On a minimization problem with a mass constraint in dimension two

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Abstract

We continue our study initialized in [1] of a singular perturbation type minimization problem with a mass constraint, involving a potential vanishing on two curves in the plane. In the case of a two dimensional nonconvex domain (and under some additional assumptions) we are able to prove a convergence result for the minimizers and characterize the limit as a solution of a mixed Dirichlet-Neumann boundary condition problem with a mass constraint.

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1 Introduction

The following problem was introduced and studied by Sternberg [10]. Let Γ_1 and Γ_2 be two disjoint, smooth and simple closed curves in $\mathbb{R}^2$ of lengths $l_1 = l(\Gamma_1)$ and $l_2 = l(\Gamma_2)$, respectively, such that Γ_1 lies inside Γ_2 and the origin 0 lies inside Γ_1. Let $W : \mathbb{R}^2 \to [0, \infty)$ be a smooth function (i.e., at least of class C^4) satisfying

$$W > 0 \text{ on } \mathbb{R}^2 \setminus (\Gamma_1 \cup \Gamma_2) \quad \text{and} \quad W = 0 \text{ on } \Gamma_1 \cup \Gamma_2\,. \tag{H_1}$$

Since W attains its minimal value zero on $\Gamma_1 \cup \Gamma_2$, the normal derivative W_n equals zero on $\Gamma_1 \cup \Gamma_2$. We make the generic assumption that

$$W_{nn} > 0 \text{ on } \Gamma_1 \cup \Gamma_2\,. \tag{H_2}$$

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We shall also assume a coercivity condition on W at infinity: there exist constants $R_0 > 0$ and $\tilde{c}_0 > 0$ such that

$$W(y) \geq \tilde{c}_0|y| \text{ for } |y| \geq R_0\,. \tag{H_3}$$

Let G be a bounded smooth domain in $\mathbb{R}^N$. In order to simplify notation, we shall assume in the sequel, without loss of generality, that G has unit volume:

$$\mu(G) = 1\,. \tag{1.1}$$

For each $\varepsilon > 0$ consider the energy functional

$$E_\varepsilon(u) = \int_G |\nabla u|^2 + \frac{W(u)}{\varepsilon^2} \quad \text{over } u \in H^1(G, \mathbb{R}^2)\,. \tag{1.2}$$

Let R_c be a positive number such that the circle $\{|x| = R_c\}$ lies between the two curves Γ_1 and Γ_2. The number R_c represents the constraint in the following minimization problem:

$$\min\{E_\varepsilon(u)\ :\ u \in H^1(G, \mathbb{R}^2),\ \int_G |u| = R_c\}\,. \tag{P_ε}$$

Denoting by u_ε a minimizer in (P_ε), we are interested in the asymptotic behavior of the minimizers $\{u_\varepsilon\}$ and their energies $E_\varepsilon(u_\varepsilon)$, as ε goes to 0.

In [1] it was shown that the asymptotic behavior of the minimizers depends in a crucial way on the geometry of G through a certain problem, involving the *isoperimetric profile* of G. Recall that the isoperimetric profile of G satisfying (1.1) is the function $I = I_G : (0, 1) \to \mathbb{R}$ defined by

$$I(t) = \min\{\text{Per}_G\,\Omega\ :\ \Omega \subset G \text{ s.t. } \chi_\Omega \in BV(G) \text{ and } |\Omega| = t\}\,. \tag{1.3}$$

Clearly, $I(t)$ is a symmetric function w.r.t. the middle point $1/2$ and we refer the reader to [9] and the references therein for more information on it. As in [1] we define the interval $\mathcal{I}_0 = [\beta_1, \beta_2]$ by

$$\mathcal{I}_0 := [\frac{m_2 - R_c}{m_2 - m_1}, \frac{M_2 - R_c}{M_2 - M_1}]\,, \tag{1.4}$$

where

$$m_j = \min_{x\in\Gamma_j} |x| \quad \text{and} \quad M_j = \max_{x\in\Gamma_j} |x|\ (j = 1, 2)\,. \tag{1.5}$$

Finally, the geometric problem relevant to our study which was introduced in [1] is

$$\min\{I(t)\ :\ t \in \mathcal{I}_0\}\,. \tag{1.6}$$

In [1] it was shown that when the minimum in (1.6) is attained (only) at one of the end points, β_j of $\mathcal{I}_0$ (and under more technical assumptions), there exists a subsequence s.t. $u_{\varepsilon_n} \to u_*$ in $L^1(G)$, where u_* has the simple form

$$u_*(x) = \begin{cases} x^{(1)} & x \in G_1\,, \\ x^{(2)} & x \in G \setminus G_1\,, \end{cases} \tag{1.7}$$

with G_1 a set of minimal perimeter $I(\beta_j)$ among all sets of volume β_j. It was noted in [1] that the above is always the case for a *convex* G. On the other hand, for a nonconvex G, the minimum in (1.6) may be attained only at an interior point $\alpha \in (\beta_1, \beta_2)$, and then the behavior of the minimizers is expected to be quite different. This latter case will be characterized by "property (NC)":

Definition 1. We shall say that the pair G and $\mathcal{I}_0$ have property (NC) if the minimum in (1.6) is attained at a unique interior point α of the interval $\mathcal{I}_0$.

The domain drawn in Figure 1 (with an appropriate $\mathcal{I}_0$) is an example where property (NC) holds. Another, more explicit example, consists of a domain of unit area, symmetric with respect to both the x and y axes, given by

$$G = \{(x,y) \,:\, -f(x) < y < f(x),\, x \in (-a,a)\},$$

such that $0 < m := \min_{[-a,a]} f = f(0) < a$ (0 is the unique minimum of f), with $\frac{1}{2}$ lying in the interior of $\mathcal{I}_0$. We remark that the uniqueness of α is assumed only for the sake of simplicity. The difference between the two cases (the "convex" and "nonconvex") is manifested already by the following energy estimate, [1, Theorem 3] that we recall below:

Theorem 1. *If property (NC) holds and W satisfies $(H_1) - (H_3)$, then there exists a constant C, independent of ε, such that*

$$\left|E_\varepsilon(u_\varepsilon) - \frac{2D}{\varepsilon} I(\alpha)\right| \le C\,. \tag{1.8}$$

Above, the constant D, representing a "distance" between the two curves Γ_1 and Γ_2, is defined, following Sternberg [10], by

$$D := \inf_{\substack{\gamma \in \mathrm{Lip}([0,1],\mathbb{R}^2),\\ \gamma(0)\in\Gamma_1, \gamma(1)\in\Gamma_2}} \int_0^1 \big(W(\gamma(t))\big)^{1/2} |\gamma'(t)|\, dt\,. \tag{1.9}$$

More generally, for any pair of points $x, y \in \mathbb{R}^2$ we define

$$d_W(x,y) = \inf_{\substack{\gamma \in \mathrm{Lip}([0,1],\mathbb{R}^2),\\ \gamma(0)=x,\gamma(1)=y}} L(\gamma)\,, \tag{1.10}$$

where

$$L(\gamma) = \int_0^1 \big(W(\gamma(t))\big)^{1/2}\,|\gamma'(t)|\,dt\,. \tag{1.11}$$

Note that thanks to scaling invariance we may replace the interval $[0,1]$ in (1.10)–(1.11) by any interval $[0,A]$, $A > 0$. We define the corresponding distance functions to the curves Γ_1, Γ_2 as follows:

$$\Psi_j(\zeta) = d_W(\zeta,\Gamma_j) := \inf_{x\in\Gamma_j} d_W(\zeta,x)\,,\ j = 1,2\,,\ \zeta \in \mathbb{R}^2\,. \tag{1.12}$$

We also set

$$\widetilde{\Psi} = \min(\Psi_1,\Psi_2)\,. \tag{1.13}$$

It is well known (c.f. [10, 6]) that for $j = 1, 2$, $\Psi_j \in \mathrm{Lip}(\mathbb{R}^2)$ is a solution of the eikonal-type equation

$$|\nabla\Psi_j(\zeta)|^2 = W(\zeta) \quad \text{a.e. on } \mathbb{R}^2\,. \tag{1.14}$$

It was further shown in [2] that Ψ_j is regular in a neighborhood of Γ_j, i.e.,

$$\exists d_0 > 0 \text{ s.t. } \Psi_j \text{ is of class } C^2 \text{ in } \{x\,:\,\Psi_j(x) < d_0\}\,,\ j = 1,2\,. \tag{1.15}$$

Moreover, we have

$$\Psi_j(x) \sim W(x) \sim \mathrm{dist}^2(x,\Gamma_j) \quad \text{on } \{\Psi_j(x) < d_0\}\,, \tag{1.16}$$

where "dist" stands for the euclidean distance. As in [1], we denote the set of *geodesics* w.r.t. the degenerate metric d_W ((1.10)), realizing the infimum in (1.9), by

$$\mathcal{G} = \{\underline{\gamma}^{(i)}\,:\, i \in \mathcal{I}\}\,, \tag{1.17}$$

where $\mathcal{I}$ is a set of indices.

The main result of this paper is a *convergence result* that characterizes the limit of $\{u_\varepsilon\}$, when ε goes to zero. Unfortunately, we were able to establish this result only in dimension two, although we conjecture that it should be valid in any dimension. Therefore, we shall assume in the sequel that $N = 2$. In addition, we shall also make the simplifying assumption that G is *simply connected*. When $N = 2$ it is well known that if G_1 is a set realizing the minimum in (1.3) for $t = \alpha$, then the separation surface

$$\Sigma := \overline{\partial G_1 \cap G}\,, \tag{1.18}$$

consists of a finite union of segments (see Figure 1),

$$\Sigma = \bigcup_{j=1}^{J} \Sigma^{(j)}\,, \tag{1.19}$$

that are all orthogonal to the boundary ∂G.

