Distances between homotopy classes of $W^{s,p}(\mathbb{S}^N;\mathbb{S}^N)$

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Dedicated to Jean-Michel Coron with esteem and affection

1 Introduction

In [9], J.-M. Coron and the first author (H. B.) have investigated the existence of multiple $\mathbb{S}^2$-valued harmonic maps. In the process they were led to introduce a concept of topological degree for maps $f \in H^1(\mathbb{S}^2;\mathbb{S}^2)$. Note that such maps need not be continuous and thus the standard degree (defined for continuous maps) is not well-defined. Instead they used Kronecker's formula

$$\deg f = \fint_{\mathbb{S}^2} \det(\nabla f) \tag{1.1}$$

valid for $f \in C^1(\mathbb{S}^2;\mathbb{S}^2)$, and a density argument ($C^1(\mathbb{S}^2;\mathbb{S}^2)$ is dense in $H^1(\mathbb{S}^2;\mathbb{S}^2)$) due to R. Schoen and K. Uhlenbeck [16], to assert that $\deg f$, defined by (1.1), belongs to $\mathbb{Z}$ for every $f \in H^1(\mathbb{S}^2;\mathbb{S}^2)$.

They also used the technique of "bubble insertion" which allows to modify the degree d_1 of a given (smooth) map $f : \mathbb{S}^2 \to \mathbb{S}^2$ by changing its values in a small disc $B_\varepsilon(x_0)$. More precisely (see [9] and [7]), for any $\varepsilon > 0$ and $d_2 \in \mathbb{Z}$ one can construct some $g \in H^1(\mathbb{S}^2;\mathbb{S}^2)$ such that $g = f$ outside $B_\varepsilon(x_0)$, $\deg g = d_2$, and

$$\int_{\mathbb{S}^2} |\nabla g - \nabla f|^2 \le 8\pi\, |d_2 - d_1| + o(1) \text{ as } \varepsilon \to 0 \tag{1.2}$$

(in fact [9] contains a more refined estimate in the spirit of Lemma 3.4 below). This kind of argument serves as a major source of inspiration for several proofs in this paper. As we are going to see, estimate (1.2) provides a useful upper bound for the Hausdorff distance between homotopy classes in $H^1(\mathbb{S}^2;\mathbb{S}^2)$.

Subsequently the first author and L. Nirenberg [13] (following a suggestion of L. Boutet de Monvel and O. Gabber [5, Appendix]) developed a concept of topological degree for map in $\mathrm{VMO}(\mathbb{S}^N;\mathbb{S}^N)$ which applies in particular to the (integer or fractional) Sobolev spaces $W^{s,p}(\mathbb{S}^N;\mathbb{S}^N)$ with

$$s > 0,\ 1 \le p < \infty \text{ and } sp \ge N. \tag{1.3}$$

This degree is stable with respect to strong convergence in BMO and coincides with the usual degree when maps are smooth.

In the remaining cases, i.e., when $sp < N$, there is no natural notion of degree. Indeed, one may construct a sequence of smooth maps $f_n : \mathbb{S}^N \to \mathbb{S}^N$ such that $f_n \to P$ (with $P \in \mathbb{S}^N$ a fixed point) in $W^{s,p}$ and $\deg f_n \to \infty$ [4, Lemma 1.1]. Therefore, in what follows we make the assumption (1.3).

Given any $d \in \mathbb{Z}$, consider the classes

$$\mathscr{E}_d := \{f \in W^{s,p}(\mathbb{S}^N;\mathbb{S}^N); \deg f = d\}; \tag{1.4}$$

these classes depend not only on d, but also on s and p, but in order to keep notation simple we do not mention the dependence on s and p.

These classes are precisely the connected or path-connected components of $W^{s,p}(\mathbb{S}^N;\mathbb{S}^N)$. [This was proved in [13] in the VMO context, but the proof can be adapted to $W^{s,p}$.] Moreover if $N = 1$ we have (see Section 2)

$$\mathscr{E}_d = \left\{f; f(z) = e^{\imath\varphi(z)} z^d, \text{ with } \varphi \in W^{s,p}(\mathbb{S}^1;\mathbb{R})\right\}. \tag{1.5}$$

Our purpose is to investigate the usual distance and the Hausdorff distance (in $W^{s,p}$) between the classes $\mathscr{E}_d$. For that matter we introduce the $W^{s,p}$-distance between two maps $f, g \in W^{s,p}(\mathbb{S}^N;\mathbb{S}^N)$ by

$$d_{W^{s,p}}(f,g) := |f-g|_{W^{s,p}}, \tag{1.6}$$

where for $h \in W^{s,p}(\mathbb{S}^N;\mathbb{R}^{N+1})$ we let

$$|h|_{W^{s,p}} := \left\| h - \fint_{\mathbb{S}^N} h \right\|_{W^{s,p}},$$

and $\| \ \|_{W^{s,p}}$ is any one of the standard norms on $W^{s,p}$. Let $d_1 \neq d_2$ and define the following two quantities:

$$\operatorname{dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) := \inf_{f \in \mathscr{E}_{d_1}} \inf_{g \in \mathscr{E}_{d_2}} d_{W^{s,p}}(f,g), \tag{1.7}$$

and

$$\operatorname{Dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) := \sup_{f \in \mathscr{E}_{d_1}} \inf_{g \in \mathscr{E}_{d_2}} d_{W^{s,p}}(f,g). \tag{1.8}$$

It is conceivable that

$$\operatorname{Dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) = \operatorname{Dist}_{W^{s,p}}(\mathscr{E}_{d_2}, \mathscr{E}_{d_1}), \forall\, d_1, d_2 \in \mathbb{Z}, \tag{1.9}$$

but we have not been able to prove this equality (see Open Problem 1 below). Therefore we consider also the symmetric version of (1.8), which is nothing but the Hausdorff distance between the two classes:

$$H - \operatorname{dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) = \max\left\{\operatorname{Dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}), \operatorname{Dist}_{W^{s,p}}(\mathscr{E}_{d_2}, \mathscr{E}_{d_1})\right\}. \tag{1.10}$$

We should mention that even in cases where we know that (1.9) holds true, the qualitative properties of the two quantities in (1.9) might be quite different. Consider for example the classes $\mathscr{E}_{d_1}, \mathscr{E}_{d_2}$ in $W^{1,1}(\mathbb{S}^1;\mathbb{S}^1)$ when $0 < d_1 < d_2$. It is shown in Proposition 3.2 that $\operatorname{Dist}_{W^{1,1}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2})$ is attained by some f and g, while $\operatorname{Dist}_{W^{1,1}}(\mathscr{E}_{d_2}, \mathscr{E}_{d_1})$ is not.

It turns out that in general the analysis of the usual distance $\text{dist}_{W^{s,p}}$ is simpler than that of $\text{Dist}_{W^{s,p}}$, so we start with it. Note that we clearly have

$$\text{dist}_{C^0}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) = 2, \ \forall d_1 \neq d_2. \tag{1.11}$$

Indeed, on the one hand we have $\|f-g\|_{C^0} \leq 2$, $\forall f, g$, and on the other hand if $\|f-g\|_{C^0} < 2$ then $\deg f = \deg g$. [This is obtained by considering the homotopy $H_t = \dfrac{tf+(1-t)g}{|tf+(1-t)g|}$, $t \in [0,1]$.] By contrast, it was established in [13] that surprisingly, when $s = 1/2$, $p = 2$ and $N = 1$ one has $\text{dist}_{H^{1/2}}(\mathscr{E}_1, \mathscr{E}_0) = 0$, and thus $\text{dist}_{\text{VMO}}(\mathscr{E}_1, \mathscr{E}_0) = 0$. The usual distance $\text{dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2})$ in certain (non-fractional) Sobolev spaces was investigated in works by J. Rubinstein and I. Shafrir [15], when $s = 1$, $p \geq N = 1$, and S. Levi and I. Shafrir [14], when $s = 1$, $p \geq N \geq 2$. In particular, they obtained exact formulas for the distance (see [15, Remark 2.1], [14, Theorem 3.4]) and tackled the question whether this distance is achieved (see [15, Theorem 1], [14, Theorem 3.4]). Another motivation comes from the forthcoming paper [12], where we consider a natural notion of class in $W^{1,1}(\Omega; \mathbb{S}^1)$ (with $\Omega \subset \mathbb{R}^N$) and determine the distance between these classes. In particular, Theorem 4 is used in [12].

Throughout most of the paper we assume that $N = 1$. It is only in the last two sections that we consider $N \geq 2$.

We pay special attention to the case where $s = 1$. In this case, we have several sharp results when we take

$$d_{W^{1,p}}(f,g) = |f-g|_{W^{1,p}} := \left(\int_{\mathbb{S}^1} |\dot{f} - \dot{g}|^p\right)^{1/p}. \tag{1.12}$$

The following result was obtained in [15].

