,. \tag{3.31}$$

Combining (3.30) with (3.31) we deduce that $F(\mathbf{m}^0) \le F(\mathbf{m})$, $\forall \mathbf{m} \in H^1((0,1);S^1)$, i.e., $\mathbf{m}^0$ is a minimizer for $I(\mu)$. It also follows that the convergence $\mathbf{m}^{\varepsilon_n} \to \mathbf{m}^0$ is actually strong in $H^1(0,1)$. □

We are now in position to state our main result for the minimization problem (3.2).

Theorem 3.7
(i) *For each $\mu < \lambda_2/2$ there exists $\varepsilon_0(\mu) > 0$ such that for $\varepsilon \le \varepsilon_0(\mu)$ we have $\mathcal{F}_{\mu,\varepsilon} = 0$ and the only minimizers for* (3.2) *are constant functions $\mathbf{m}^\varepsilon \equiv \boldsymbol{\alpha} \in S^1$.*
(ii) *For $\mu > \lambda_2/2$ we have $\mathcal{F}_{\mu,\varepsilon} < 0$ for every $\varepsilon > 0$. For each $\varepsilon > 0$ we may choose a representative for the minimizer $\mathbf{m}^\varepsilon$ (by replacing $\mathbf{m}^\varepsilon$ with $\mathcal{S}_i(\mathbf{m}^\varepsilon)$, see* (3.3)*) such that $\lim_{\varepsilon\to 0}\mathbf{m}^\varepsilon = \mathbf{m}^0$ in $H^1(0,1)$ and in $C[0,1]$, where $\mathbf{m}^0 \in C^\infty([0,1];S^1)$ is a non-trivial minimizer for $I(\mu)$.*
(iii) *In the limiting case $\mu = \lambda_2/2$, we have for a subsequence, $\lim_{\varepsilon_n\to 0}\mathbf{m}^{\varepsilon_n} = \boldsymbol{\alpha}$ in $H^1(0,1)$ and in $C[0,1]$, for some constant $\boldsymbol{\alpha} \in S^1$.*

PROOF. (i) By Proposition 3.6 we have, in particular, that $\lim_{\varepsilon\to 0}|\mathbf{m}^\varepsilon| = 1$, uniformly on $[0,1]$. Hence, for any $\delta > 0$ we have, for $\varepsilon \le \varepsilon_1(\mu)$,

$$1-\delta \le |\mathbf{m}^\varepsilon(x)| \le 1+\delta\,, \quad x \in [0,1]\,. \tag{3.32}$$

In particular, if $\delta \le 1/2$, say, then we may write $\mathbf{m}^\varepsilon = \rho e^{i\phi}$, with $\rho = |\mathbf{m}^\varepsilon|$. A simple computation gives

$$F(\mathbf{m}^\varepsilon) = \frac{1}{2}\int_0^1 (\rho^2|\phi_x|^2 + |\rho_x|^2)\,dx + \frac{1}{4\varepsilon}\int_0^1 (1-\rho^2)^2\,dx \\ - \frac{\mu}{4}\int_0^1 \Big(\rho^2\sin 2\phi - \int_0^1 \rho^2 \sin 2\phi\,dt\Big)^2\,dx\,. \tag{3.33}$$

By the Cauchy-Schwarz inequality we get,

$$\begin{aligned}
&\int_0^1 \Big(\rho^2\sin 2\phi - \int_0^1 \rho^2\sin 2\phi\,dt\Big)^2\,dx \\
&\quad = \int_0^1 \Big(\big(\sin 2\phi - \int_0^1 \sin 2\phi\,dt\big) + (\rho^2-1)\sin 2\phi - \int_0^1 (\rho^2-1)\sin 2\phi\,dt\Big)^2\,dx \\
&\quad\quad \le (1+\delta)\int_0^1 \big(\sin 2\phi - \int_0^1 \sin 2\phi\,dt\big)^2\,dx \\
&\quad\quad + 2(1+\frac{1}{\delta})\Big(\int_0^1 (1-\rho^2)^2\sin^2 2\phi\,dx + \big(\int_0^1 (1-\rho^2)\sin 2\phi\,dx\big)^2\Big) \\
&\quad\quad \le (1+\delta)\int_0^1 \big(\sin 2\phi - \int_0^1 \sin 2\phi\,dt\big)^2\,dx + 4(1+\frac{1}{\delta})\int_0^1 (1-\rho^2)^2\,dx\,.
\end{aligned} \tag{3.34}$$