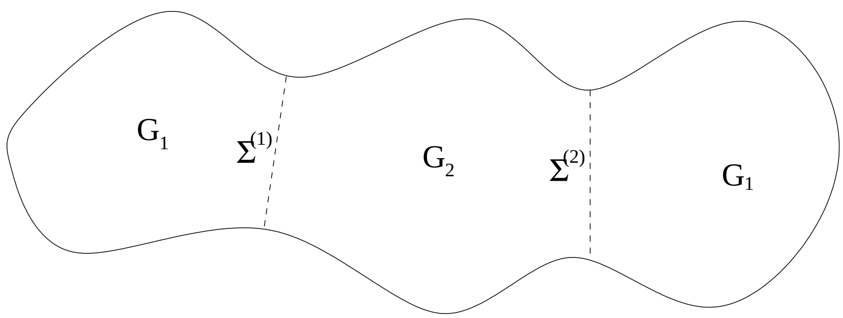

Figure 1: In this example Σ is composed of two segments

We denote for each $\lambda > 0$,

$$\Sigma_\lambda = \{x \in G \,:\, \mathrm{dist}(x, \Sigma) \le \lambda\} := \bigcup_{j=1}^{J} \Sigma_\lambda^{(j)}\,. \tag{1.20}$$

Our main result is the following:

Theorem 2. *Assume the hypotheses of Theorem 1, and in addition that $N = 2$, G is simply connected and that W satisfies*

$$a_1 W(y) \le \nabla W(y) \cdot y \le a_2 W(y)\,, \quad |y| \ge R_1\,, \tag{H_3'}$$

for some positive constants a_1, a_2 and R_1 (we may take $R_1 > R_0$). Suppose (for simplicity) that there is a unique minimizing $\alpha \in I_0$ in (1.6). Assume also that the set $\mathcal{G}$ consists of a single geodesic $\underline{\gamma}$ with endpoints $p_1 \in \Gamma_1$ and $p_2 \in \Gamma_2$. Then, there is a set G_1 realizing the minimum for $I(\alpha)$ in (1.3) such that for every $\lambda > 0$ we have

$$u_\varepsilon\big|_{G_j} \to U_j \text{ in } H^1(G_j \setminus \Sigma_\lambda) \cap C(G_j \setminus \Sigma_\lambda)\,,\ j = 1, 2\,, \tag{1.21}$$

where $G_2 = G \setminus \bar{G}_1$, *and the pair* $(U_1, U_2) \in H^1(G_1, \Gamma_1) \times H^1(G_2, \Gamma_2)$ *is a minimizer for the following problem:*

$$E_0 := \min \Big\{ \int_{G_1} |\nabla v_1|^2 + \int_{G_2} |\nabla v_2|^2 \,:\, v_j \in H^1(G_j, \Gamma_j)\,,\ \mathrm{Tr}\, v_j\big|_{\partial G_j \cap G} = p_j\,,\ j = 1, 2\,, \\ \int_{G_1} |v_1| + \int_{G_2} |v_2| = R_c \Big\}\,. \quad (1.22)$$

Furthermore, the following more precise formula than (1.8) *for the energy holds:*

$$E_\varepsilon(u_\varepsilon) = \frac{2D}{\varepsilon} I(\alpha) + E_0 + o(1)\,. \quad (1.23)$$

Remark 1.1. Existence of a minimizer $(U_1, U_2) \in H^1(G_1, \Gamma_1) \times H^1(G_2, \Gamma_2)$ to the problem (1.22) follows easily by the direct method of the calculus of variations. It will be convenient to associate with U_k ($k = 1, 2$) the S^1-valued map $\widetilde{U}_k = g_k^{-1} \circ U_k$ where g_k is a diffeomorphism of S^1 onto Γ_k satisfying $|\frac{dg_k(e^{i\phi})}{d\phi}| = \frac{l_k}{2\pi}$, $\forall \phi \in [0, 2\pi)$, $k = 1, 2$. For each $\Sigma^{(j)}$ let $G_k^{(j)}$ denote the component of G_k for which $\Sigma^{(j)} \subseteq \partial G_k^{(j)} \cap G$, $k = 1, 2$. Since $\widetilde{U}_k \in H^1(G_k^{(j)}, S^1)$ and $G_k^{(j)}$ is simply connected there exists a lifting $\phi_k^{(j)} \in H^1(G_k^{(j)}, \mathbb{R})$ such that $\widetilde{U}_k = e^{i\phi_k^{(j)}}$ on $G_k^{(j)}$ (see [4, 5]). Define a function $h_k(\phi)$ by $h_k(\phi) = |g_k(e^{i\phi})|$. The function $\phi_k^{(j)}$ is a minimizer for the problem

$$\min\{ \int_{G_k^{(j)}} |\nabla \phi|^2 \,:\, \phi \in H^1(G_k^{(j)})\,,\ \int_{G_k^{(j)}} h_k(\phi) = \alpha_{j,k} \}\,,$$

for some constant $\alpha_{j,k}$ ($\alpha_{j,k} = \int_{G_k^{(j)}} |U_k|$). Therefore, it solves the equation

$$-\Delta \phi_k^{(j)} = a_{j,k} h_k'(\phi_k^{(j)})\,, \quad (1.24)$$

for some constant $a_{j,k}$, with zero Neumann condition on $\partial G_k^{(j)} \cap \partial G$ and (constant) Dirichlet condition on $\partial G_k^{(j)} \cap G$. It follows that $\phi_k^{(j)}$ is a smooth function in $G_k^{(j)}$. Therefore, U_1 and U_2 are smooth maps away from Σ (continuity of U_k in $\bar{G}_k$ holds as well as one can see by a reflection argument near the corners).

Remark 1.2. The uniqueness assumptions on α and $\mathcal{G}$ in Theorem 2 are not essential and are made only for the sake of simplicity of the statement of the theorem.

Remark 1.3. After completing our work we learned about a recent interesting article by Lin, Pan and Wang [8]. They study a different, but related problem: the *Dirichlet problem* for a similar type of energy (for very special and suitably chosen boundary data $\{g_\varepsilon\}$). Their framework is more general than ours since they allow for arbitrary dimension both in the domain and the target (we allow arbitrary dimension for the domain only in the energy estimate of Theorem 1, proven in [1]). We mention that our two dimensional result provides convergence in stronger norms, namely in H^1 and uniform convergence (that probably can be improved to C^k-convergence) away from Σ. It is interesting to note that a nontrivial limit as we found in our Theorem 2 is obtained under simple and natural mass constraint (compare with Remark 1.2(5) in [8]).

The paper is organized as follows. In Section 2 we prove some preliminary estimates needed for the proof of Theorem 2 (the proof of two technical results is postponed to the Appendix). Finally, the proof of Theorem 2 is completed in Section 3.

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2 Preliminary results

Let $\{u_\varepsilon\}$ be a family of minimizers for (P_ε), assuming that the assumptions of Theorem 2 hold. The next lemma shows that the sets were u_ε takes values close to Γ_1 and to Γ_2, have measures close to α and $1-\alpha$, respectively.

Lemma 2.1. *We have*

$$\lim_{\varepsilon\to 0} |\mu(\Psi_1(u_\varepsilon)\leq \varepsilon^{1/2})-\alpha| + |\mu(\Psi_2(u_\varepsilon)\leq \varepsilon^{1/2})-(1-\alpha)| = 0\,. \tag{2.1}$$

Furthermore, for a subsequence $\varepsilon_n\to 0$ we have for every $t\in(0,d_0]$,

$$\chi_{\{\Psi_1(u_{\varepsilon_n})<t\}}\to\chi_{G_1}\quad\textit{and}\quad \chi_{\{\Psi_2(u_{\varepsilon_n})<t\}}\to\chi_{G_2}\ \textit{in}\ L^1(G)\,, \tag{2.2}$$

where G_1 realizes the minimum for $I(\alpha)$ in (1.3) and $G_2=G\setminus\bar{G}_1$.

Proof. The proof uses similar arguments to those used in [1]. First, there exists $\alpha_\varepsilon\in(0,1)$ such that

$$\mu(\{\Psi_1(u_\varepsilon)<d_0\})-\alpha_\varepsilon=O(\varepsilon)\quad\text{and}\quad \mu(\{\Psi_2(u_\varepsilon)<d_0\})-(1-\alpha_\varepsilon)=O(\varepsilon)\,. \tag{2.3}$$

In the proof of [1, Theorem 3] it was shown that for some constant k we have

$$\mathrm{Per}_G\{\Psi_j(u_\varepsilon)<a\}\geq I(\mu(\{\Psi_j(u_\varepsilon)<a\}))\geq I(\alpha)\,,\ a\in[k\varepsilon,D/2],\ j=1,2\,. \tag{2.4}$$

From (2.4), the Cauchy-Schwarz inequality and the coarea formula, we obtain the lower-bound part of (1.8), namely,

$$\begin{aligned} E_\varepsilon(u_\varepsilon) &\geq \frac{2}{\varepsilon}\int_G \sqrt{W(u_\varepsilon)}|\nabla u_\varepsilon| \\ &\geq \frac{2}{\varepsilon}\Big(\int_{\{k\varepsilon\leq\Psi_1(u_\varepsilon)\leq D/2\}}|\nabla(\Psi_1(u_\varepsilon))| + \int_{\{k\varepsilon\leq\Psi_2(u_\varepsilon)\leq D/2\}}|\nabla(\Psi_2(u_\varepsilon))|\Big) \\ &= \frac{2}{\varepsilon}\int_{k\varepsilon}^{D/2}\mathrm{Per}_G\{\Psi_1(u_\varepsilon)<a\}\,da + \frac{2}{\varepsilon}\int_{k\varepsilon}^{D/2}\mathrm{Per}_G\{\Psi_2(u_\varepsilon)<a\}\,da \\ &\geq \frac{2}{\varepsilon}(D-2k\varepsilon)I(\alpha) \geq \frac{2DI(\alpha)}{\varepsilon} - C\,. \end{aligned} \tag{2.5}$$

It follows from (2.4)–(2.5) and (1.8) that there exists $a_\varepsilon \in (d_0/2, d_0)$ such that

$$\mathrm{Per}_G\{\Psi_1(u_\varepsilon) < a_\varepsilon\} - I(\alpha) = O(\varepsilon)\,. \tag{2.6}$$

Set

$$\beta_\varepsilon = \mu(\{\Psi_1(u_\varepsilon) < a_\varepsilon\})\,.$$

Denoting respectively by $\tilde{s}_1$ and $\tilde{s}_2$ the euclidean nearest point projections on Γ_1 and Γ_2, and

$$m_\varepsilon^{(i)} = \frac{1}{\mu(\{\Psi_i(u_\varepsilon) < a_\varepsilon\})}\int_{\{\Psi_i(u_\varepsilon)<a_\varepsilon\}} |\tilde{s}_i(u_\varepsilon)|\,, \quad \text{for } i = 1, 2\,, \tag{2.7}$$

we obtain by the argument of [1, Proposition 2.2] that

$$m_\varepsilon^{(1)}\mu(\{\Psi_1(u_\varepsilon) < a_\varepsilon\}) + m_\varepsilon^{(2)}\mu(\{\Psi_2(u_\varepsilon) < a_\varepsilon\}) = R_c + O(\varepsilon^{1/2})\,. \tag{2.8}$$

It follows that for some $K = K_\varepsilon$ which is uniformly bounded as a function of ε, we have

$$m_\varepsilon^{(1)}(\beta_\varepsilon + K\varepsilon^{1/2}) + m_\varepsilon^{(2)}(1 - \beta_\varepsilon - K\varepsilon^{1/2}) = R_c\,. \tag{2.9}$$

It follows from (2.9) that $\beta_\varepsilon + K\varepsilon^{1/2} \in \mathcal{I}_0$. Since I is a Lipschitz function we deduce that

$$\mathrm{Per}_G\{\Psi_1(u_\varepsilon) < a_\varepsilon\} \geq I(\beta_\varepsilon) \geq I(\beta_\varepsilon + K\varepsilon^{1/2}) - c\varepsilon^{1/2} \geq I(\alpha) - c\varepsilon^{1/2}\,. \tag{2.10}$$

Combining (2.10) with (2.6) yields

$$I(\beta_\varepsilon + K\varepsilon^{1/2}) - I(\alpha) = O(\varepsilon^{1/2})\,. \tag{2.11}$$

Since $\beta_\varepsilon + K\varepsilon^{1/2} \in \mathcal{I}_0$, we get from (2.11) and the uniqueness of α that $\lim_{\varepsilon\to 0}\beta_\varepsilon = \alpha$. But clearly

$$|\alpha_\varepsilon - \beta_\varepsilon| \leq \mu(\{\Psi_1(u_\varepsilon) \geq d_0/2\}) + O(\varepsilon) = O(\varepsilon)\,,$$

implying that also $\lim_{\varepsilon\to 0}\alpha_\varepsilon = \alpha$, and (2.1) follows.