Theorem 0. *Let $1 \leq p < \infty$. We have*

$$\text{dist}_{W^{1,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) = 2^{(1/p)+1}\pi^{(1/p)-1}|d_1 - d_2|, \ \forall d_1, d_2 \in \mathbb{Z}. \tag{1.13}$$

In particular

$$\text{dist}_{W^{1,1}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) = 4|d_1 - d_2|, \ \forall d_1, d_2 \in \mathbb{Z}. \tag{1.14}$$

For the convenience of the reader, and also because it is used in the proof of Theorem 1, the proof of Theorem 0 is presented in Sections 3 and 4.

In view of (1.13), it is natural to ask whether, given $d_1 \neq d_2$, the infimum

$$\inf_{f \in \mathscr{E}_{d_1}} \inf_{g \in \mathscr{E}_{d_2}} d_{W^{1,p}}(f,g) = 2^{(1/p)+1}\pi^{(1/p)-1}|d_1 - d_2| \tag{1.15}$$

is achieved. The answer is given by the following result, proved in [15] when $p = 2$.

Theorem 1. *Let $d_1, d_2 \in \mathbb{Z}$, $d_1 \neq d_2$.*

1. *When $p = 1$, the infimum in (1.15) is always achieved.*
2. *When $1 < p < 2$, the infimum in (1.15) is achieved if and only if $d_2 = -d_1$.*
3. *When $p \geq 2$, the infimum in (1.15) is not achieved.*

We now turn to the case $s \neq 1$. Here, we will only obtain the order of magnitude of the distances $\text{dist}_{W^{s,p}}$, and thus our results are not sensitive to the choice of a specific distance among various equivalent ones. [However, we will occasionally obtain sharp results for $H^{1/2}(\mathbb{S}^1;\mathbb{S}^1)$ equipped with the Gagliardo distance defined below.] When $0 < s < 1$ a standard distance is associated with the Gagliardo $W^{s,p}$ semi-norm

$$d_{W^{s,p}}(f,g) := \left(\int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{|[f(x) - g(x)] - [f(y) - g(y)]|^p}{|x-y|^{1+sp}} dx dy \right)^{1/p}. \tag{1.16}$$

Theorem 2. *We have*

1. *Let* $1 < p < \infty$. *Then*

$$\text{dist}_{W^{1/p,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) = 0, \ \forall d_1, d_2 \in \mathbb{Z}. \tag{1.17}$$

2. *Let* $s > 0$ *and* $1 \le p < \infty$ *be such that* $sp > 1$. *Then*

$$C'_{s,p} |d_1 - d_2|^s \le \text{dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) \le C_{s,p} |d_1 - d_2|^s \tag{1.18}$$

for some constants $C_{s,p}, C'_{s,p} > 0$.

We next investigate the Hausdorff distance $H - \text{dist}_{W^{s,p}}$ (still with $N = 1$).

Theorem 3. *We have*

1. *In* $W^{1,1}$,

$$\text{Dist}_{W^{1,1}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) = 2\pi |d_1 - d_2|, \ \forall d_1, d_2 \in \mathbb{Z}. \tag{1.19}$$

2. *If* $1 < p < \infty$, *then*

$$H - \text{dist}_{W^{1/p,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) \le C_p |d_1 - d_2|^{1/p}, \ \forall d_1, d_2 \in \mathbb{Z}. \tag{1.20}$$

3. *If* $s > 0$ *and* $1 \le p < \infty$ *are such that* $sp > 1$, *then*

$$\text{Dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) = \infty, \ \forall d_1, d_2 \in \mathbb{Z} \text{ such that } d_1 \neq d_2. \tag{1.21}$$

We do not know whether (1.20) is optimal in the sense that for every $1 < p < \infty$ we have

$$\text{Dist}_{W^{1/p,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) \ge C'_p |d_1 - d_2|^{1/p}, \ \forall d_1, d_2 \in \mathbb{Z}, \tag{1.22}$$

for some positive constant C'_p. See Open Problem 2 below for a more general question. See also Section 7 for some partial positive answers.

We now discuss similar questions when $N \ge 2$. We define $\text{dist}_{W^{s,p}}$ and $H - \text{dist}_{W^{s,p}}$ using one of the usual $W^{s,p}$ (semi-)norms.

For $s = 1$, $N \ge 2$, $p \ge N$, and for the semi-norm $|f - g|_{W^{1,p}} = \|\nabla f - \nabla g\|_{L^p}$, the exact value of the $W^{1,p}$ distance $\text{dist}_{W^{1,p}}$ between the classes $\mathscr{E}_{d_1}$ and $\mathscr{E}_{d_2}$, $d_1 \neq d_2$, has been computed by S. Levi and I. Shafrir [14]. A striking fact is that this distance does not depend on d_1 and d_2, but only on p (and N).

We start with $\text{dist}_{W^{s,p}}$.

Theorem 4. *We have*

1. *If $N \geq 1$ and $1 < p < \infty$, then*

$$\mathrm{dist}_{W^{N/p,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) = 0, \ \forall\, d_1, d_2 \in \mathbb{Z}. \tag{1.23}$$

2. *If $[1 < p < \infty$ and $s > N/p]$ or $[p = 1$ and $s \geq N]$, there exist constants $C_{s,p,N}, C'_{s,p,N} > 0$ such that*

$$C'_{s,p,N} \leq \mathrm{dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) \leq C_{s,p,N}, \ \forall\, d_1, d_2 \in \mathbb{Z} \text{ such that } d_1 \neq d_2, \tag{1.24}$$

(here $N \geq 2$ is essential).

Remark 1.1. We do not know whether, under the assumptions of Theorem 4, item *2*, it is true that $\mathrm{dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) = C''_{s,p,N}$, $\forall\, d_1, d_2 \in \mathbb{Z}$ such that $d_1 \neq d_2$, for some appropriate choice of the $W^{s,p}$ semi-norm. [Recall that the answer is positive when $s = 1$ [14].]

We now turn to the Hausdorff distance.

Theorem 5. *Let $N \geq 1$. We have*

1. *For every $1 \leq p < \infty$*

$$H-\mathrm{dist}_{W^{N/p,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) \leq C_{p,N}|d_1 - d_2|^{1/p}, \ \forall\, d_1, d_2 \in \mathbb{Z}. \tag{1.25}$$

2. *If $s > 0$ and $1 \leq p < \infty$ are such that $sp > N$, then*

$$\mathrm{Dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) = \infty, \ \forall\, d_1, d_2 \in \mathbb{Z} \text{ such that } d_1 \neq d_2. \tag{1.26}$$

We conclude with three questions.

Open Problem 1. Is it true that for every d_1, d_2, N, s, p

$$\mathrm{Dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) = \mathrm{Dist}_{W^{s,p}}(\mathscr{E}_{d_2}, \mathscr{E}_{d_1})? \tag{1.27}$$

(recall that $\mathrm{Dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2})$ has been defined in (1.8)). Or even better:

$$\text{Does } \mathrm{Dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) \text{ depend only on } |d_1 - d_2| \text{ (and } s, p, N)? \tag{1.28}$$

There are several cases where we have an explicit formula for $\mathrm{Dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2})$ and in all such cases (1.28) holds. See e.g. the proofs of Theorem 3, items *1* and *3*, and Theorem 5, item *2*. We may also ask questions similar to (1.28) for $\mathrm{dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2})$ and for $H-\mathrm{dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2})$ (assuming the answer to (1.28) is negative); again, the answer is positive in many cases. A striking special case still open when $N = 1$ is: does $\mathrm{dist}_{W^{2,1}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2})$ depend only on $|d_1 - d_2|$?

Open Problem 2. Is it true that for every $N \geq 1$ and every $1 \leq p < \infty$, there exists some $C'_{p,N} > 0$ such that

$$\mathrm{Dist}_{W^{N/p,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) \geq C'_{p,N}|d_1 - d_2|^{1/p}, \ \forall\, d_1, d_2 \in \mathbb{Z}? \tag{1.29}$$

A weaker version of Open Problem 2 is obtained when we replace $\mathrm{Dist}_{W^{N/p,p}}$ by $H-\mathrm{dist}\, W^{N/p,p}$ (there will be no difference of course in case the answer to Open Problem 1 is positive):

Open Problem 2′. With the same assumptions as in Open Problem 2, is it true that

$$H-\operatorname{dist}_{W^{N/p,p}}(\mathscr{E}_{d_1},\mathscr{E}_{d_2})\geq C'_{p,N}|d_1-d_2|^{1/p},\ \forall d_1,d_2\in\mathbb{Z}? \tag{1.30}$$

The only case for which Open Problem 2 is settled is $[N=1,p=1]$ (see Theorem 3, item *1*). We emphasize three cases of special interest: *1.* $[N=1,p=2]$, *2.* $[N=2,p=2]$ and *3.* $[N=2,p=1]$. In case *1*, the answer to Open Problem 2′ is positive (see Corollary 7.6). See also Section 7 where further partial answers are presented.