Combining (3.34) with (3.33) and (3.32) yields

$$F(\mathbf{m}^\varepsilon) \geq \frac{(1-\delta)^2}{2} \int_0^1 |\phi_x|^2 - \frac{\mu(1+\delta)}{4} \int_0^1 \big(\sin 2\phi - \int_0^1 \sin 2\phi \, dt\big)^2 + \big(\frac{1}{4\varepsilon} - \mu(1+\frac{1}{\delta})\big) \int_0^1 (1-\rho^2)^2 \, dx \,. \quad (3.35)$$

Since $\mu < \lambda_2/2$ we can fix δ small enough so that

$$\tilde{\mu} := \frac{1+\delta}{(1-\delta)^2}\mu < \frac{\lambda_2}{2} \,.$$

For ε small enough such that $\frac{1}{4\varepsilon} \geq \mu(1+1/\delta)$ we obtain from (3.35)

$$0 \geq F(\mathbf{m}^\varepsilon) \geq (1-\delta)^2 \Big\{ \frac{1}{2} \int_0^1 |\phi_x|^2 \, dx - \frac{\tilde{\mu}}{4} \int_0^1 \big(\sin 2\phi - \int_0^1 \sin 2\phi \, dt\big)^2 \, dx \Big\} + \frac{1}{8\varepsilon} \int_0^1 (1-\rho^2)^2 \, dx \geq 0 \,. \quad (3.36)$$

By Proposition 3.4 strict inequality holds for the last inequality on the r.h.s. of (3.36), unless $\mathbf{m}^\varepsilon$ equals identically a constant of modulus one, hence the result.

(ii) By Proposition 3.4 we have in this case,

$$\mathcal{F}_{\mu,\varepsilon} \leq I(\mu) < 0 \,.$$

The convergence assertion follows from Proposition 3.6 and the uniqueness result Proposition 3.5.

(iii) This part is a direct consequence of Proposition 3.6 and Proposition 3.4. $\square$

Remark 3.8 *We do not know whether in the the limiting case $\mu = \lambda_2/2$ (case (iii)) the minimizer $\mathbf{m}^\varepsilon$ is necessarily a constant for ε small enough, as in case (i).*

4 The analysis of the gradient flow equation

Let T be a positive number, we define $Q = \Omega \times (0,T)$ and consider the initial boundary value problem

$$\mathbf{u}_t = \mathbf{u}_{xx} - \varepsilon^{-1}(|\mathbf{u}|^2 - 1)\mathbf{u} + \mu\Lambda(\mathbf{u})[\Lambda(\mathbf{u}) \cdot \mathbf{u} - \int_0^1 \Lambda(\mathbf{u}) \cdot \mathbf{u} dx] \quad (4.1)$$

with the boundary conditions

$$\mathbf{u}_x(0,t) = \mathbf{u}_x(1,t) = 0, \qquad t \in (0,T) \quad (4.2)$$

and the initial condition

$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}^0(\mathbf{x}), \qquad \mathbf{x}\in\Omega\equiv(0,1) \tag{4.3}$$

Provided the solution $\mathbf{u}(t)$ of (4.1), (4.2), (4.3) exists for all t, we show that $\lim_{t\to\infty}\mathbf{u}(t)=\mathbf{u}_\infty$ exists and, for suitable choice of the initial datum $\mathbf{u}^0$, the function $\mathbf{u}_\infty$ is a negative energy solution to (2.12), (2.13).

The following existence and uniqueness theorem holds

Theorem 4.1 *Let $\mathbf{u}^0(\mathbf{x})\in C^2([0,1];S^1)$ (that is $\mathbf{u}^0(\mathbf{x})\in C^2[0,1]$ and $|\mathbf{u}^0|=1$) which satisfies the condition $\mathbf{u}^0_x(0)=\mathbf{u}^0_x(1)=0$, then for each μ and for each positive ε small enough, that is $\varepsilon^{-1}>2\mu$, there exists a unique solution $\mathbf{u}\in C^2(Q)$ to the problem (4.1), (4.2), (4.3). Moreover the following estimates hold*

$$\int_0^T\int_0^1|\mathbf{u}_t|^2dxdt+\frac{1}{2}\int_0^1|\mathbf{u}_x|^2dx\le C(\mathbf{u}^0) \tag{4.4}$$

$$\|\mathbf{u}\|^2_{L^\infty(Q)}\le C(\mathbf{u}^0) \tag{4.5}$$

where $C(\mathbf{u}^0)$ is a positive constant depending on the initial datum $\mathbf{u}^0$.