For the proof of (2.2) we take a_ε as in (2.6). Since

$$\mathrm{Per}_G\{\Psi_1(u_\varepsilon) < a_\varepsilon\} = \|\chi_{\{\Psi_1(u_\varepsilon)<a_\varepsilon\}}\|_{BV} = I(\alpha) + O(\varepsilon)\,, \tag{2.12}$$

where $\|\cdot\|_{BV}$ stands for the BV-seminorm, the family of functions $\{\chi_{\{\Psi_1(u_\varepsilon)<a_\varepsilon\}}\}_{\varepsilon>0}$ is bounded in BV and for a subsequence $\varepsilon_n \to 0$ we have $\chi_{\{\Psi_1(u_{\varepsilon_n})<a_{\varepsilon_n}\}} \to \chi_E$ in $L^1(G)$ for some measurable set E, with

$$\mu(E) = \lim_{n\to\infty}\mu(\{\Psi_1(u_{\varepsilon_n}) < a_{\varepsilon_n}\}) = \alpha\,. \tag{2.13}$$

By the lower semicontinuity of the BV-seminorm w.r.t. the L^1-norm and (2.12) we have

$$\mathrm{Per}_G\,E = \|\chi_E\|_{BV} \le \liminf_{n\to\infty}\mathrm{Per}_G\{\Psi_1(u_{\varepsilon_n}) < a_{\varepsilon_n}\} = I(\alpha)\,. \tag{2.14}$$

By (2.13)–(2.14) we deduce that $E = G_1$ realizes the minimum for $I(\alpha)$. Finally, (2.2) follows since for every $t \in (0, d_0]$,

$$\|\chi_{\{\Psi_1(u_{\varepsilon_n})<a_{\varepsilon_n}\}} - \chi_{\{\Psi_1(u_{\varepsilon_n})<t\}}\|_{L^1(G)} \le \mu(\{\Psi_1(u_{\varepsilon_n}) > \min(t, d_0/2)\}) = O(\varepsilon)\,.$$

□

In the sequel we shall continue the study of the subsequence $\{u_{\varepsilon_n}\}$ given by Lemma 2.1, denoting occasionally $\varepsilon = \varepsilon_n$ for short. From property (NC) it follows that there exists $d_1 \in (0, d_0/2)$ such that

$$I(\alpha) \le I(\alpha + a)\,, \quad \forall a \in (-d_1, d_1)\,. \tag{2.15}$$

By Lemma 2.1 we have, for n large,

$$|\mu(\Psi_1(u_{\varepsilon_n}) < d) - \alpha| < \frac{d_1}{3} \ \text{ and } \ |\mu(\Psi_2(u_{\varepsilon_n}) < d) - (1-\alpha)| < \frac{d_1}{3},\ \forall d \in [\varepsilon_n^{\frac{1}{2}}, d_0]\,. \tag{2.16}$$

Fix any $\lambda \in (0, \frac{d_1}{3})$. By (2.5), (1.8) and Fubini Theorem, there exists $t \in (\lambda, 2\lambda)$ such that

$$|\,\mathrm{Per}_G\{\Psi_j(u_\varepsilon) > t\} - I(\alpha)| \le C\varepsilon, \quad j = 1, 2\,. \tag{2.17}$$

Furthermore, by Sard Lemma we may assume that t is a regular value for both maps $\Psi_1 \circ u_\varepsilon$ and $\Psi_2 \circ u_\varepsilon$. Put

$$A_{\varepsilon,t} = \{x \in G \,:\, \Psi_2(u_\varepsilon(x)) > t\}\,,$$

and write it as a disjoint union of its connected components

$$A_{\varepsilon,t} = \bigcup_{i\in K} C_i \,,$$

where K is a set of indices (countable, at most). From the choice of t we know that the boundary of each C_i is a C^1 closed curve. We also denote (see (1.13))

$$M_\varepsilon = \{x \in G \,:\, \widetilde{\Psi}(u_\varepsilon(x)) \geq d_0\} \,. \tag{2.18}$$

The next lemma shows that, roughly speaking, small components of $A_{\varepsilon,t}$ must be "very small".

Lemma 2.2. *There exists a constant $\tilde{c}_0$ such that, if for some $J \subset K$, $\mu(\bigcup_{j\in J} C_j) \leq \lambda$, then*

$$\mu\big(\bigcup_{j\in J} C_j\big) \leq \tilde{c}_0\varepsilon^2 \quad \textit{and} \quad \mathrm{Per}_G\,\big(\bigcup_{j\in J} C_j\big) \leq \tilde{c}_0\varepsilon \,. \tag{2.19}$$

Proof. Assume that $\mu(\bigcup_{j\in J} C_j) := m \leq \lambda$. Since $\lambda < \frac{d_1}{3}$ and $|\mu(A_{\varepsilon,t}) - \alpha| \leq \frac{d_1}{3}$ for ε small (by Lemma 2.1) we deduce from (2.15) that

$$\mathrm{Per}_G(\bigcup_{j\in K\setminus J} C_j) \geq I(\alpha) \,. \tag{2.20}$$

By (2.17) and (2.20) we get that

$$\mathrm{Per}_G(\bigcup_{j\in J} C_j) \leq C\varepsilon \,. \tag{2.21}$$

Since G is smooth, $\mathrm{Per}_G\, V \geq c\mu(V)^{1/2}$ for every subset $V \subset G$ of finite perimeter, and we conclude that $\mu(\bigcup_{j\in J} C_j) \leq \tilde{c}_0\varepsilon^2$ from (2.21). □

In view of Lemma 2.2 we may write $A_{\varepsilon,t}$ as disjoint union of two unions of components

$$A_{\varepsilon,t} = \bigcup_{j\in K_0} C_j \cup \bigcup_{j\in K_1} C_j \,, \tag{2.22}$$

where K_0 is a finite set consisting of all the j's for which $\mu(C_j) > \lambda$, and

$$\mu(\bigcup_{j\in K_1} C_j) \leq \tilde{c}_0\varepsilon^2 \,. \tag{2.23}$$

In an analogous manner, we may write the set $B_{\varepsilon,t} = \{x \in G : \Psi_1(u_\varepsilon(x)) > t\}$ as a disjoint union of its components $B_{\varepsilon,t} = \bigcup_{i\in L} D_i$, and then decompose

$$B_{\varepsilon,t} = \bigcup_{j\in L_0} D_j \cup \bigcup_{j\in L_1} D_j\,, \tag{2.24}$$

where L_0 is a finite set consisting of all the j's for which $\mu(D_j) > \lambda$, and $\mu(\bigcup_{j\in L_1} D_j) \le \tilde{c}_0\varepsilon^2$.

In the proof of the next lemma we shall need the following gradient estimate. This is the only point where we use the assumption (H_3') . The proof uses an argument that was shown to us by Petru Mironescu and is given in the appendix.

Proposition 2.1. *There exists a constant $\tilde{c} > 0$, independent of ε, such that for $\varepsilon < \varepsilon_0$,*

$$|\nabla u_\varepsilon| \le \frac{\tilde{c}}{\varepsilon} \text{ in } G\,. \tag{2.25}$$

Next we obtain a more precise information on the set M_ε (see (2.18)).

Lemma 2.3. *There exist finite subsets, $K_2 \subset K_1$ and $L_2 \subset L_1$, such that*

$$M_\varepsilon \subset (\bigcup_{k\in K_0} C_k) \cup (\bigcup_{l\in L_0} D_l) \cup (\bigcup_{k\in K_2} C_k) \cup (\bigcup_{l\in L_2} D_l)\,. \tag{2.26}$$

Furthermore, for each $k \in K_2$ there exists a disc $B_{c_1\varepsilon}(x)$ with $x \in M_\varepsilon$ such that $B_{c_1\varepsilon}(x) \subset C_k$. An analogous statement holds for each $l \in L_2$.

Proof. Take any $x \in M_\varepsilon$ such that $x \in C_k$ for some $k \in K_1$ (the same argument applies if $x \in D_l$ for some $l \in L_1$). Thanks to the L^∞ gradient bound (2.25) and the Lipschitz property of Ψ_2 we obtain, for some constant c_0,

$$|\nabla(\Psi_2(u_\varepsilon))| \le \frac{c_0}{\varepsilon} \text{ in } G\,. \tag{2.27}$$

Since $\Psi_2(u_\varepsilon) = t \le \frac{d_0}{3}$ on $\partial C_k \cap G$ while $\Psi_2(u_\varepsilon(x)) \ge d_0$, there exists c_1 such that $B_{c_1\varepsilon}(x) \cap G \subset C_k$ (we can take $c_1 = \frac{2d_0}{3c_0}$). Since G is smooth, there exists $\alpha = \alpha(G) > 0$ such that $\mu(B_{c_1\varepsilon}(x) \cap G) \ge \alpha\pi(c_1\varepsilon)^2$. By (2.23) we deduce that the number of the C_k's with $k \in K_1$, having a nonempty intersection with M_ε, is bounded by $\frac{\tilde{c}_0\varepsilon^2}{\alpha\pi(c_1\varepsilon)^2} = \frac{\tilde{c}_0}{\alpha\pi c_1^2}$. □

Remark 2.1. Our techniques are limited to dimension $N = 2$ since in higher dimension we are not able to prove an analogous result to Lemma 2.3. In fact, there is an analog to Lemma 2.2 and the second inequality in (2.19) holds in any dimension, i.e., we have,

$$\mu(\bigcup_{j\in J} C_j) \le \lambda \implies \mathrm{Per}_G\,(\bigcup_{j\in J} C_j) \le \tilde{c}_0\varepsilon\,.$$

However, as for the first inequality in (2.19), the isoperimetric inequality gives in dimension n, $\mu\big(\bigcup_{j\in J} C_j\big) \leq c\varepsilon^{\frac{N}{N-1}}$. The argument of Lemma 2.3 implies that $\mu(C_k) \geq c\varepsilon^N$, as a lower bound for the measure of a component C_k with $k \in K_1$ that intersects M_ε. Only in dimension two the powers of ε in the lower and upper bounds match each other, implying finiteness of the number of "small components" intersecting M_ε. It is possible that the techniques of [8] can be used to prove L^1-convergence in higher dimensions, but we didn't investigate it.