Here is another natural open problem. Recall that for any $f\in W^{N/p,p}(\mathbb{S}^N;\mathbb{S}^N)$ and any sequence $(f_n)\subset W^{N/p,p}(\mathbb{S}^N;\mathbb{S}^N)$ such that $|f_n-f|_{W^{N/p,p}}\to 0$, we have $\deg f_n\to\deg f$. We also know (Theorem 4, item *1*) that there exist sequences (f_n), (g_n) in $W^{N/p,p}(\mathbb{S}^N;\mathbb{S}^N)$ such that $|f_n-g_n|_{W^{N/p,p}}\to 0$ but $|\deg f_n-\deg g_n|=1$, $\forall n$.

Open Problem 3. Is it true that $|\deg f_n-\deg g_n|\to 0$ for any sequences (f_n), (g_n) in $W^{N/p,p}(\mathbb{S}^N;\mathbb{S}^N)$ such that

$$|f_n-g_n|_{W^{N/p,p}}\to 0 \text{ as } n\to\infty$$

and

$$|f_n|_{W^{N/p,p}}+|g_n|_{W^{N/p,p}} \text{ remains bounded as } n\to\infty?$$

Our paper is organized as follows. In Section 2 we recall some known properties of $W^{s,p}(\mathbb{S}^N;\mathbb{S}^N)$. Sections 3–5 concern only the case $N=1$, while Sections 6–7 deal with $N\geq 1$. The proofs of Theorems 0 and 1 are presented in Sections 3 and 4. Theorem 2, item *1* and Theorem 3, items *2–3*, are special cases of, respectively, Theorem 4, item *1* and Theorem 5, items *1–2*; their proofs are presented in Section 6. Theorem 2, item *2* is established in Section 5. The proof of Theorem 3, item *1* appears in Section 3. Theorems 4 and 5 belong to Section 6. Partial solutions to the open problems are given in Section 7. A final Appendix gathers various auxiliary results.

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2 Some standard properties of maps $f : \mathbb{S}^N \to \mathbb{S}^N$

In this section, we alway assume that (1.3) holds.

Lemma 2.1. *$C^\infty(\mathbb{S}^N; \mathbb{S}^N)$ is dense in $W^{s,p}(\mathbb{S}^N; \mathbb{S}^N)$.*

When $s = 1$, $p = 2$, $N = 2$, the above was proved in [16]. The argument there extends to the general case.

When

$$[0 \le s - N/p < 1] \text{ or } [s - N/p = 1 \text{ and } p > 1], \tag{2.1}$$

we can complement Lemma 2.1 as follows.

Lemma 2.2. *Assume that (2.1) holds. Then every map $f \in W^{s,p}(\mathbb{S}^N; \mathbb{S}^N)$ can be approximated by a sequence $(f_n) \subset C^\infty(\mathbb{S}^N; \mathbb{S}^N)$ such that every f_n is constant near some point.*

We note that condition (2.1) is equivalent to (1.3) + the non embedding $W^{s,p} \not\hookrightarrow C^1$. The non embedding is also necessary for the validity of the conclusion of Lemma 2.2. Indeed, a C^1 function f, say on the real line, whose derivative does not vanish, cannot be approximated in C^1 by a sequence (f_n) such that each f_n is constant near some point.

The proof of Lemma 2.2 is postponed to the Appendix.

Theorem 2.3 ([13])**.** *For $1 \le p < \infty$, the degree of smooth maps $f : \mathbb{S}^N \to \mathbb{S}^N$ is continuous with respect to the $W^{N/p,p}$ convergence.*

As a consequence, under assumption (1.3) the degree extends to maps in $W^{N/p,p}(\mathbb{S}^N; \mathbb{S}^N)$. Moreover, if (f_n) and f are in $W^{N/p,p}$ and $|f_n - f|_{W^{N/p,p}} \to 0$, then $\deg f_n \to \deg f$.

This follows from the corresponding assertion for the BMO convergence [13] and the fact that $W^{N/p,p} \hookrightarrow \text{BMO}$.

When $N = 1$, an alternative equivalent definition of the degree can be obtained via lifting [11, 10]. In this case, given $f \in W^{s,p}(\mathbb{S}^1; \mathbb{S}^1)$, it is always possible to write

$$f(e^{\imath\theta}) = e^{\imath\varphi(\theta)},\ \forall \theta \in \mathbb{R}, \text{ for some } \varphi \in W^{s,p}_{loc}(\mathbb{R}; \mathbb{R}) \tag{2.2}$$

(no condition on s and p [2]).

If, in addition, (1.3) holds, then the function $\varphi(\cdot+2\pi)-\varphi(\cdot)$ is constant a.e. [2], and we have

$$\deg f = \frac{1}{2\pi}(\varphi(\cdot+2\pi)-\varphi(\cdot)). \tag{2.3}$$

If instead of (1.3) we assume that either $[sp>1]$ or $[s=1$ and $p=1]$, then φ is continuous and (2.3) becomes

$$\deg f = \frac{1}{2\pi}(\varphi(2\pi)-\varphi(0)) = \frac{1}{2\pi}(\varphi(\pi)-\varphi(-\pi)). \tag{2.4}$$

Finally, we mention the formula

$$\deg f = \frac{1}{2\pi}\int_{\mathbb{S}^1} f\wedge\dot f,\ \forall f\in W^{1,1}(\mathbb{S}^1;\mathbb{S}^1). \tag{2.5}$$

3 $W^{1,1}$ maps

Proof of Theorem 0 for $p=1$, and Theorem 1, item 1.
Step 1. Proof of "$\le$" in (1.14)
With no loss of generality we may assume that $d_1>d_2$ and $d_1>0$. Set $d:=d_1-d_2$ and $L:=d+1$. We define $f(e^{\imath\theta}):=e^{\imath\varphi(\theta)}\in\mathscr{E}_{d_1}$, $g(e^{\imath\theta}):=e^{\imath\psi(\theta)}\in\mathscr{E}_{d_2}$, where $\varphi,\psi\in W^{1,1}((0,2\pi))$ are defined as follows:

$$\varphi(\theta):=\begin{cases} L\theta, & \text{if } \theta\in[0,2d\pi/L)\\ Ld_2\theta+2(d_1-Ld_2)\pi, & \text{if } \theta\in[2d\pi/L,2\pi)\end{cases},$$

and

$$\psi(\theta):=\begin{cases} L\,\mathrm{dist}(\theta,2\pi\mathbb{Z}/L), & \text{if } \theta\in[0,2d\pi/L)\\ \varphi(\theta)-2d\pi, & \text{if } \theta\in[2d\pi/L,2\pi)\end{cases}$$

(and thus on $[0,2d\pi/L]$ the graph of ψ is a zigzag consisting of d triangles).

For $k\in\mathbb{Z}$, $0\le k\le d-1$, set

$$I_k=\left[\frac{2k\pi}{L},\frac{(2k+1)\pi}{L}\right] \text{ and } J_k=\left[\frac{(2k+1)\pi}{L},\frac{(2k+2)\pi}{L}\right].$$

Note that

$$\psi(\theta)=\begin{cases} L\theta-2k\pi, & \text{if } \theta\in I_k\\ 2(k+1)\pi-L\theta, & \text{if } \theta\in J_k\end{cases},$$

so that $g=f$ on I_k and $g=\overline{f}$ on J_k. Hence

$$|\dot f-\dot g|=\begin{cases} 0, & \text{on } I_k\\ -2(\sin\varphi)\varphi', & \text{on } J_k\end{cases}.$$

Therefore

$$\int_{\mathbb{S}^1}|\dot f-\dot g| = 2\sum_{k=0}^{d-1}\int_{J_k}(\cos\varphi)'(\theta)\,d\theta = 4d = 4(d_1-d_2).$$

Step 2. Proof of "≥" in (1.14)
We may assume that $d := d_1 - d_2 > 0$. We prove that when $f \in \mathscr{E}_{d_1}$ and $g \in \mathscr{E}_{d_2}$ we have $\int_{\mathbb{S}^1} |\dot f - \dot g| \geq 4d$. The map f/g is onto (since its degree is $d \neq 0$), and thus with no loss of generality we may assume that $f(1) = g(1)$. Write $f(e^{\imath\theta}) = e^{\imath\varphi(\theta)} g(e^{\imath\theta})$, with $\varphi \in W^{1,1}((0,2\pi))$. We have $\varphi(2\pi) - \varphi(0) = 2d\pi$, and we may assume that $\varphi(0) = 0$. Consider $0 = t_0 < \tau_0 < t_1 < \cdots < \tau_{d-1} < t_d = 2\pi$ such that $\varphi(t_j) = 2\pi j$, $j = 0, \ldots, d$, and $\varphi(\tau_j) = 2\pi j + \pi$, $j = 0, \ldots, d-1$. Thus the function $w := |f - g|$ satisfies $w(e^{\imath t_j}) = 0$ and $w(e^{\imath\tau_j}) = 2$. Therefore, we have $\int_{\mathbb{S}^1} |\dot w| \geq 4d$. In order to conclude, it suffices to note the inequality $|\dot w| \leq |\dot f - \dot g|$ a.e. □

We now turn to the properties of the Hausdorff distance in $W^{1,1}$.