Proof. The existence of a classical solution follows from well known results established in [7, chap.VII] and a fixed point argument. Indeed we first consider the quasilinear parabolic system depending on a fixed parameter $a\in\mathbb{R}$ i.e.

$$\mathbf{u}^a_t=\mathbf{u}^a_{xx}-\varepsilon^{-1}(|\mathbf{u}^a|^2-1)\mathbf{u}^a+\mu\Lambda(\mathbf{u}^a)[\Lambda(\mathbf{u}^a)\cdot\mathbf{u}^a-a], \qquad \text{in } Q \tag{4.6}$$

with the boundary condition

$$\mathbf{u}^a_x(0,t)=\mathbf{u}^a_x(1,t)=0, \qquad t\in(0,T) \tag{4.7}$$

and the initial condition

$$\mathbf{u}^a(\mathbf{x},0)=\mathbf{u}^0(\mathbf{x}), \qquad \mathbf{x}\in\Omega \tag{4.8}$$

Applying Theorem 7.1 established in [7, chap.VII] we easily get the existence and uniqueness of the solution $\mathbf{u}^a\in C^2(Q)$ to the problem (4.6), (4.7), (4.8). Multiplying the equation (4.6) by $\mathbf{u}^a$ one has

$$\frac{1}{2}\frac{d}{dt}|\mathbf{u}^a|^2=\frac{1}{2}\frac{d^2}{dx^2}|\mathbf{u}^a|^2-|\mathbf{u}^a_x|^2-\varepsilon^{-1}(|\mathbf{u}^a|^2-1)|\mathbf{u}^a|^2+\mu\Lambda(\mathbf{u}^a)\cdot\mathbf{u}^a[\Lambda(\mathbf{u}^a)\cdot\mathbf{u}^a-a]$$

recalling that $\Lambda(\mathbf{u})\cdot\mathbf{u}\le|\mathbf{u}|^2$ we get the following inequality for $|\mathbf{u}^a|^2$, that is

$$\frac{1}{2}\frac{d}{dt}|\mathbf{u}^a|^2-\frac{1}{2}\frac{d^2}{dx^2}|\mathbf{u}^a|^2+\varepsilon^{-1}(|\mathbf{u}^a|^2-1)|\mathbf{u}^a|^2-\mu|\mathbf{u}^a|^2[|\mathbf{u}^a|^2+|a|]\le 0 \tag{4.9}$$

Now putting $K_1 = 1 - \mu\varepsilon$ and $K_2 = 1 + |a|\mu\varepsilon$ we obtain

$$\frac{1}{2}\frac{d}{dt}|\mathbf{u}^a|^2 - \frac{1}{2}\frac{d^2}{dx^2}|\mathbf{u}^a|^2 + \varepsilon^{-1}(K_1|\mathbf{u}^a|^2 - K_2)|\mathbf{u}^a|^2 \leq 0 \tag{4.10}$$

We assume ε small enough, more precisely $\varepsilon^{-1} > \mu$ and introduce the function $\mathbf{y} = K_1|\mathbf{u}^a|^2 - K_2$. From the maximum principle applied to

$$\frac{1}{2}\frac{d\mathbf{y}}{dt} - \frac{1}{2}\frac{d^2\mathbf{y}}{dx^2} + \varepsilon^{-1}K_1|\mathbf{u}^a|^2\,\mathbf{y} \leq 0$$

we deduce that the function $\mathbf{y}$ can not assume a positive maximum in Q and since $K_1|\mathbf{u}^0|^2 - K_2 = K_1 - K_2 \leq 0$ we have $\mathbf{y} \leq 0$ in $\bar{Q}$ and so we get to the estimate

$$|\mathbf{u}^a|^2 \leq K = \frac{1 + |a|\mu\varepsilon}{1 - \mu\varepsilon}. \tag{4.11}$$

We consider then the map

$$g : a \mapsto \int_0^1 (\Lambda u^a \cdot u^a)\,dx.$$

It is a continuous map. Moreover for $\varepsilon\mu < \frac{1}{2}$ if we choose $|a|$ large enough such that

$$\mu\varepsilon < \frac{1}{2} - \frac{1}{2|a|}$$

we have by (4.11)

$$\left|\int_0^1 \Lambda u^a \cdot u^a\,dx\right| \leq |u^a|^2 \leq \frac{1 + |a|\mu\varepsilon}{1 - \mu\varepsilon} \leq |a|.$$

Thus g maps $(-|a|, |a|)$ into itself and has a fixed point.