3 Proof of Theorem 2

Lemma 3.1. *If $k_0 \in K_0$ is such that $C_{k_0} \cap G_2 \neq \emptyset$, then*

$$0 \leq \operatorname{Per}_{G_2} C_{k_0} - \operatorname{Per}_{C_{k_0}} G_1 \cap C_{k_0} \leq c_2\varepsilon\,. \tag{3.1}$$

Proof. Note first that since our choice of t ensures the regularity of the boundary of $A_{t,\varepsilon}$, all the perimeters in the statement of the lemma can be represented as the lengths of C^1-curves, denoted by the one dimensional Hausdorff measure in the sequel. By (2.17) we have

$$I(\alpha) \leq \operatorname{Per}_G A_{\varepsilon,t} = \operatorname{Per}_G A_{\varepsilon,t} \setminus C_{k_0} + \operatorname{Per}_G C_{k_0} \leq I(\alpha) + c_2\varepsilon\,. \tag{3.2}$$

Since $\mu(C_{k_0} \cap G_2) \to 0$, we have $\operatorname{Per}_G A_{\varepsilon,t} \setminus (C_{k_0} \cap G_2) \geq I(\alpha)$. Therefore,

$$\begin{aligned} I(\alpha) \leq \operatorname{Per}_G A_{\varepsilon,t} \setminus (C_{k_0} \cap G_2) &= \operatorname{Per}_G A_{\varepsilon,t} \setminus C_{k_0} + \operatorname{Per}_G C_{k_0} \\ &\quad - \mathcal{H}^1(\partial C_{k_0} \cap G_2) + \mathcal{H}^1(\partial G_1 \cap C_{k_0}) \\ &\leq I(\alpha) + c_2\varepsilon - \mathcal{H}^1(\partial C_{k_0} \cap G_2) + \mathcal{H}^1(\partial G_1 \cap C_{k_0})\,, \end{aligned}$$

i.e.,

$$0 \leq \mathcal{H}^1(\partial C_{k_0} \cap G_2) - \mathcal{H}^1(\partial G_1 \cap C_{k_0}) \leq c_2\varepsilon\,,$$

which is (3.1). □

An analogous statement to Lemma 3.1 holds for the components $D_l, l \in L_0$. Next we define several quantities that will be useful in the sequel. First we set, for each $r > 0$,

$$T(r) = \sup\{s \in (0, d_1) \,:\, I(\alpha + s) - I(\alpha) \leq r\}\,. \tag{3.3}$$

For each C_{k_0} as in Lemma 3.1, we define

$$d_M = d_M^{k_0} = \min_{1\leq j\leq J} \max_{x\in\partial C_{k_0}\cap G_2} \operatorname{dist}(x, \Sigma^{(j)})\,. \tag{3.4}$$

Proposition 3.1. *There exists a constant c_3 such that, for C_{k_0} as in Lemma 3.1, we have*

$$d_M \le \delta(\varepsilon) := c_3 \max\left(\varepsilon^{\frac{1}{2}}, T(c_2\varepsilon)\right). \tag{3.5}$$

Proof. Assume that d_M is attained at x_0 for $j = j_0$ and let $\mathcal{T}$ denote the closure of the component of $\partial C_{k_0} \cap G_2$ containing x_0. Set

$$d_m = \min_{x\in\mathcal{T}} \operatorname{dist}(x, \Sigma^{(j_0)}) \text{ and } \eta = \frac{d_m}{d_M}.$$

We distinguish two possibilities:

(i) $\eta \le \frac{1}{2}$.
(ii) $\eta > \frac{1}{2}$.

Case (i): Let d_m be attained at the point y_0. It may happen that $y_0 \in \Sigma^{(j_0)}$, i.e., $\eta = 0$, as shown in Figure 2. Suppose an arclength parametrization of the part of the curve $\mathcal{T}$ joining x_0 to y_0 is given by γ on the interval $[a, b]$, with $\gamma(a) = x_0$ and $\gamma(b) = y_0$. Denote by γ_1 the projection of γ on $\Sigma^{(j_0)}$ and by γ_2 the projection in the orthogonal direction. From Lemma 3.1 it follows that

$$\begin{aligned} c_2\varepsilon \ge \int_a^b |\gamma'(t)|\, dt - \Big|\int_a^b \gamma_1'(t)\, dt\Big| \ge \int_a^b (1 - |\gamma_1'(t)|)\, dt = \int_a^b \frac{|\gamma_2'(t)|^2}{1 + |\gamma_1'(t)|}\, dt \\ \ge \frac{1}{2(b-a)}\Big|\int_a^b \gamma_2'(t)\Big|^2 = \frac{(d_M - d_m)^2}{2(b-a)}, \end{aligned}$$

implying that

$$\frac{d_M}{2} \le d_M - d_m = O(\sqrt{\varepsilon}). \tag{3.6}$$

Case (ii): The assumption $\eta > \frac{1}{2}$ implies in particular that $\mathcal{T} = \partial C_{k_0} \cap G_2$, $\operatorname{dist}(\mathcal{T}, \Sigma) > 0$ and

$$\mu(C_{k_0} \cup G_1) - \mu(G_1) = \mu(C_{k_0} \cap G_2) \ge \frac{d_M l(\Sigma^{(j_0)})}{2}, \tag{3.7}$$

where $l(\Sigma^{(j_0)})$ denotes the length of the component (segment) $\Sigma^{(j_0)}$, see Figure 3. Applying Lemma 3.1 yields

$$I(\mu(C_{k_0} \cup G_1)) - I(\alpha) \le \operatorname{Per}_G(C_{k_0} \cup G_1) - I(\alpha) = l(\mathcal{T}) - l(\Sigma^{(j_0)}) \le c_2\varepsilon. \tag{3.8}$$

By (3.7)–(3.8) and the definition (3.3) of $T(r)$ we obtain

$$\frac{d_M l(\Sigma^{(j_0)})}{2} \le T(c_2\varepsilon). \tag{3.9}$$

Combining (3.6) and (3.9) yields (3.5). □

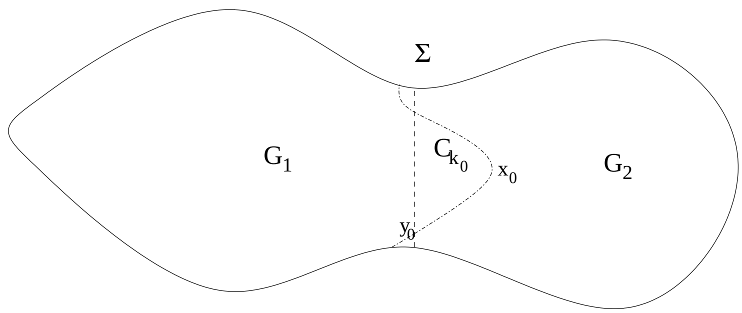

Figure 2: An example of Case (i) with $d_m = 0$, hence $\eta = 0$

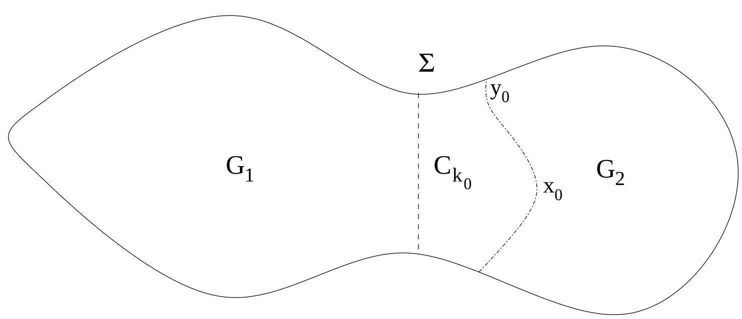

Figure 3: An example of Case (ii)

Obviously, an analogous statement holds for the components $D_l, l \in L_0$. Before moving to the next proposition we introduce some more notation. We denote by Ω the domain lying between Γ_1 and Γ_2. First we define the functions θ_1, θ_2 by

$$\begin{aligned}\theta_1(x) &= d_W(x, \Gamma_2) - (D - d_0), \quad \text{for } x \in \Omega \text{ with } \Psi_1(x) = d_0\,,\\ \theta_2(x) &= d_W(x, \Gamma_1) - (D - d_0), \quad \text{for } x \in \Omega \text{ with } \Psi_2(x) = d_0\,.\end{aligned}$$

Then we define the functions $\tilde\theta_1, \tilde\theta_2$ on Γ_1 and Γ_2, respectively, by

$$\tilde\theta_j(x) = \min\{\theta_j(y) \,:\, y \in \Omega,\ \Psi_j(y) = d_0,\ d_{\Gamma_j}(\tilde s_j(y), p_j) \geq d_{\Gamma_j}(x, p_j)\},\ j = 1, 2\,, \tag{3.10}$$

where d_{Γ_j} stands for the geodesic distance on Γ_j. Consecutively, we define the functions

$$Q_j(t) = \sup\{d_{\Gamma_j}(x, p_j) \,:\, x \in \Gamma_j,\ \tilde\theta_j(x) \leq t\}\,,\ j = 1, 2. \tag{3.11}$$

Finally, put

$$\tilde\delta(\varepsilon) = \delta(\varepsilon) + Q_1(\frac{1}{4} l_1^2 \varepsilon^{\frac{1}{2}}) + Q_2(\frac{1}{4} l_2^2 \varepsilon^{\frac{1}{2}}) + \varepsilon^{\frac{1}{2}}\,. \tag{3.12}$$

The motivation for the above definitions will become clearer in the course of the proof of Proposition 3.2 below. In the sequel we shall use in a neighborhood of each component $\Sigma^{(j)}$ ($j = 1, \ldots, J$) of Σ the system of coordinates $(\sigma, \tau) = (\sigma_j, \tau_j)$ obtained by projecting the point x on $\Sigma^{(j)}$. More precisely, $\tau_j = \tau_j(x)$ is the signed distance from x to $\Sigma^{(j)}$ (with the convention that points with $\tau < 0$ belong to G_1) and $\sigma_j = \sigma_j(x) \in [0, l(\Sigma^{(j)})]$ is the arclength parameter on the segment $\Sigma^{(j)}$ corresponding to the nearest point projection of x on $\Sigma^{(j)}$. Notice that both σ_j and τ_j are affine functions of the original coordinates (x_1, x_2). We next prove a lower bound for the energy in the neighborhood $\Sigma_{2\tilde\delta(\varepsilon)+4\varepsilon^{\frac{1}{2}}}$ of Σ (see (1.20)). We start with a lemma.