Proof of Theorem 3, item 1. Step 1. Proof of "≤" in (1.19)
By symmetry, it suffices to prove that for every $f \in \mathscr{E}_{d_1}$ and every $\varepsilon > 0$ there exists some $g \in \mathscr{E}_{d_2}$ satisfying

$$\int_{\mathbb{S}^1} |\dot f - \dot g| \leq 2\pi |d_1 - d_2| + \varepsilon. \tag{3.1}$$

By density of $C^\infty(\mathbb{S}^1; \mathbb{S}^1)$ in $W^{1,1}(\mathbb{S}^1; \mathbb{S}^1)$ it suffices to prove (3.1) for smooth f. Moreover, we may assume that f is constant near some point, say 1 (see Lemma 2.2). We may thus write $f(e^{\imath\theta}) = e^{\imath\varphi(\theta)}$, $\theta \in [0,2\pi]$, for some smooth φ satisfying $\varphi(2\pi) - \varphi(0) = 2\pi d_1$ and constant near 0. For a small $\lambda > 0$ define $\psi = \psi^{(\lambda)}$ on $[0,2\pi]$ by

$$\psi(\theta) := \begin{cases} \varphi(\theta) - \dfrac{2d\pi}{\lambda}\theta, & \text{if } \theta \in [0,\lambda] \\ \varphi(\theta) - 2d\pi, & \text{if } \theta \in (\lambda, 2\pi] \end{cases} \tag{3.2}$$

(where $d := d_1 - d_2$), and then set $g(e^{\imath\theta}) := e^{\imath\psi(\theta)} \in \mathscr{E}_{d_2}$. Clearly,

$$\int_{\mathbb{S}^1} |\dot f - \dot g| = \int_0^\lambda \left|(e^{\imath\psi} - e^{\imath\varphi})'\right| = 2|d|\pi = 2\pi |d_1 - d_2|.$$

Step 2. Proof of

$$\mathrm{Dist}_{W^{1,1}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) \geq 2\pi |d_1 - d_2|, \ \forall\, d_1, d_2 \text{ with } 0 \leq d_1 < d_2. \tag{3.3}$$

Let $f(z) := z^{d_1} \in \mathscr{E}_{d_1}$. It suffices to prove that

$$|f - g|_{W^{1,1}} \geq 2\pi (d_2 - d_1), \ \forall\, g \in \mathscr{E}_{d_2}.$$

By the triangle inequality, for any such g, we have

$$\int_{\mathbb{S}^1} |\dot f - \dot g| \geq \int_{\mathbb{S}^1} [|\dot g| - |\dot f|] \geq \left| \int_{\mathbb{S}^1} g \wedge \dot g \right| - 2\pi d_1 = 2\pi(|d_2| - d_1) = 2\pi(d_2 - d_1), \tag{3.4}$$

since $|\dot f| = d_1$ on $\mathbb{S}^1$.
Step 3. Proof of

$$\mathrm{Dist}_{W^{1,1}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) \geq 2\pi |d_1 - d_2|, \ \forall\, d_1 \geq 0, \ \forall\, d_2 \in \mathbb{Z} \text{ with } d_2 < d_1. \tag{3.5}$$

The case $d_1 = 0$ is trivial since we may take as above $f(z) := z^0 = 1$ and apply (3.4).

We now turn to the case $d_1 > 0$ and $d_2 < d_1$ which is quite involved. Inequality (3.5) is a direct consequence of the following

Lemma 3.1. *Assume that $d_1 > 0$ and $d_2 < d_1$. Then for each $\delta > 0$ there exists $f \in \mathscr{E}_{d_1}$ such that*

$$\int_{\mathbb{S}^1} |\dot f - \dot g| \geq (2\pi - \delta)(d_1 - d_2),\ \forall\, g \in \mathscr{E}_{d_2}. \tag{3.6}$$

Proof. For large n (to be chosen later) let $f_n(e^{\imath\theta}) = e^{\imath\varphi_n(\theta)} \in \mathscr{E}_{d_1}$, with $\varphi_n \in W^{1,1}((0,2\pi))$ defined by setting $\varphi_n(0) = 0$ and

$$\varphi_n'(\theta) = \begin{cases} d_1 n, & \theta \in [2j\pi/n^2),(2j+1)\pi/n^2] \\ -d_1(n-2), & \theta \in ((2j+1)\pi/n^2),(2j+2)\pi/n^2] \end{cases},\ j = 0,1,\dots,n^2-1. \tag{3.7}$$

Therefore, the graph of φ_n is a zigzag of n^2 triangles. Note that the average gradient of φ_n is d_1, since

$$\int_{2j\pi/n^2}^{(2j+2)\pi/n^2} \varphi_n' = 2\pi\frac{d_1}{n^2},\ j = 0,1,\dots,n^2-1. \tag{3.8}$$

Hence $\int_0^{2\pi} \varphi_n' = 2\pi d_1$ (so indeed $f_n \in \mathscr{E}_{d_1}$). On the other hand, note that

$$\int_{2j\pi/n^2}^{(2j+2)\pi/n^2)} |\varphi_n'| = 2(n-1)\pi\frac{d_1}{n^2},\ j = 0,1,\dots,n^2-1 \implies \int_0^{2\pi} |\varphi_n'| = 2(n-1)\pi d_1,$$

i.e., $\lim_{n\to\infty} \|\dot f_n\|_{L^1(\mathbb{S}^1)} = \infty$.

Consider now any $g \in \mathscr{E}_{d_2}$ and write $g(e^{\imath\theta}) = e^{\imath\psi(\theta)}$ with $\psi \in W^{1,1}((0,2\pi))$ satisfying $\psi(2\pi) - \psi(0) = 2\pi d_2$. For convenience we extend both φ_n and ψ to all of $\mathbb{R}$ in such a way that the extensions are continuous functions whose derivatives are 2π-periodic. Set $h = f_n\overline{g} \in \mathscr{E}_d$ with $d := d_1 - d_2 > 0$. Hence, $h(e^{\imath\theta}) = e^{\imath\eta(\theta)}$ with $\eta := \varphi_n - \psi$. We can find d (closed) arcs on $\mathbb{S}^1$, $I_1,\dots,I_d$, with disjoint interiors such that:

$$I_j = \{e^{\imath\theta};\theta \in [s_j,t_j]\},\ h(e^{\imath s_j}) = h(e^{\imath t_j}) = 1 \text{ and } \int_{s_j}^{t_j} \eta' = 2\pi,\ \text{for } j = 1,\dots,d.$$

For small $\varepsilon > 0$ define, for each $j = 1,\dots,d$:

$$\begin{aligned} \alpha_j^- &= \max\{\theta \in [s_j,t_j]; h(e^{\imath\theta}) = e^{\imath\varepsilon}\}, & \beta_j^- &= \min\left\{\theta \in [\alpha_j^-,t_j]; h(e^{\imath\theta}) = e^{\imath(\pi-\varepsilon)}\right\}, \\ \alpha_j^+ &= \max\left\{\theta \in [\beta_j^-,t_j]; h(e^{\imath\theta}) = e^{\imath(\pi+\varepsilon)}\right\}, & \beta_j^+ &= \min\left\{\theta \in [\alpha_j^+,t_j]; h(e^{\imath\theta}) = e^{\imath(2\pi-\varepsilon)}\right\}. \end{aligned} \tag{3.9}$$

Then, set $I_j^\pm := \{e^{\imath\theta};\theta \in [\alpha_j^\pm,\beta_j^\pm]\}$. Using the equality

$$f_n - g = e^{\imath\varphi_n} - e^{\imath\psi} = 2\imath\sin\left(\frac{\varphi_n - \psi}{2}\right) e^{\imath(\varphi_n+\psi)/2},$$

we obtain

$$|\dot f_n - \dot g|^2 = \cos^2\left(\frac{\varphi_n - \psi}{2}\right)(\varphi_n' - \psi')^2 + \sin^2\left(\frac{\varphi_n - \psi}{2}\right)(\varphi_n' + \psi')^2. \tag{3.10}$$