Now multiplying the equation (4.1) by $\mathbf{u}_t$ and integrating on Q one has the following energy estimate

$$F(\mathbf{u}) = F(\mathbf{u}^0) - \int_0^T \int_0^1 |\mathbf{u}_t|^2 dxdt \tag{4.12}$$

where $F(\mathbf{u}^0)$ is a constant depending on the initial datum but it is independent of ε since $|\mathbf{u}^0| = 1$. From the above equality we deduce in particular

$$\frac{1}{2}\int_0^1 |\mathbf{u}_x|^2 dx + \frac{\varepsilon^{-1}}{4}\int_0^1 (|\mathbf{u}|^2 - 1)^2 dx - \frac{\mu}{4}\int_0^1 (\Lambda(\mathbf{u})\cdot\mathbf{u})^2 dx + \frac{\mu}{4}\left(\int_0^1 \Lambda(\mathbf{u})\cdot\mathbf{u}dx\right)^2 \leq F(\mathbf{u}^0)$$

that is

$$\frac{1}{2}\int_0^1 |\mathbf{u}_x|^2 dx + \frac{\varepsilon^{-1}}{4}\int_0^1 (|\mathbf{u}|^2 - 1)^2 dx - \frac{\mu}{4}\int_0^1 (|\mathbf{u}|^2 - 1 + 1)^2 dx + \frac{\mu}{4}\left(\int_0^1 \Lambda(\mathbf{u})\cdot\mathbf{u}dx\right)^2 \leq F(\mathbf{u}^0)$$

and hence

$$\frac{1}{2}\int_0^1 |\mathbf{u}_x|^2 dx + \frac{\varepsilon^{-1}-2\mu}{4}\int_0^1 (|\mathbf{u}|^2-1)^2 dx + \frac{\mu}{4}\left(\int_0^1 \Lambda(\mathbf{u})\cdot\mathbf{u}dx\right)^2 \leq F(\mathbf{u}^0) + \frac{\mu}{2}$$

Since we have assumed $\varepsilon^{-1} \geq 4\mu$, the above inequality implies in particular

$$\frac{\mu}{4}\left(\int_0^1 \Lambda(\mathbf{u})\cdot\mathbf{u}dx\right)^2 \leq F(\mathbf{u}^0) + \frac{\mu}{2}$$

We use this estimate for computing the L^∞-norm of the solution to (4.1), (4.2), (4.3). Indeed setting

$$|a| = |\int_0^1 \Lambda(\mathbf{u})\cdot\mathbf{u}dx| \leq 2\,\mu^{-1/2}\left(F(\mathbf{u}^0) + \frac{\mu}{2}\right)^{1/2}$$

and putting this in (4.11) we obtain the desired estimate.

$$|\mathbf{u}|^2 \leq \frac{1 + 2\sqrt{\mu}\,\left(F(\mathbf{u}^0)+\mu/2\right)^{1/2}\varepsilon}{1-\mu\varepsilon}\,, \qquad \varepsilon^{-1} \geq 4\mu \tag{4.13}$$

The uniqueness of the solution follows from the estimate (4.13). Indeed using the condition (4.13) one can easily prove that the function

$$N(\mathbf{u}) = -\varepsilon^{-1}(|\mathbf{u}|^2-1)\mathbf{u} + \mu\Lambda(\mathbf{u})[\Lambda(\mathbf{u})\cdot\mathbf{u} - \int_0^1 \Lambda(\mathbf{u})\cdot\mathbf{u}dx]$$

is Lipschitz continuous, that is there exists a positive constant L such that

$$|N(\mathbf{u}^1) - N(\mathbf{u}^2)| \leq L\,|\mathbf{u}^1 - \mathbf{u}^2|$$

Now as a common praxis we assume that there are two solutions $\mathbf{u}^1$ and $\mathbf{u}^2$ of the problem (4.1), (4.2), (4.3), then

$$\mathbf{u}^1_t - \mathbf{u}^2_t = \mathbf{u}^1_{xx} - \mathbf{u}^2_{xx} + N(\mathbf{u}^1) - N(\mathbf{u}^2)$$

and hence multiplying the above equation by the function $\mathbf{y} = \mathbf{u}^1 - \mathbf{u}^2$, integrating in Ω and taking into account the boundary and initial conditions one has

$$|\mathbf{y}|^2_t \leq 2L|\mathbf{y}|^2, \qquad \mathbf{y}(\cdot,0) = 0$$

that implies $\mathbf{y} \equiv 0$ in Q. □

Lemma 4.2 *Let* $\mathbf{u}$ *the solution of the problem (4.1), (4.2), (4.3) then there exists a positive constant* K *such that the following estimate holds*

$$\int_0^\infty \left|\frac{d}{dt}\|\mathbf{u}_t\|^2_{L^2(0,1)}\right| dt \,\leq\, K$$