Lemma 3.2. *There exist $c_4 > 0$ and $d(\varepsilon) \in (\tilde\delta(\varepsilon), 2\tilde\delta(\varepsilon) + 4\varepsilon^{\frac{1}{2}})$ such that*

$$\sum_{j=1}^{J} \int_{\{\tau_j = d(\varepsilon)\} \cup \{\tau_j = -d(\varepsilon)\}} |\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon^2} \leq \frac{c_4}{d(\varepsilon)}\,. \tag{3.13}$$

Proof. By Proposition 3.1 and Lemma 2.3, if $x \in M_\varepsilon$ and $\mathrm{dist}(x, \Sigma) > \delta(\varepsilon)$, then x belongs to one of the components C_i, $i \in K_2$, or D_i, $i \in L_2$. Since the total perimeter of these components is at most $O(\varepsilon)$, we may find $\delta_1(\varepsilon) \in (\tilde\delta(\varepsilon), \tilde\delta(\varepsilon) + \varepsilon^{\frac{1}{2}})$ such that both $l_1^{(j)} = \{\tau_j = \delta_1(\varepsilon)\}$ and $l_2^{(j)} = \{\tau_j = -\delta_1(\varepsilon)\}$ do not intersect M_ε and such that

$\int_{l_1^{(j)}\cup l_2^{(j)}} W(u_\varepsilon) \le C\varepsilon^{\frac{1}{2}}$, for $j = 1, \ldots, J$. By (1.16) it follows that also $\int_{l_1^{(j)}} \Psi_1(u_\varepsilon) \le C\varepsilon^{\frac{1}{2}}$ and $\int_{l_2^{(j)}} \Psi_2(u_\varepsilon) \le C\varepsilon^{\frac{1}{2}}$. Therefore, for each $j = 1, \ldots, J$,

$$\begin{aligned} E_\varepsilon(u_\varepsilon; \{|\tau_j| \le \delta_1(\varepsilon)\}) &\ge \frac{2}{\varepsilon} \int_{\{|\tau_j|\le\delta_1(\varepsilon), \sigma_j\in[0,l(\Sigma^{(j)})]\}} |\nabla(\Psi_1(u_\varepsilon))| \\ &\ge \frac{2}{\varepsilon} \left| \int_{\{|\tau_j|\le\delta_1(\varepsilon), \sigma_j\in[0,l(\Sigma^{(j)})]\}} \nabla(\Psi_1(u_\varepsilon)) \cdot \nabla\tau \right| \\ &= \frac{2}{\varepsilon} \int_0^{l(\Sigma^{(j)})} \big(\Psi_1(u_\varepsilon(\sigma_j, \delta_1(\varepsilon))) - \Psi_1(u_\varepsilon(\sigma_j, -\delta_1(\varepsilon)))\big)\, d\sigma \\ &\ge \frac{2}{\varepsilon}(D - c\varepsilon^{\frac{1}{2}}) l(\Sigma^{(j)}) \end{aligned} \tag{3.14}$$

Summing (3.14) for $j = 1, \ldots, J$ yields

$$E_\varepsilon(u_\varepsilon; \Sigma_{\delta_1(\varepsilon)}) \ge \frac{2DI(\alpha)}{\varepsilon} - C\varepsilon^{-\frac{1}{2}}\,. \tag{3.15}$$

By (1.8) and (3.15) we conclude that $\int_{G\setminus\Sigma_{\delta_1(\varepsilon)}} W(u_\varepsilon) \le C\varepsilon^{\frac{3}{2}}$. Therefore, there exists $\delta_2(\varepsilon) \in (\delta_1(\varepsilon), \delta_1(\varepsilon) + \varepsilon^{\frac{1}{2}})$ such that

$$\int_{\{\tau_j=-\delta_2(\varepsilon)\}\cup\{\tau_j=\delta_2(\varepsilon)\}} W(u_\varepsilon) \le C\varepsilon\,,\ j = 1, \ldots, J\,. \tag{3.16}$$

We can now repeat the argument used above in (3.14)–(3.15), with $\delta_2(\varepsilon)$ replacing $\delta_1(\varepsilon)$, to obtain

$$E_\varepsilon(u_\varepsilon; \Sigma_{\delta_2(\varepsilon)}) \ge \frac{2DI(\alpha)}{\varepsilon} - C\,. \tag{3.17}$$

By (3.17) and (1.8) it follows that $E_\varepsilon(u_\varepsilon; G \setminus \Sigma_{\delta_2(\varepsilon)}) \le C$, hence there exists $d(\varepsilon) \in (\delta_2(\varepsilon), 2\delta_2(\varepsilon))$ satisfying (3.13). □

Proposition 3.2. *There exists $c_5 > 0$ such that*

$$\begin{aligned} &E_\varepsilon(u_\varepsilon; \Sigma_{d(\varepsilon)}) - \frac{2DI(\alpha)}{\varepsilon} \ge \\ &\frac{c_5}{d(\varepsilon)} \sum_{j=1}^{J} \int_0^{l(\Sigma^{(j)})} \Big(d^2_{\Gamma_1}(\tilde{s}_1(u_\varepsilon(s, -d(\varepsilon)), p_1) + d^2_{\Gamma_2}(\tilde{s}_2(u_\varepsilon(s, d(\varepsilon)), p_2)\Big)\, ds + o(1)\,. \end{aligned} \tag{3.18}$$

Proof. We claim: there exists $c_5 > 0$ such that for every $j = 1, \ldots, J$ and every $\sigma_j \in$

$(0,l(\Sigma^{(j)}))$ we have

$$\int_{-d(\varepsilon)}^{d(\varepsilon)}\Big(|\frac{\partial u_\varepsilon}{\partial\tau}(\sigma_j,\tau)|^2+\frac{W(u_\varepsilon(\sigma_j,\tau))}{\varepsilon^2}\Big)\,d\tau$$
$$\geq\frac{2D}{\varepsilon}-\frac{2}{\varepsilon}\big(\Psi_1(u_\varepsilon(\sigma_j,-d(\varepsilon)))+\Psi_2(u_\varepsilon(\sigma_j,d(\varepsilon)))\big)$$
$$+\frac{c_5}{d(\varepsilon)}\big(d^2_{\Gamma_1}(\tilde{s}_1(u_\varepsilon(\sigma_j,-d(\varepsilon)),p_1)+d^2_{\Gamma_2}(\tilde{s}_2(u_\varepsilon(\sigma_j,d(\varepsilon)),p_2)\big)+o(1)\,. \quad (3.19)$$

To prove the claim it suffices to show that

$$\int_{-d(\varepsilon)}^{d(\varepsilon)}\Big(|\frac{\partial u_\varepsilon}{\partial\tau}(\sigma_j,\tau)|^2+\frac{W(u_\varepsilon(\sigma_j,\tau))}{\varepsilon^2}\Big)\,d\tau$$
$$\geq\frac{2D}{\varepsilon}-\frac{2}{\varepsilon}\Big(\Psi_1\big(u_\varepsilon(\sigma_j,-d(\varepsilon))\big)+\Psi_2\big(u_\varepsilon(\sigma_j,d(\varepsilon))\big)\Big)$$
$$+\frac{c_5}{d(\varepsilon)}d^2_{\Gamma_1}\big(\tilde{s}_1(u_\varepsilon(\sigma_j,-d(\varepsilon))),p_1\big)+o(1)\,, \quad (3.20)$$

since then an analogous estimate to (3.20) holds when we replace $\tilde{s}_1$ and p_1 by $\tilde{s}_2$ and p_2, respectively. Finally (3.19) would follow by adding the two estimates and dividing by 2. To prove (3.20) we begin by setting

$$\eta_1=\inf\{\tau\in(-d(\varepsilon),d(\varepsilon))\,:\,\Psi_1(u_\varepsilon(\sigma_j,\tau))=d_0\}\,,\ y_0=u_\varepsilon(\sigma_j,-d(\varepsilon))\,,\ y_1=u_\varepsilon(\sigma_j,\eta_1)\,.$$

We first consider the contribution to the integral on the l.h.s. of (3.20) from the interval $(\eta_1,d(\varepsilon))$. We have

$$\int_{\eta_1}^{d(\varepsilon)}\Big(|\frac{\partial u_\varepsilon}{\partial\tau}(\sigma_j,\tau)|^2+\frac{W(u_\varepsilon(\sigma_j,\tau))}{\varepsilon^2}\Big)\,d\tau\geq\frac{2}{\varepsilon}\int_{\eta_1}^{d(\varepsilon)}\Big(W(u_\varepsilon(\sigma_j,\tau))\Big)^{1/2}\big|\frac{\partial u_\varepsilon}{\partial\tau}(\sigma,\tau)\big|\,d\tau$$
$$\geq\frac{2}{\varepsilon}d_W(y_1,u_\varepsilon(\sigma_j,d(\varepsilon)))\geq\frac{2}{\varepsilon}\Big(d_W(y_1,\Gamma_2)-\Psi_2(u_\varepsilon(\sigma_j,d(\varepsilon)))\Big)$$
$$=\frac{2}{\varepsilon}\Big(\theta_1(y_1)+D-d_0-\Psi_2(u_\varepsilon(\sigma_j,d(\varepsilon)))\Big)\,. \quad (3.21)$$