Note that by the definition of $I_j^\pm$ we have

$$z = e^{\imath\theta} \in I_j^\pm \implies \left|\sin\left(\frac{\varphi_n(\theta) - \psi(\theta)}{2}\right)\right|, \left|\cos\left(\frac{\varphi_n(\theta) - \psi(\theta)}{2}\right)\right| \geq \sin(\varepsilon/2). \tag{3.11}$$

Combining (3.11) with (3.10) and (3.7) gives

$$\int_{I_j^\pm} |\dot{f}_n - \dot{g}| \geq \sin(\varepsilon/2) \int_{\alpha_j^\pm}^{\beta_j^\pm} \sqrt{(\varphi_n' - \psi')^2 + (\varphi_n' + \psi')^2} \geq \sqrt{2}\sin(\varepsilon/2) \int_{\alpha_j^\pm}^{\beta_j^\pm} |\varphi_n'| \geq \sqrt{2}\sin(\varepsilon/2)\, d_1(n-2)|I_j^\pm|, \tag{3.12}$$

where $|I_j^\pm| := \beta_j^\pm - \alpha_j^\pm$. If for one of the arcs $I_j^\pm$ there holds

$$\sqrt{2}\sin(\varepsilon/2)\, d_1(n-2)|I_j^\pm| > 2\pi d,$$

then we clearly have $\int_{\mathbb{S}^1} |\dot{f} - \dot{g}| > 2\pi d$ by (3.12), and (3.6) follows. Therefore, we are left with the case where

$$|I_j^-|, |I_j^+| \leq \frac{c_0}{n},\ j = 1, \ldots, d, \tag{3.13}$$

where $c_0 = c_0(d_1, d_2, \varepsilon)$.

While in the previous case the lower bound followed from the fact that $|\varphi_n'|$ is large (i.e., of the order of n), the argument under assumption (3.13) uses another property of φ_n. Namely, thanks to (3.8), the change of φ_n on an interval of length $O(1/n)$ (like is the case for $I_j^\pm$) is only of the order $O(1/n)$. It follows that f_n is "almost" a constant on the corresponding arc and an important contribution to the BV norm of $f_n - g$ comes from the change of the phase ψ on the corresponding interval. The latter equals approximately $\pi - 2\varepsilon$, and summing the contributions from all the arcs yields the desired lower bound. The details are given below.

In the sequel we will denote by c different constants depending on d_1, d_2 and ε alone. A direct consequence of (3.8) that will play a key role in the sequel is the following:

$$\left| \int_J \varphi_n' \right| \leq \frac{c}{n}, \text{ for every interval } J \subset \mathbb{R} \text{ with } |J| \leq \frac{c_0}{n}. \tag{3.14}$$

This implies that

$$|f_n(z_1) - f_n(z_2)| \leq \frac{c}{n},\ \forall z_1, z_2 \in I_j^\pm,\ j = 1, \ldots, d.$$

Therefore, for each $I_j^\pm$ there exists $v_j^\pm \in \mathbb{S}^1$ such that

$$|f_n(z) - v_j^\pm| \leq \frac{c}{n},\ \forall z \in I_j^\pm,\ j = 1, \ldots, d. \tag{3.15}$$

By (3.15) we have

$$\Big| 1 - |g(z) - (f_n(z) - v_j^\pm)| \Big| \leq \frac{c}{n},\ \forall z \in I_j^\pm,\ j = 1, \ldots, d. \tag{3.16}$$

Fix an arc $I_j^\pm$. By (3.16), we can define on $[\alpha_j^\pm, \beta_j^\pm]$ a $W^{1,1}$-function $\psi_n = \psi_{n,j,\pm}$, determined uniquely up to addition of an integer multiple of 2π, by

$$g(e^{\imath\theta}) - (f_n(e^{\imath\theta}) - v_j^\pm) = |g(e^{\imath\theta}) - (f_n(e^{\imath\theta}) - v_j^\pm)|\, e^{\imath\psi_n(\theta)}. \tag{3.17}$$

From (3.15)–(3.17) we have

$$|e^{\imath\psi(\theta)} - e^{\imath\psi_n(\theta)}| \leq \frac{c}{n},\ \forall \theta \in [\alpha_j^\pm, \beta_j^\pm], \tag{3.18}$$

and

$$|\dot g(e^{\imath\theta})-\dot f_n(e^{\imath\theta})|\geq|g(e^{\imath\theta})-(f_n(e^{\imath\theta})-v_j^{\pm})|\,|\psi_n'(\theta)|\geq\left(1-\frac{c}{n}\right)|\psi_n'(\theta)|. \tag{3.19}$$

By (3.19), we have

$$\int_{I_j^{\pm}}|\dot g-\dot f_n|\geq\left(1-\frac{c}{n}\right)\int_{\alpha_j^{\pm}}^{\beta_j^{\pm}}|\psi_n'|\geq\left(1-\frac{c}{n}\right)|\psi_n(\beta_j^{\pm})-\psi_n(\alpha_j^{\pm})|. \tag{3.20}$$

By (3.18), (3.20),(3.14) and (3.9), we obtain

$$\int_{I_j^{\pm}}|\dot g-\dot f_n|\geq\left(1-\frac{c}{n}\right)|\psi(\beta_j^{\pm})-\psi(\alpha_j^{\pm})|-\frac{c}{n}\geq\left(1-\frac{c}{n}\right)|\eta(\beta_j^{\pm})-\eta(\alpha_j^{\pm})|-\frac{c}{n}\geq\left(1-\frac{c}{n}\right)(\pi-2\varepsilon). \tag{3.21}$$

Summing (3.21) over the $2d$ arcs I_j^-, I_j^+, $j=1,\dots,d$ yields

$$\int_{I_j^{\pm}}|\dot g-\dot f_n|\geq\left(1-\frac{c}{n}\right)(2\pi d-4\varepsilon d). \tag{3.22}$$

Finally we choose $\varepsilon=\delta/8$ and $n\geq\dfrac{4\pi}{\delta}c(d_1,d_2,\varepsilon)$ and deduce from (3.22) that (3.6) holds. □

Step 4. Proof of (1.19) completed
Combining Steps 1, 2 and 3 we find that

$$\mathrm{Dist}_{W^{1,1}}(\mathscr{E}_{d_1},\mathscr{E}_{d_2})=2\pi|d_1-d_2|,\ \forall\, d_1\geq 0,\ \forall\, d_2\in\mathbb{Z},$$

which yields directly

$$\mathrm{Dist}_{W^{1,1}}(\mathscr{E}_{d_1},\mathscr{E}_{d_2})=2\pi|d_1-d_2|,\ \forall\, d_1\in\mathbb{Z},\ \forall\, d_2\in\mathbb{Z}.$$ □

We close this section with some results concerning the attainability of $\mathrm{Dist}_{W^{1,1}}(\mathscr{E}_{d_1},\mathscr{E}_{d_2})$. For any $d_1\neq d_2$ we may ask (question 1) whether there exists $f\in\mathscr{E}_{d_1}$ such that

$$d_{W^{1,1}}(f,\mathscr{E}_{d_2}):=\inf_{g\in\mathscr{E}_{d_2}}|f-g|_{W^{1,1}}=\mathrm{Dist}_{W^{1,1}}(\mathscr{E}_{d_1},\mathscr{E}_{d_2}), \tag{3.23}$$

and in case the answer to question 1 is positive for some $f\in\mathscr{E}_{d_1}$, we may ask (question 2) whether the infimum $\inf_{g\in\mathscr{E}_{d_2}}|f-g|_{W^{1,1}}$ is actually a minimum, i.e., for some $g\in\mathscr{E}_{d_2}$,

$$|f-g|_{W^{1,1}}=d_{W^{1,1}}(f,\mathscr{E}_{d_2})=\mathrm{Dist}_{W^{1,1}}(\mathscr{E}_{d_1},\mathscr{E}_{d_2})\,. \tag{3.24}$$

There is a trivial case where the answer to both questions is affirmative, namely, when $0=d_1\neq d_2$. Indeed, for $f=1$ and $g(z)=z^{d_2}$ we clearly have,

$$|f-g|_{W^{1,1}}=\int_{S^1}|\dot g|=2\pi|d_2|=\mathrm{Dist}_{W^{1,1}}(\mathscr{E}_0,\mathscr{E}_{d_2}).$$

The next proposition provides answers to these attainability questions, demonstrating different behaviors according to the sign of $d_1(d_2-d_1)$.

Proposition 3.2. *We have*

1. *If $d_1(d_2-d_1)>0$ then $f\in\mathscr{E}_{d_1}$ satisfies (3.23) if and only if*

$$d_1(f\wedge\dot f)\ge 0 \ \text{ a.e. in } \mathbb{S}^1. \tag{3.25}$$

   Among all maps satisfying (3.23), some satisfy (3.24) and others do not.