Proof. We write the equation (4.1) in the form

$$\mathbf{u}_t = \mathbf{u}_{xx} + N(\mathbf{u})$$

derivating with respect to t and multiplying by $\mathbf{u}_t$ we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 |\mathbf{u}_t|^2 dx + \int_0^1 |\mathbf{u}_{xt}|^2 dx = \int_0^1 N'(\mathbf{u})|\mathbf{u}_t|^2 dx. \tag{4.14}$$

Hence from (4.14), (4.4), (4.5)

$$\int_0^T \left|\frac{d}{dt}\|\mathbf{u}_t\|^2_{L^2(0,1)}\right| dt \leq 2\left|\int_Q N'(\mathbf{u})|\mathbf{u}_t|^2 dx\right| \leq K$$

and the proof of the lemma easily follows. □

From Lemma 4.2 follows that

$$\lim_{t\to\infty} \|\mathbf{u}_t\|_{L^2(0,1)} = 0. \tag{4.15}$$

Now we can prove the following theorem

Theorem 4.3 *Let* $\mathbf{u}$ *the solution of the problem (4.1), (4.2), (4.3) then as* $t \to \infty$ *there exists a sequence* t_k *such that*

$$\mathbf{u}(\mathbf{x}, t_k) \rightharpoonup \mathbf{u}_\infty(\mathbf{x}) \quad \text{in } H^1(0,1) \tag{4.16}$$

where $\mathbf{u}_\infty(\mathbf{x})$ *is a stationary point of (4.1). Moreover all the weakly convergent sequences converge to stationary points.*

Proof. Let $\mathbf{u}^k = \mathbf{u}(\cdot, t_k)$ be the solution of (4.1), (4.2), (4.3) at time t_k. From the estimate (4.4) it follows that, passing to a subsequence if it is necessary,

$$\mathbf{u}^k \to \mathbf{u}_\infty \quad \text{weakly} \quad \text{in } H^1(0,1) \tag{4.17}$$

$$\mathbf{u}^k \to \mathbf{u}_\infty \quad \text{strongly} \quad \text{in } L^2(0,1) \tag{4.18}$$

$$\mathbf{u}^k \cdot \Lambda(\mathbf{u}^k) \to \mathbf{u}_\infty \Lambda(\mathbf{u}_\infty) \quad \text{strongly} \quad \text{in } L^2(0,1) \tag{4.19}$$

$$|\mathbf{u}^k|^2 \to |\mathbf{u}_\infty|^2 \quad \text{strongly} \quad \text{in } L^2(0,1) \tag{4.20}$$

Now we have to prove that $\mathbf{u}_\infty$ is a solution of the stationary problem. For this we multiply the equation (4.1) by $\mathbf{v} \in \mathbf{H}^1(0,1)$ one has

$$\int_0^1 \mathbf{u}_t^k \cdot \mathbf{v}\, dx = -\int_0^1 \mathbf{u}_x^k \cdot \mathbf{v}_x\, dx - \varepsilon^{-1}\int_0^1 (|\mathbf{u}^k|^2 - 1)\mathbf{u}^k \cdot \mathbf{v} dx + \tag{4.21}$$

$$+\mu \int_0^1 \Lambda(\mathbf{u}^k) \cdot \mathbf{v}\left[\Lambda(\mathbf{u}^k) \cdot \mathbf{u}^k - \int_0^1 \Lambda(\mathbf{u}^k) \cdot \mathbf{u}^k dx\right] \tag{4.22}$$

¿From Lemma 4.2 and from the convergence established above it follows that $\mathbf{u}_\infty$ is a weak solution of the stationary problem. □

Corollary 4.4 *Let* $\mathbf{u}^0$ *be a function verifying the hypotheses of Theorem 4.1. If* $F(\mathbf{u}^0) < 0$ *then the limit function* $\mathbf{u}_\infty(\mathbf{x})$ *defined in Theorem 4.3 is a negative energy stationary point of (3.1).*

PROOF. The proof easily follows from the energy estimate (4.12). Indeed since the system is dissipative we have

$$F(\mathbf{u}_\infty) \leq F(\mathbf{u}^0)$$

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