On the interval $(-d(\varepsilon),\eta_1)$, $u_\varepsilon(\sigma_j,\cdot)$ takes values close to Γ_1 (since $\Psi_1(u_\varepsilon(\sigma_j,\tau))\leq d_0$). This enables us to get a more precise estimate, taking into account also the contribution of the component of the gradient which is "tangential" to Γ_1 (a similar argument was used in the proof of [2, Lemma 3.3]). At each point (σ_j,τ) with $\tau\in(-d(\varepsilon),\eta_1)$ we write

$$|\partial_\tau u_\varepsilon|^2=|\partial_\tau^{(\tilde{t})}u_\varepsilon|^2+|\partial_\tau^{(\tilde{s})}u_\varepsilon|^2\,,$$

where

$$\partial_\tau^{(\tilde t)} u_\varepsilon = \frac{\partial_\tau\big(\Psi_1(u_\varepsilon)\big)}{|\nabla\Psi_1(u_\varepsilon)|} = \partial_\tau u_\varepsilon\cdot\frac{\nabla\Psi_1(u_\varepsilon)}{|\nabla\Psi_1(u_\varepsilon)|} \quad\text{and}\quad \partial_\tau^{(\tilde s)} u_\varepsilon = \frac{\partial_\tau\big(\tilde s_1(u_\varepsilon)\big)}{|\nabla\tilde s(u_\varepsilon)|} = \partial_\tau u_\varepsilon\cdot\frac{\nabla\tilde s(u_\varepsilon)}{|\nabla\tilde s(u_\varepsilon)|}\,.$$

Since for some $\beta>0$, $|\nabla\tilde s_1(y)|\le\frac{1}{\sqrt\beta}$ whenever $\Psi_1(y)\le d_0$, we conclude that

$$|\partial_\tau^{(\tilde s)} u_\varepsilon|^2 \ge \beta|\partial_\tau\big(\tilde s_1(u_\varepsilon)\big)|^2\,,\ \text{for } \tau\in(-d(\varepsilon),\eta_1)\,.$$

From the above we conclude that

$$\begin{aligned}\int_{-d(\varepsilon)}^{\eta_1}\Big(|\frac{\partial u_\varepsilon}{\partial\tau}(\sigma_j,\tau)|^2+\frac{W(u_\varepsilon(\sigma_j,\tau))}{\varepsilon^2}\Big)\,d\tau &\ge \int_{-d(\varepsilon)}^{\eta_1}\frac{|\partial_\tau(\Psi_1(u_\varepsilon))|^2}{W(u_\varepsilon)}+\frac{W(u_\varepsilon)}{\varepsilon^2}+\beta|\partial_\tau\big(\tilde s_1(u_\varepsilon)\big)|^2\\ &\ge \frac{2}{\varepsilon}\Big|\int_{-d(\varepsilon)}^{\eta_1}\partial_\tau(\Psi_1(u_\varepsilon))\Big| + \beta\frac{d^2_{\Gamma_1}(\tilde s_1(y_0),\tilde s_1(y_1))}{d(\varepsilon)+\eta_1}\\ &= \frac{2}{\varepsilon}(d_0-\Psi_1(y_0)) + \beta\frac{d^2_{\Gamma_1}(\tilde s_1(y_0),\tilde s_1(y_1))}{d(\varepsilon)+\eta_1}\,. \end{aligned}\tag{3.22}$$

Adding together (3.21) and (3.22) gives

$$\begin{aligned}\int_{-d(\varepsilon)}^{d(\varepsilon)}\Big(|\frac{\partial u_\varepsilon}{\partial\tau}(\sigma,\tau)|^2+\frac{W(u_\varepsilon(\sigma,\tau))}{\varepsilon^2}\Big)\,d\tau &\ge \beta\frac{d^2_{\Gamma_1}(\tilde s_1(y_0),\tilde s_1(y_1))}{d(\varepsilon)+\eta_1}\\ &\quad+\frac{2}{\varepsilon}\Big(D-\Psi_1(u_\varepsilon(-d(\varepsilon),\sigma))-\Psi_2(u_\varepsilon(d(\varepsilon),\sigma))+\theta_1(y_1)\Big)\,.\end{aligned}\tag{3.23}$$

We consider two cases:

(i) $d_{\Gamma_1}(\tilde s_1(y_0),\tilde s_1(y_1)) > \frac{d_{\Gamma_1}(\tilde s_1(y_0),p_1)}{2}$,

(ii) $d_{\Gamma_1}(\tilde s_1(y_0),\tilde s_1(y_1)) \le \frac{d_{\Gamma_1}(\tilde s_1(y_0),p_1)}{2}$.

In case (i), (3.20) clearly follows from (3.23), so it remains to consider case (ii). Denote by z_1 a point on Γ_1 satisfying

$$d_{\Gamma_1}(z_1,p_1) = d_{\Gamma_1}(z_1,\tilde s_1(y_0)) = \frac12 d_{\Gamma_1}(p_1,\tilde s_1(y_0))\,.$$

We distinguish two possibilities. If $\frac{\tilde\theta_1(z_1)}{\varepsilon} > \frac{d^2_{\Gamma_1}(p_1,\tilde s_1(y_0))}{d(\varepsilon)}$ then (3.20) follows again from (3.23). Indeed, this follows from (3.10) by the inequality $\theta_1(y_1)\ge\tilde\theta_1(z_1)$ that holds since by (ii)

$$d_{\Gamma_1}(\tilde s_1(y_1),p_1) \ge d_{\Gamma_1}(\tilde s_1(y_0),p_1) - d_{\Gamma_1}(\tilde s_1(y_1),\tilde s_1(y_0)) \ge \frac12 d_{\Gamma_1}(\tilde s_1(y_0),p_1) = d_{\Gamma_1}(z_1,p_1)\,.$$

We assume then that

$$\frac{\tilde{\theta}_1(z_1)}{\varepsilon} \leq \frac{d^2_{\Gamma_1}(p_1, \tilde{s}_1(y_0))}{d(\varepsilon)} \,. \tag{3.24}$$

We claim that the above inequality implies that

$$\frac{d^2_{\Gamma_1}(\tilde{s}_1(y_0), p_1)}{d(\varepsilon)} = o(1) \,. \tag{3.25}$$

Clearly, combining (3.25) with (3.23) yields (3.20). It suffices then to prove (3.25). First, note that (3.24) implies that $\tilde{\theta}_1(z_1) \leq \frac{l^2(\Gamma_1)\varepsilon}{4d(\varepsilon)} \leq \frac{1}{4}l^2(\Gamma_1)\varepsilon^{\frac{1}{2}}$. By the definition (3.11) of Q_1 it follows that

$$d_{\Gamma_1}(\tilde{s}_1(y_0), p_1) = 2d_{\Gamma_1}(z_1, p_1) \leq 2Q_1(\frac{1}{4}l^2(\Gamma_1)\varepsilon^{\frac{1}{2}}) \,.$$

Since $d(\varepsilon) \geq Q_1(\frac{1}{4}l^2(\Gamma_1)\varepsilon^{\frac{1}{2}})$, we obtain that

$$\frac{d^2_{\Gamma_1}(\tilde{s}_1(y_0), p_1)}{d(\varepsilon)} \leq 4Q_1(\frac{1}{4}l^2(\Gamma_1)\varepsilon^{\frac{1}{2}}) \,,$$

and (3.25) follows. The validity of the claim (3.19) is now established.

Finally, we note that by (3.13) and (1.16) we have for each $j \in \{1, \dots, J\}$

$$\int_0^{l(\Sigma^{(j)})} \Psi_1(u_\varepsilon(\sigma_j, -d(\varepsilon)) + \Psi_2(u_\varepsilon(\sigma_j, d(\varepsilon)) \leq C \int_0^{l(\Sigma^{(j)})} W(u_\varepsilon) \leq \frac{C\varepsilon^2}{d(\varepsilon)} \leq C\varepsilon^{3/2} \,,$$

so integrating (3.19) for $\sigma_j \in (0, l(\Sigma^{(j)}))$ and summing over $j = 1, \dots, J$ yields (3.18). □

Thanks to Proposition 3.2 we can now conclude the convergence of $\{u_{\varepsilon_n}\}$ away from Σ.

Proposition 3.3. *For a subsequence we have*

$$u_{\varepsilon_n} \rightharpoonup u_* \text{ in } H^1(G \setminus \Sigma_\lambda), \ \forall \lambda > 0 \,,$$

where $u_* : G \to \Gamma_1 \cup \Gamma_2$ *satisfies*

$$\int_G |u_*| = R_c \,, \ u_*|_{G_j} \in H^1(G_j, \Gamma_j) \text{ and } \mathrm{Tr}(u_*, \partial G_j \cap G) = p_j, \ j = 1, 2. \tag{3.26}$$

Proof. For each $\lambda > 0$ we have by Proposition 3.2 and the upper-bound,

$$E_\varepsilon(u_\varepsilon, G \setminus \Sigma_\lambda) \leq C \,,$$

hence, passing to a diagonal subsequence, we may extract a subsequence satisfying

$$u_{\varepsilon_n} \rightharpoonup u_* \text{ in } H^1_{\mathrm{loc}}(\overline{G} \setminus \Sigma)\,, \quad \text{with} \;\; u_*|_{G_j} \in H^1(G_j, \Gamma_j),\; j = 1, 2\,.$$

We shall next verify that u_* satisfies both the constraint and the boundary conditions in (3.26).

Claim 1: $\displaystyle\int_G |u_*| = R_c$.

First, notice that for any $\lambda > 0$,

$$R_c = \int_G |u_{\varepsilon_n}| = \int_{\Sigma_\lambda} |u_{\varepsilon_n}| + \int_{G\setminus\Sigma_\lambda} |u_{\varepsilon_n}|\,.$$

Since $u_{\varepsilon_n} \to u_*$ in $L^1(G \setminus \Sigma_\lambda)$ we have $\int_{G\setminus\Sigma_\lambda} |u_*| = \int_{G\setminus\Sigma_\lambda} |u_{\varepsilon_n}| + o^{(\lambda)}_{\varepsilon_n}(1)$. Here and in the sequel $o^{(\lambda)}_{\varepsilon_n}(1)$ denotes a quantity that goes to zero with ε_n, for every fixed $\lambda > 0$. By (H_3) and (1.8),

$$\begin{aligned}\int_{\Sigma_\lambda} |u_{\varepsilon_n}| &= \int_{\Sigma_\lambda\cap\{|u_{\varepsilon_n}|>R_0\}} |u_{\varepsilon_n}| + \int_{\Sigma_\lambda\cap\{|u_{\varepsilon_n}|\le R_0\}} |u_{\varepsilon_n}| \\ &\le \tilde{c}_0 \int_{\Sigma_\lambda\cap\{|u_{\varepsilon_n}|>R_0\}} W(u_{\varepsilon_n}) + C\lambda R_0 \le C(\varepsilon_n + \lambda R_0)\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_G |u_*| &= \int_{G\setminus\Sigma_\lambda} |u_*| + \int_{\Sigma_\lambda} |u_*| = \int_{G\setminus\Sigma_\lambda} |u_*| + o_\lambda(1) = \int_{G\setminus\Sigma_\lambda} |u_{\varepsilon_n}| + o^{(\lambda)}_{\varepsilon_n}(1) + o_\lambda(1) \\ &= R_c - \int_{\Sigma_\lambda} |u_{\varepsilon_n}| + o^{(\lambda)}_{\varepsilon_n}(1) + o_\lambda(1) = R_c + o^{(\lambda)}_{\varepsilon_n}(1) + o_\lambda(1)\,.\end{aligned}$$

Letting $\varepsilon_n \to 0$ and then $\lambda \to 0$ yields the claim.