2. *If $d_1(d_2-d_1)<0$ then for every $f\in\mathscr{E}_{d_1}$ we have $d_{W^{1,1}}(f,\mathscr{E}_{d_2})<\mathrm{Dist}_{W^{1,1}}(\mathscr{E}_{d_1},\mathscr{E}_{d_2})$, so (3.23) is never satisfied.*

The proof relies on several lemmas.

Lemma 3.3. *Let $d_1,d_2\in\mathbb{Z}$ be such that $d_1(d_2-d_1)>0$. If $f\in\mathscr{E}_{d_1}$ satisfies (3.25) then*

$$\int_{\mathbb{S}^1}|\dot f-\dot g|\ge 2\pi|d_1-d_2|,\ \forall g\in\mathscr{E}_{d_2}. \tag{3.26}$$

If the stronger condition

$$d_1(f\wedge\dot f)>0 \text{ a.e. in } \mathbb{S}^1, \tag{3.27}$$

holds, then

$$\int_{\mathbb{S}^1}|\dot f-\dot g|> 2\pi|d_1-d_2|,\ \forall g\in\mathscr{E}_{d_2}. \tag{3.28}$$

Proof of Lemma 3.3. It suffices to consider the case $0<d_1<d_2$. Note that (3.25) is equivalent to $\int_{\mathbb{S}^1}|\dot f|=\int_0^{2\pi}f\wedge\dot f=2\pi d_1$, i.e., to f being a minimizer for $\int_{\mathbb{S}^1}|v'|$ over $\mathscr{E}_{d_1}$ ((4.3) for $p=1$). Therefore the same computation as in (3.4) yields (3.26).

Next assume the stronger condition (3.27). Writing $f(e^{\imath\theta})=e^{\imath\varphi(\theta)}$, with $\varphi\in W^{1,1}((0,2\pi))$, we then have $\varphi'>0$ a.e. in $(0,2\pi)$. Suppose by contradiction that for some $g\in\mathscr{E}_{d_2}$ equality holds in (3.26). Then (3.4) yields

$$|\dot g-\dot f|=|\dot g|-|\dot f|, \text{ a.e. in } \mathbb{S}^1. \tag{3.29}$$

Writing $g(e^{\imath\theta})=e^{\imath\psi(\theta)}$, with $\psi\in W^{1,1}((0,2\pi))$, the same computation as in (3.10), gives

$$\left|(e^{\imath\psi}-e^{\imath\varphi})'\right|^2=\cos^2\left(\frac{\varphi-\psi}{2}\right)(\varphi'-\psi')^2+\sin^2\left(\frac{\varphi-\psi}{2}\right)(\varphi'+\psi')^2. \tag{3.30}$$

Combining (3.29) with (3.30) leads to

$$\sin^2\left(\frac{\psi-\varphi}{2}\right)(\psi'-\varphi')^2=\sin^2\left(\frac{\psi-\varphi}{2}\right)(\psi'+\varphi')^2. \tag{3.31}$$

The equality (3.31) clearly implies that $\varphi'=0$ a.e. on the set $\{f\ne g\}$. Since this set has positive measure, we reached a contradiction to (3.27). □

Lemma 3.4. *If $d_1(d_2-d_1)<0$ then for every $f\in\mathscr{E}_{d_1}$ there exists $g\in\mathscr{E}_{d_2}$ such that*

$$\int_{\mathbb{S}^1}|\dot f-\dot g|<2\pi|d_1-d_2|. \tag{3.32}$$

The proof of Lemma 3.4 is quite involved. It is inspired by the work of H. Brezis and J.-M. Coron (see [9, 7]) in a two-dimensional setting, where the importance of a strict inequality like (3.32) was emphasized. The heart of the estimate is the following lemma.

Lemma 3.5. *Consider any $f \in \mathscr{E}_{d_1}$ and a point $\zeta \in \mathbb{S}^1$, which is a Lebesgue point of $\dot{f}$ with $(f \wedge \dot{f})(\zeta) \neq 0$. Then for every d_2 such that*

$$(d_2 - d_1) \cdot (f \wedge \dot{f})(\zeta) < 0 \tag{3.33}$$

there exists $g \in \mathscr{E}_{d_2}$ satisfying (3.32).

Proof of Lemma 3.5. We may assume that condition (3.33) is satisfied by $\zeta = 1$. Write $f(e^{\imath\theta}) = e^{\imath\varphi(\theta)}$ with $\varphi \in W^{1,1}((0,2\pi))$ satisfying $\varphi(2\pi) - \varphi(0) = 2\pi d_1$. By assumption, $\theta_0 = 0$ is a Lebesgue point of $\varphi' = f \wedge \dot{f}$ with $\varphi'(0) := \alpha \neq 0$ and we have

$$\lim_{\delta \to 0} \frac{1}{\delta} \int_0^\delta |\varphi' - \alpha| = 0. \tag{3.34}$$

Denote $d = d_1 - d_2$ and note that, by (3.33), we have $\alpha d > 0$. For each small $\varepsilon > 0$ set $g = e^{\imath\psi}$, where $\psi = \psi^\varepsilon$ is defined by

$$\psi(\theta) = \begin{cases} \varphi(\theta) - \dfrac{2d\pi}{\varepsilon}\theta, & \text{if } \theta \in [0,\varepsilon] \\ \varphi(\theta) - 2d\pi, & \text{if } \theta \in [\varepsilon, 2\pi] \end{cases}.$$

By (3.30), we have

$$\int_{\mathbb{S}^1} |\dot{g} - \dot{f}| = \left(\frac{2|d|\pi}{\varepsilon}\right) \int_0^\varepsilon h(\theta)\, d\theta, \tag{3.35}$$

where

$$h(\theta) = h_\varepsilon(\theta) := \left[1 + 4\sin^2\left(\frac{d\pi\theta}{\varepsilon}\right)\left\{-\frac{\varepsilon\varphi'(\theta)}{2d\pi} + \left(\frac{\varepsilon\varphi'(\theta)}{2d\pi}\right)^2\right\}\right]^{1/2}. \tag{3.36}$$

Set $F := \varphi' - \alpha$ and write

$$(h_\varepsilon(\theta))^2 = X_\varepsilon + Y_\varepsilon + Z_\varepsilon, \tag{3.37}$$

where

$$X_\varepsilon = X_\varepsilon(\theta) := 1 - \frac{2\varepsilon\alpha}{d\pi}\left(1 - \frac{\varepsilon\alpha}{2d\pi}\right)\sin^2\left(\frac{d\pi\theta}{\varepsilon}\right) = 1 - \frac{2\varepsilon\alpha}{d\pi}\sin^2\left(\frac{d\pi\theta}{\varepsilon}\right) + O(\varepsilon^2), \tag{3.38}$$

$$Y_\varepsilon = Y_\varepsilon(\theta) := \frac{2\varepsilon F}{d\pi}\left(-1 + \frac{\varepsilon\alpha}{d\pi}\right)\sin^2\left(\frac{d\pi\theta}{\varepsilon}\right) = O(\varepsilon F), \tag{3.39}$$

and

$$Z_\varepsilon = Z_\varepsilon(\theta) := \frac{\varepsilon^2 F^2}{(d\pi)^2}\sin^2\left(\frac{d\pi\theta}{\varepsilon}\right) = O(\varepsilon^2 F^2). \tag{3.40}$$

Since $X_\varepsilon \geq 1/4$ for small ε, for such ε we deduce from (3.37) that

$$h_\varepsilon(\theta) \leq (X_\varepsilon)^{1/2} + |Y_\varepsilon| + (Z_\varepsilon)^{1/2}. \tag{3.41}$$

Integrating (3.41) over $(0,\varepsilon)$ and using (3.34), (3.39) and (3.40) yields

$$\int_0^\varepsilon h_\varepsilon(\theta)\, d\theta \leq \int_0^\varepsilon (X_\varepsilon(\theta))^{1/2}\, d\theta + o(\varepsilon^2). \tag{3.42}$$

From (3.38) we have

$$(X_\varepsilon)^{1/2} = 1 - \frac{\varepsilon\alpha}{d\pi}\sin^2\left(\frac{d\pi\theta}{\varepsilon}\right) + O(\varepsilon^2). \tag{3.43}$$

Combining (3.35), (3.42) and (3.43) we obtain

$$\int_{\mathbb{S}^1} |\dot g - \dot f| \le \frac{2|d|\pi}{\varepsilon}\left(\varepsilon - \frac{\varepsilon\alpha}{d\pi}\int_0^\varepsilon \sin^2\left(\frac{d\pi\theta}{\varepsilon}\right) + o(\varepsilon^2)\right) = 2|d|\pi - \varepsilon|\alpha| + o(\varepsilon),$$

so that (3.32) holds for sufficiently small ε. □

Proof of Lemma 3.4. It suffices to consider the case where $d_1 > 0$, so by assumption $d_2 - d_1 < 0$. Since $\int_{\mathbb{S}^1}(f \wedge \dot f) = 2\pi d_1 > 0$, the set

$$A := \{\zeta \in \mathbb{S}^1; \zeta \text{ is a Lebesgue point of } \dot f \text{ with } (f \wedge \dot f)(\zeta) > 0\},$$

has positive measure. Applying Lemma 3.5 to any point $\zeta \in A$ we conclude that there exists $g \in \mathscr{E}_{d_2}$ for which (3.32) holds. □