Claim 1: $\mathrm{Tr}(u_*, \partial G_j \cap G) = p_j, j = 1, 2.$

In the sequel we shall write again for the sake of simplicity u_ε instead of u_{ε_n}. Since the geodesic distance on Γ_j is equivalent to the euclidean distance, we get from Lemma 3.2 and Proposition 3.2 that

$$\sum_{j=1}^{J} \int_{\{\tau_j = \pm d(\varepsilon)\}} |\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon^2} + \frac{|\tilde{s}_1(u_\varepsilon) - p_1|^2}{d^2(\varepsilon)} \le \frac{C}{d(\varepsilon)}\,. \tag{3.27}$$

By (3.27) and (1.16) we have for every j,

$$\frac{1}{d(\varepsilon)}\int_{\{\tau_j=-d(\varepsilon)\}}|u_\varepsilon-p_1|^2\le\frac{2}{d(\varepsilon)}\int_{\{\tau_j=-d(\varepsilon)\}}|u_\varepsilon-\tilde{s}_1(u_\varepsilon)|^2+|\tilde{s}_1(u_\varepsilon)-p_1|^2$$

$$\le\frac{C}{d(\varepsilon)}\int_{\{\tau_j=-d(\varepsilon)\}}W(u_\varepsilon)+C\le\frac{C\varepsilon^2}{d^2(\varepsilon)}+C\le C\,. \quad (3.28)$$

We now define a new map w_ε on G_1 by

$$w_\varepsilon(x)=\begin{cases}u_\varepsilon(x) & x\in G_1\setminus\Sigma_{d(\varepsilon)},\\ (1-\frac{\tau_j}{d(\varepsilon)})p_1+\frac{\tau_j}{d(\varepsilon)}u_\varepsilon(\sigma_j,d(\varepsilon)) & x\in\Sigma^{(j)}_{d(\varepsilon)}\,,j=1,\dots,J\,.\end{cases}$$

Using (1.8),(3.18) and (3.27)–(3.28) we get by a direct computation that

$$\int_{\Sigma_{d(\varepsilon)}}|\nabla w_\varepsilon|^2+|w_\varepsilon|^2\le C\,,$$

so we have $w_\varepsilon\rightharpoonup u_*$ in $H^1(G_1)$, implying that $\mathrm{Tr}(u_*,\partial G_1\cap G)=p_1$. The same argument shows that also $\mathrm{Tr}(u_*,\partial G_2\cap G)=p_2$. □

Next, we prove a lower bound for the energy.

Proposition 3.4. $E_\varepsilon(u_\varepsilon)\ge\frac{2DI(\alpha)}{\varepsilon}+\int_G|\nabla u_*|^2+o(1)\,.$

Proof. It suffices to consider a subsequence u_{ε_n} as in Proposition 3.3. Fix any $\lambda>0$. By Proposition 3.2,

$$E_{\varepsilon_n}(u_{\varepsilon_n};\Sigma_\lambda)\ge\frac{2DI(\alpha)}{\varepsilon_n}+o(1)\,. \quad (3.29)$$

By Proposition 3.3,

$$\liminf_{\varepsilon_n\to0}E_{\varepsilon_n}(u_{\varepsilon_n};G\setminus\Sigma_\lambda)\ge\int_{G\setminus\Sigma_\lambda}|\nabla u_*|^2\,. \quad (3.30)$$

Since λ is arbitrary, the result follows by combining (3.29) and (3.30). □

In order to identify the limit u_* we shall need the following upper-bound which improves the estimate of (1.8).

Proposition 3.5. *There exists a family of functions* $\{v_\varepsilon\}\subset H^1(G,\mathbb{R}^2)$*, each satisfying the constraint* $\int_G|v_\varepsilon|=R_c$*, such that*

$$E_\varepsilon(v_\varepsilon)\le\frac{2DI(\alpha)}{\varepsilon}+E_0+o(1)\,, \quad (3.31)$$

where E_0 *is defined in* (1.22).

Proof. We shall need some notation associated with the geodesic $\underline{\gamma}$ that were used in [1]. We associate with $\underline{\gamma}$ an arclength parametrization on $[0, L]$ and define $z(s)$ as the solution of the ODE

$$\frac{dz}{ds} = \sqrt{W(\underline{\gamma}(z(s)))}\,, \quad z(0) = L/2\,, \tag{3.32}$$

which is defined on the whole real line and satisfies

$$\lim_{s\to-\infty} z(s) = 0 \text{ and } \lim_{s\to\infty} z(s) = L\,.$$

Since

$$\sqrt{W(\underline{\gamma}(z(s)))} \sim |\underline{\gamma}(z(s)) - p_1| \sim z(s) \quad \text{as } s \to -\infty \tag{3.33}$$

and

$$\sqrt{W(\underline{\gamma}(z(s)))} \sim |\underline{\gamma}(z(s)) - p_2| \sim L - z(s) \quad \text{as } s \to \infty\,, \tag{3.34}$$

we have, for some positive constants $\tilde{c}_1, \tilde{c}_2$,

$$0 \le z(s) \le Ce^{\tilde{c}_1 s} \qquad \text{for } s < 0\,, \tag{3.35}$$

$$0 \le L - z(s) \le Ce^{-\tilde{c}_2 s} \qquad \text{for } s > 0\,. \tag{3.36}$$

Recall that (U_1, U_2) denote minimizers for the problem in (1.22). It will be convenient to denote by u_0 the map defined on the whole domain $G \setminus \Sigma$ by $u_0|_{G_k} = U_k$, $k = 1, 2$. Our construction of a test function $v_\varepsilon^{(T)}$ will depend on a real parameter $T = T_\varepsilon$, whose value will be determined later by the constraint.

Fix a small $\lambda > 0$. In $G \setminus \Sigma_\lambda$ we set $v_\varepsilon^{(T)} = u_0$, so it remains to define $v_\varepsilon^{(T)}$ in each $\Sigma_\lambda^{(j)}$ (see (1.20)). Consider first the case $T \ge 0$. We set

$$S_\varepsilon = T\varepsilon \ln\frac{1}{\varepsilon} \text{ and } K_\varepsilon = S_\varepsilon + 2(\frac{1}{\tilde{c}_1} + \frac{1}{\tilde{c}_2})\varepsilon \ln\frac{1}{\varepsilon}\,.$$

For each $j = 1, \ldots, J$ we next define $v_\varepsilon^{(T)}$ for $x \in \Sigma_\lambda^{(j)}$, using the coordinates $(\sigma_j, \tau_j) = (\sigma_j(x), \tau_j(x))$ as follows. Recall the definition of $\phi_k^{(j)}$ in Remark 1.1 and let π_2 be the constant which is the trace of $\phi_2^{(j)}$ on $\partial G_2^{(j)} \cap G$ (so that $g_2(e^{i\pi_2}) = p_2$). Then define, writing for short $\tilde{g}_2(\phi) := g_2(e^{i\phi})$,

$$v_\varepsilon^{(T)}(x) = \begin{cases} u_0(x) & \tau_j < 0 \\ p_1 & \tau_j \in [0, S_\varepsilon] \\ \text{affine func. of } \tau_j \text{ joining } p_1 \text{ to } \underline{\gamma}(z(-2\frac{\ln 1/\varepsilon}{\tilde{c}_1})) & \tau_j \in [S_\varepsilon, S_\varepsilon + \varepsilon] \\ \underline{\gamma}(z(\frac{\tau_j - \varepsilon - S_\varepsilon - 2\varepsilon\frac{\ln 1/\varepsilon}{\tilde{c}_1}}{\varepsilon})) & \tau_j \in (S_\varepsilon + \varepsilon, K_\varepsilon + \varepsilon) \\ \text{affine func. of } \tau_j \text{ joining } \underline{\gamma}(z(2\frac{\ln 1/\varepsilon}{\tilde{c}_2})) \text{ to } p_2 & \tau_j \in [K_\varepsilon + \varepsilon, K_\varepsilon + 2\varepsilon] \\ \tilde{g}_2\left((\frac{\tau_j - (K_\varepsilon + 2\varepsilon)}{\varepsilon \ln \frac{1}{\varepsilon}})\phi_k^{(j)}(x) + \left(1 - (\frac{\tau_j - (K_\varepsilon + 2\varepsilon)}{\varepsilon \ln \frac{1}{\varepsilon}})\right)\pi_2\right) & \tau_j - K_\varepsilon - 2\varepsilon \in (0, \varepsilon \ln \frac{1}{\varepsilon}] \\ u_0(x) & \tau_j > K_\varepsilon + 2\varepsilon + \varepsilon \ln \frac{1}{\varepsilon}\,. \end{cases} \tag{3.37}$$

Since $\mu(\Sigma_\lambda^{(j)}) = \lambda l(\Sigma^{(j)}) + O(\lambda^2)$, we find by a direct computation,

$$\Big| \int_G (|v_\varepsilon^{(T)}| - |u_0|) - \sum_{j=1}^J \int_{\Sigma^{(j)}_{T\varepsilon \ln \frac{1}{\varepsilon}} \cap \{\tau_j > 0\}} (|p_1| - |u_0|) \Big| \le C\varepsilon \ln \frac{1}{\varepsilon}\,, \tag{3.38}$$

for some constant C independent of T (provided that $T\varepsilon \ln \frac{1}{\varepsilon} << 1$). It follows from (3.38) and (1.5) that for another constant C_1 we have,

$$\int_G |v_\varepsilon^{(T)}| - R_c \le (C_1 - T(m_2 - |p_1|))\varepsilon \ln \frac{1}{\varepsilon}\,. \tag{3.39}$$

For $T < 0$ we use an analogous construction to the one in (3.37), but where the transition layer between Γ_1 and Γ_2 is shifted from Σ by a distance $|T|\varepsilon \ln \frac{1}{\varepsilon}$ into G_1, so that a layer of the same width where $v_\varepsilon^{(T)} \equiv p_2$ is created. This yields

$$\int_G |v_\varepsilon^{(T)}| - R_c \ge (|T|(|p_2| - m_1) - C_2)\varepsilon \ln \frac{1}{\varepsilon}\,, \tag{3.40}$$

where C_2 is a constant independent of T. Clearly, the value of $\int_G |v_\varepsilon^{(T)}|$ is a continuous function of T, so from (3.39)–(3.40) it follows that there exists a critical value $\tilde{T} = \tilde{T}_\varepsilon$ for which $v_\varepsilon := v_\varepsilon^{(\tilde{T})}$ satisfies $\int_G |v_\varepsilon| = R_c$. □

Proof of Theorem 2. The weak convergence of u_{ε_n} towards u_* in $H^1(G_j \setminus \Sigma_\lambda)$ was established in Proposition 3.3. The identification of u_* as a minimizer for (1.22) follows from combining Proposition 3.4 and Proposition 3.5. Finally, the strong convergence in $H^1(G_j \setminus \Sigma_\lambda)$ and the uniform convergence follow as in [2], by adapting the methods of [3].