Proof of Proposition 3.2.
Step 1. Proof of item *1*
Assume without loss of generality that $0 < d_1 < d_2$. Let $f \in \mathscr{E}_{d_1}$ satisfy (3.25). By (3.26), $d_{W^{1,1}}(f, \mathscr{E}_{d_2}) \ge 2\pi(d_2 - d_1)$. Since $\mathrm{Dist}_{W^{1,1}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) = 2\pi(d_2 - d_1)$ (by (1.19)) we obtain that f satisfies (3.23). On the other hand, for $f \in \mathscr{E}_{d_1}$ for which (3.25) does not hold we conclude from Lemma 3.5 that $d_{W^{1,1}}(f, \mathscr{E}_{d_2}) < \mathrm{Dist}_{W^{1,1}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) = 2\pi(d_2 - d_1)$, so (3.23) does not hold for f.

For $f \in \mathscr{E}_{d_1}$ satisfying condition (3.27) (we may take for example $f(\zeta) = \zeta^{d_1}$) we get from (3.28) that (3.24) is violated (although (3.23) holds). Finally to show that (3.24) occurs for some f, choose $\varphi \in W^{1,1}((0,2\pi))$ such that for some $a \in (0,2\pi)$ we have:
(i) $\varphi' \ge 0$ on $[0,a]$.
(ii) $\varphi(0) = 0, \varphi(a) = 2\pi d_1$.
(iii) $\varphi = 2\pi d_1$ on $[a, 2\pi]$.
Next define ψ on $[0,2\pi]$ by:

$$\psi(\theta) = \begin{cases} \varphi(\theta), & \text{for } \theta \in [0,a] \\ 2\pi d_1 + 2\pi(d_2 - d_1)\dfrac{\theta - a}{2\pi - a}, & \text{for } \theta \in (a, 2\pi] \end{cases}.$$

Setting $f(e^{\imath\theta}) = e^{\imath\varphi(\theta)}$ and $g(e^{\imath\theta}) = e^{\imath\psi(\theta)}$ we clearly have $f \in \mathscr{E}_{d_1}$ and $g \in \mathscr{E}_{d_2}$. Since f satisfies (3.25) we know that $d_{W^{1,1}}(f, \mathscr{E}_{d_2}) = 2\pi(d_2 - d_1)$. But clearly also $|f - g|_{W^{1,1}} = 2\pi(d_2 - d_1)$.
Step 2. Proof of item *2*
The result follows directly from Lemma 3.4 and (1.19). □

Remark 3.6. If $d_1 = 0$ and $d_2 \ne 0$ then for every *non constant* $f \in \mathscr{E}_0$ we have $d_{W^{1,1}}(f, \mathscr{E}_{d_2}) < \mathrm{Dist}_{W^{1,1}}(\mathscr{E}_0, \mathscr{E}_{d_2}) = 2\pi|d_2|$. This implies that a constant map is the only map for which (3.23) holds. Indeed, since $\int_{\mathbb{S}^1}(f \wedge \dot f) = 0$, there are Lebesgue points of $f \wedge \dot f$ of both positive and negative sign. Hence, for every $d_2 \ne 0$ we can find a Lebesgue point for which (3.33) is satisfied, and the result follows from Lemma 3.5.

4 $W^{1,p}$ maps, with $1 < p < \infty$

Proof of Theorem 0 when $1 < p < \infty$. We first sketch the proof of the inequality "≥" in (1.15). Given any $f \in \mathscr{E}_{d_1}$ and $g \in \mathscr{E}_{d_2}$, set $w := f\overline{g} \in \mathscr{E}_d$, with $d := d_1 - d_2$. Let $\widetilde{w} := T \circ w \in \mathscr{E}_d$ where, as in [15, 12], $T : \mathbb{S}^1 \to \mathbb{S}^1$ is defined by

$$T(e^{\imath\theta}) = e^{\imath\varphi} \text{ with } \varphi = \varphi(\theta) = \pi \sin(\theta/2),\ \forall\, \theta \in (-\pi, \pi]. \tag{4.1}$$

Noting that $|e^{\imath\theta} - 1| = \dfrac{2}{\pi}|\varphi|$, we obtain as in [15, 12] (with ∂_τ standing for the tangential derivative)

$$\begin{aligned} \int_{\mathbb{S}^1} |\partial_\tau(f-g)|^p \ge \int_{\mathbb{S}^1} |\partial_\tau|f-g||^p = \int_{\mathbb{S}^1} \left|\partial_\tau|f\overline{g}-1|\right|^p = \int_{\mathbb{S}^1} |\partial_\tau|w-1||^p \\ = \left(\frac{2}{\pi}\right)^p \int_{\mathbb{S}^1} |\partial_\tau \widetilde{w}|^p \ge \left(\frac{2}{\pi}\right)^p \inf_{v \in \mathscr{E}_d} \int_{\mathbb{S}^1} |\dot{v}|^p. \end{aligned} \tag{4.2}$$

The inequality "≥" in (1.15) clearly follows from (4.2) and the next claim:

$$\min_{v \in \mathscr{E}_d} \int_{\mathbb{S}^1} |\dot{v}|^p = 2|d|^p \pi. \tag{4.3}$$

To verify (4.3) we first associate to each $v \in \mathscr{E}_d$ a function $\psi \in W^{1,p}((-\pi,\pi);\mathbb{R})$ such that $v(e^{\imath\theta}) = e^{\imath\psi(\theta)}$, $\theta \in [-\pi,\pi]$, with $\psi(\pi) - \psi(-\pi) = 2d\pi$. We then have, invoking Hölder inequality,

$$\int_{\mathbb{S}^1} |\dot{v}|^p = \int_{-\pi}^{\pi} |\psi'|^p \ge \frac{(2|d|\pi)^p}{(2\pi)^{p-1}},$$

whence the inequality "≥" in (4.3). On the other hand, the function $\widetilde{w}(e^{\imath\theta}) = e^{\imath d\theta}$ clearly gives equality in (4.3), completing the proof of (4.3). Note that $\widetilde{w}$ is the unique minimizer in (4.3), up to rotations. The proof of the inequality "≤" in (1.15) can be carried out using an explicit construction, like the proof in [15] for $p = 2$. □

Next we turn to the question of attainment of the infimum in (1.15).

Proof of Theorem 1, items 2 and 3. The proof of the case $p \ge 2$ is identical to the one given in [15] for $p = 2$, so we consider here only item *3* (i.e., we let $1 < p < 2$).

Step 1. The infimum in (1.15) is achieved when $d_2 = -d_1$

Assume that $d_2 = -d_1$. Let $d := d_1 - d_2 = 2d_1$. We saw above that $\widetilde{w}(e^{\imath\theta}) = e^{\imath d\theta}$ realizes the minimum in (4.3). Consider $S := T^{-1} : \mathbb{S}^1 \to \mathbb{S}^1$ (see (4.1)), given explicitly by

$$S(e^{\imath\theta}) = e^{\imath\psi}, \text{ with } \psi(\theta) = 2\arcsin(\theta/\pi),\ \forall\theta \in [-\pi,\pi].$$

Although S is not Lipschitz, we do have $w := S \circ \widetilde{w} \in W^{1,p}(\mathbb{S}^1;\mathbb{S}^1)$ (i.e., $w \in \mathscr{E}_d$). Indeed, this amounts to $\dfrac{1}{\sqrt{1-t^2}} \in L^p((1-\delta,1))$, which holds since $p < 2$. Since d is even and w has degree d, there exists $f \in \mathscr{E}_{d_1}$ satisfying $w = f^2$. We let $g := \overline{f} \in \mathscr{E}_{d_2}$, so that $w = f\overline{g}$. Note that $f - g$ takes only purely imaginary values, and therefore

$$|\partial_\tau(f-g)| = |\partial_\tau|f-g|| \quad \text{a.e. on } \mathbb{S}^1. \tag{4.4}$$

For these particular f, g, w and $\widetilde{w}$, we get, using (4.4) that all the inequalities in (4.2) are actually equalities, and we see that the infimum in (1.15) is attained.