□

A Proof of Proposition 2.1

We shall need the following two lemmas.

Lemma A.1. *The minimizer u_ε satisfies*

$$\begin{cases} -2\Delta u_\varepsilon + \frac{1}{\varepsilon^2}\nabla W(u_\varepsilon) = \Big(\int_G 2|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2}\nabla W(u_\varepsilon)\cdot u_\varepsilon\Big) f_\varepsilon \quad \text{in } G, \\ \qquad\qquad \frac{\partial u_\varepsilon}{\partial n} = 0 \quad \text{on } \partial G\,, \end{cases} \tag{A.1}$$

where the function f_ε satisfies

$$\|f_\varepsilon\|_{L^\infty(G)} = \frac{1}{R_c} \;\text{ and }\; f_\varepsilon(x) = \frac{1}{R_c}\frac{u_\varepsilon(x)}{|u_\varepsilon(x)|} \;\text{ whenever }\; u_\varepsilon(x) \neq 0\,. \tag{A.2}$$

Proof. For every $\phi \in C_c^\infty(G,\mathbb{R}^2)$ and every $t\in\mathbb{R}$, with $|t|$ small enough, the function

$$u^{(t)} = R_c\frac{u_\varepsilon + t\phi}{\int_G |u_\varepsilon + t\phi|}$$

is an admissible function satisfying the constraint. A direct computation gives

$$u^{(t)} = \begin{cases} u_\varepsilon + t(\phi - \alpha_+ u_\varepsilon) + o(t) & t>0, \\ u_\varepsilon + t(\phi - \alpha_- u_\varepsilon) + o(t) & t<0, \end{cases} \tag{A.3}$$

where

$$\alpha_\pm = \frac{1}{R_c}\left(\int_{\{u_\varepsilon\neq 0\}} \frac{u_\varepsilon}{|u_\varepsilon|}\cdot\phi \pm \int_{\{u_\varepsilon = 0\}} |\phi|\right)\,.$$

Using (A.3) yields

$$E_\varepsilon(u^{(t)}) - E_\varepsilon(u_\varepsilon) = 2t\int_G \nabla u_\varepsilon\cdot\nabla(\phi - \alpha_\pm u_\varepsilon) + \frac{t}{\varepsilon^2}\int_G \nabla W(u_\varepsilon)\cdot(\phi - \alpha_\pm u_\varepsilon) + o(t)\,. \tag{A.4}$$

Taking the two limits $t\to 0^+$ and $t\to 0^-$ in (A.4) and using the minimality property of u_ε we obtain that

$$\begin{aligned} \frac{\lambda_\varepsilon}{R_c}\Big\{\int_{\{u_\varepsilon\neq 0\}} \frac{u_\varepsilon}{|u_\varepsilon|}\cdot\phi + \int_{\{u_\varepsilon=0\}}|\phi|\Big\} &\le \int_G 2\nabla u_\varepsilon\cdot\nabla\phi + \frac{1}{\varepsilon^2}\nabla W(u_\varepsilon)\cdot\phi \\ &\le \frac{\lambda_\varepsilon}{R_c}\Big\{\int_{\{u_\varepsilon\neq 0\}} \frac{u_\varepsilon}{|u_\varepsilon|}\cdot\phi - \int_{\{u_\varepsilon=0\}}|\phi|\Big\}\,, \end{aligned} \tag{A.5}$$

where

$$\lambda_\varepsilon = \int_G 2|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2}\nabla W(u_\varepsilon)\cdot u_\varepsilon\,.$$

From (A.5) we deduce that $-2\Delta u_\varepsilon + \frac{1}{\varepsilon^2}\nabla W(u_\varepsilon)$, viewed as a distribution in $H^{-1}(G)$, is actually a function $\lambda_\varepsilon f_\varepsilon \in L^\infty(G,\mathbb{R}^2)$ satisfying (A.2). Finally, the Neumann boundary condition in (A.1) follows by preforming the above computation with $\phi \in C^\infty(G,\mathbb{R}^2)$. □

The next lemma provides an L^∞ bound for u_ε whose proof was shown to us by Petru Mironescu. Note that such an estimate was proved by Gurtin and Matano in [7] in the *scalar case*, but their argument does not seem to apply here.

Lemma A.2 (Mironescu)**.** *Let R_1 be given in (H_3'). Then, there exists an ε_0 such that*

$$\|u_\varepsilon\|_{L^\infty(G)} \le R_1\,,\ \varepsilon < \varepsilon_0\,. \tag{A.6}$$

Proof. We fix a function $\Phi \in C^\infty[0,\infty)$ satisfying:

(i) $\Phi(t) = 0$ on $[0,R_1]$.

(ii) $0 < \Phi'(t) \le \dfrac{c}{t^2+1}$ on (R_1,∞). Put

$$\phi = \Phi(|u_\varepsilon|)u_\varepsilon\,.$$

Using ϕ as a test function in (A.1) and the identity

$$u_\varepsilon \cdot \frac{\partial u_\varepsilon}{\partial x_i} = \frac{1}{2}\frac{\partial |u_\varepsilon|^2}{\partial x_i} = |u_\varepsilon|\frac{\partial |u_\varepsilon|}{\partial x_i}\,,$$

yields

$$\begin{aligned} 2\int_G \Phi(|u_\varepsilon|)|\nabla u_\varepsilon|^2 + \Phi'(|u_\varepsilon|)|u_\varepsilon|\big|\nabla|u_\varepsilon|\big|^2 + \int_G \frac{\Phi(|u_\varepsilon|)}{\varepsilon^2}\nabla W(u_\varepsilon)\cdot u_\varepsilon \\ = \frac{1}{R_c}\Big(\int_G 2|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2}\nabla W(u_\varepsilon)\cdot u_\varepsilon\Big)\int_G \Phi(|u_\varepsilon|)|u_\varepsilon|\,. \end{aligned} \tag{A.7}$$

Since the first integral on the L.H.S. of (A.7) is nonnegative, using the assumptions (H_3') and (H_3) yields

$$\frac{a_1\tilde{c}_0}{\varepsilon^2}\int_{\{|u_\varepsilon|>R_1\}} \Phi(|u_\varepsilon|)|u_\varepsilon| \le \frac{1}{R_c}\Big(\int_G 2|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2}\nabla W(u_\varepsilon)\cdot u_\varepsilon\Big)\int_{\{|u_\varepsilon|>R_1\}} \Phi(|u_\varepsilon|)|u_\varepsilon|\,. \tag{A.8}$$

Next we claim that

$$\int_G 2|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2}\nabla W(u_\varepsilon)\cdot u_\varepsilon = O\Big(\frac{1}{\varepsilon^{3/2}}\Big)\,. \tag{A.9}$$

First, by the energy estimate (1.8) we have

$$\int_G |\nabla u_\varepsilon|^2 = O\Big(\frac{1}{\varepsilon}\Big)\,. \tag{A.10}$$

Next, we note that the assumption (H_2) implies that $|\nabla W| \le c\sqrt{W}$ in some neighborhood of $\Gamma_1 \cup \Gamma_2$, hence the same estimate holds in $B_{R_1}(0)$ (for a different c) and we deduce, using again (1.8), that

$$\frac{1}{\varepsilon^2}\int_{\{|u_\varepsilon|\le R_1\}} |\nabla W(u_\varepsilon)\cdot u_\varepsilon| \le \frac{CR_1^2}{\varepsilon^2}\Big(\int_{\{|u_\varepsilon|\le R_1\}} W(u_\varepsilon)\Big)^{1/2} = O\Big(\frac{1}{\varepsilon^{3/2}}\Big)\,. \tag{A.11}$$

Finally, using (H_3') with (1.8) yields

$$\frac{1}{\varepsilon^2}\int_{\{|u_\varepsilon|> R_1\}} |\nabla W(u_\varepsilon)\cdot u_\varepsilon| \le \frac{a_2}{\varepsilon^2}\int_{\{|u_\varepsilon|> R_1\}} W(u_\varepsilon) = O\Big(\frac{1}{\varepsilon}\Big)\,. \tag{A.12}$$

Combining (A.10) with (A.11)–(A.12) we are led to (A.9). Plugging (A.9) in (A.8) implies that for $\varepsilon < \varepsilon_0$,

$$\int_{\{|u_\varepsilon|> R_1\}} \Phi(|u_\varepsilon|)|u_\varepsilon| = 0\,,$$

i.e., the set $\{|u_\varepsilon| > R_1\}$ is empty. □

Proof of Proposition 2.1. For each $x_0 \in G$ with $\mathrm{dist}(x_0, \partial G) \ge \varepsilon$ define the rescaled function $\tilde{u}_\varepsilon(x) = u_\varepsilon(x_0 + \varepsilon x)$ on $B_1(0)$. Using (A.1) we conclude that $\tilde{u}_\varepsilon$ satisfies the equation

$$\Delta\tilde{u}_\varepsilon = \frac{1}{2}\nabla W(\tilde{u}_\varepsilon) - \frac{\varepsilon^2}{2}\Big(\int_G 2|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2}\nabla W(u_\varepsilon)\cdot u_\varepsilon\Big) f_\varepsilon(x_0+\varepsilon x) \ \text{ in } B_1(0)\,. \tag{A.13}$$

From (A.6) and (A.9) we deduce that the R.H.S. of (A.13) is bounded in $L^\infty(B_1(0))$, uniformly in $\varepsilon < \varepsilon_0$. By standard elliptic estimates it follows that $\nabla\tilde{u}_\varepsilon$ is bounded in $L^\infty(B_{1/2}(0))$. Rescaling back we deduce the desired bound for ∇u_ε in $L^\infty_{\mathrm{loc}}(G)$. The estimate at points x_0 with $\mathrm{dist}(x_0, \partial G) < \varepsilon$ follows similarly from elliptic estimates for a Neumann boundary problem on the domain obtained by rescaling of the domain $B_{2\varepsilon}(x_0) \cap G$ □

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