Step 2. If the infimum in (1.15) is achieved, then $d_2 = -d_1$

Assume that the infimum in (1.15) is achieved by two functions $f \in \mathscr{E}_{d_1}$ and $g \in \mathscr{E}_{d_2}$. Set $d := d_1 - d_2$, $w := f\overline{g}$ and $\widetilde{w} := T \circ w$. We then have $w, \widetilde{w} \in \mathscr{E}_d$. We may assume that $d > 0$. From the fact that both inequalities in (4.2) must be equalities we deduce that

(i) $\widetilde{w}$ is a minimizer in (4.3)
and
(ii) (4.4) holds.

From (i) it follows that $\widetilde{w}(e^{\imath\theta}) = e^{\imath(d\theta + C)}$ for some constant C, and we may assume that $C = 0$. Therefore,

$$w^{-1}(1) = \widetilde{w}^{-1}(1) = \{1, \omega, \omega^2, \dots, \omega^{d-1}\}, \text{ with } \omega = e^{\imath 2\pi/d}.$$

On the small arc I_j between ω^j and ω^{j+1} we may write $f - g = \rho e^{\imath\psi}$ with $\rho = |f - g|$ and $\psi \in W^{1,p}_{loc}$, and we have

$$|\partial_\tau (f-g)|^2 = \rho^2 [\dot\psi]^2 + [\dot\rho]^2.$$

By (ii), $\dot\psi = 0$ on I_j, so that ψ is constant on I_j, say $\psi = \alpha_j$ on I_j. The equality $f - g = \rho e^{\imath\alpha_j}$ on I_j implies that $g = e^{\imath(2\alpha_j - \pi)}\overline{f}$ on I_j, and therefore $g \wedge \dot g = -f \wedge \dot f$ on each I_j. Since this is true on each I_j, we finally conclude that $d_2 = -d_1$. □

5 $W^{s,p}$ maps, with $sp > 1$

Proof of Theorem 2, item 2.
Step 1. Proof of "$\lesssim$" in (1.18)
Fix a smooth map $h \in \mathscr{E}_1$ such that $h(z) \equiv 1$ when Re $z \le 0$.

Given d_2, consider a smooth map $g \in \mathscr{E}_{d_2}$ such that $g(z) \equiv 1$ when Re $z \ge 0$. Set $f := h^{d_1 - d_2} g \in \mathscr{E}_{d_1}$. Then

$$|f - g|_{W^{s,p}} \lesssim |d_1 - d_2|^s. \tag{5.1}$$

Indeed, estimate (5.1) is clear when s is an integer, since $f - g = h^{d_1-d_2} - 1$. The general case follows via Gagliardo-Nirenberg.

Step 2. Proof of "$\gtrsim$" in (1.18) when $0 < s \le 1$
We rely on an argument similar to the one in Step 2 in the proof of Theorem 0 in Section 3. Assume that $d := d_1 - d_2 > 0$, and that $f(1) = g(1)$. Write $f(e^{\imath\theta}) = e^{\imath\varphi(\theta)} g(e^{\imath\theta})$, with $\varphi \in W^{s,p}((0,2\pi))$ and $\varphi(0) = 0$. Let $0 = t_0 < \tau_0 < \cdots < \tau_{d-1} < t_d = 2\pi$ be such that $(f-g)(e^{\imath t_j}) = 0$ and $|(f-g)(e^{\imath\tau_j})| = 2$. By scaling and the hypotheses $0 < s \le 1$ and $sp > 1$, we have

$$|h(b) - h(a)| \lesssim (b-a)^{s-1/p} |h|_{W^{s,p}((a,b))}, \ \forall\, a < b, \ \forall\, h \in W^{s,p}((a,b)). \tag{5.2}$$

Applying (5.2) to $h := (f-g)(e^{\imath\theta})$ on $(a,b) := (t_j, \tau_j)$, $j = 0, \dots, d-1$, we obtain that $|h|_{W^{s,p}((t_j,\tau_j))} \gtrsim 1/(\tau_j - t_j)^{s-1/p}$, and thus

$$|f-g|^p_{W^{s,p}} \gtrsim \sum_{j=0}^{d-1} |h|^p_{W^{s,p}((t_j,\tau_j))} \gtrsim \sum_{j=0}^{d-1} \frac{1}{(\tau_j - t_j)^{sp-1}} \gtrsim d^{sp},$$

the latter inequality following from Jensen's inequality applied to the function $x \mapsto 1/x^{sp-1}$, $x > 0$.

Step 3. Proof of "$\gtrsim$" in (1.18) when $s > 1$
The key ingredient in Step 4 is the Gagliardo-Nirenberg type inequality

$$|f|_{W^{\theta s,p/\theta}} \le C_{\theta,s,p}|f|^{\theta}_{W^{s,p}}\|f\|^{1-\theta}_{L^\infty},\ \forall s>0,\ 1\le p<\infty \text{ such that } (s,p)\neq(1,1),\ \forall\theta\in(0,1). \quad (5.3)$$

Let us note that, if $f,g:\mathbb{S}^1\to\mathbb{S}^1$ and $\deg f\neq\deg g$, then (by the argument leading to (1.11))

$$\|f-g\|_{L^\infty}=2. \quad (5.4)$$

By (5.3) and (5.4), we find that for every s, p, θ as in (5.3) we have

$$\mathrm{dist}_{W^{s,p}}(\mathscr{E}_{d_1},\mathscr{E}_{d_2})\ge C'_{\theta,s,p}[\mathrm{dist}_{W^{\theta s,p/\theta}}(\mathscr{E}_{d_1},\mathscr{E}_{d_2})]^{1/\theta},\ \forall d_1,d_2\in\mathbb{Z}. \quad (5.5)$$

If we take, in (5.5), θ such that $\theta s<1$, we obtain Step 4 from Step 3. □

6 Maps $f:\mathbb{S}^N\to\mathbb{S}^N$

6.1 A useful construction

Throughout Section 6 we will make an extensive use of special smooth maps $f:\mathbb{S}^N\to\mathbb{S}^N$, $N\ge1$. Such maps "live" on a small spherical cap, say $B_R(\sigma)$, where $B_R(\sigma)$ is the geodesic ball of radius $R<1$ centered at some point σ of $\mathbb{S}^N$, and are constant on $\mathbb{S}^N\setminus B_R(\sigma)$. Since the construction is localized we may as well work first on a flat ball $B_R(0)$ centered at 0 in $\mathbb{R}^N$ and then we will transplant f to $B_R(\sigma)$, thereby producing a map $f:\mathbb{S}^N\to\mathbb{S}^N$. On $B_R(0)$, the map f is determined by a smooth function $F:[0,R]\to\mathbb{R}$ and a smooth map $h:\mathbb{S}^{N-1}\to\mathbb{S}^{N-1}$.

For simplicity we start with the case $N\ge2$ since the case $N=1$ is somewhat "degenerate" and will be discussed later.

Fix a smooth function $F:[0,R]\to\mathbb{R}$ satisfying

$$F(r)=0 \text{ for } r \text{ near } 0. \quad (6.1)$$

$$F(r)=k\pi \text{ for } r \text{ near } R, \text{ where } k\in\mathbb{Z}. \quad (6.2)$$

We may now define $f:B_R(0)\to\mathbb{S}^N$ by

$$f(x)=(\sin F(|x|)\,h(x/|x|),(-1)^N\cos F(|x|)). \quad (6.3)$$

Note that f is well defined and smooth on $B_R(0)$ (by (6.1)) and that f is constant near $\partial B_R(0)$ (by (6.2)). More precisely

$$f(0)=(0,0,\dots,0,(-1)^N)=\begin{cases}\mathbf{N}, & \text{if } N \text{ is even}\\ \mathbf{S}, & \text{if } N \text{ is odd}\end{cases}$$

and

$$\text{for } x \text{ near } \partial B_R(0),\ f(x)=(0,0,\dots,0,(-1)^N\cos k\pi)=\mathbf{C}:=\begin{cases}\mathbf{N}, & \text{if } N+k \text{ is even}\\ \mathbf{S}, & \text{if } N+k \text{ is odd}\end{cases};$$

here $\mathbf{N}=(0,0,\dots,0,1)$ and $\mathbf{S}=(0,0,\dots0,-1)$ are the north pole and the south pole of $\mathbb{S}^N$. We transport f into $B_R(\sigma)\subset\mathbb{S}^N$ via a fixed orientation preserving diffeomorphism and extend it by the value $\mathbf{C}$ on $\mathbb{S}^N\setminus B_R(\sigma)$. In this way we have defined a smooth map $f:\mathbb{S}^N\to\mathbb{S}^N$.

For the purpose of Lemmas 6.1 and 6.2 below it suffices to assume that $F:[0,R]\to\mathbb{R}$ is merely continuous and satisfies $F(0)=0$, $F(R)=k\pi$, so that $f:\mathbb{S}^N\to\mathbb{S}^N$ is a well-defined continuous map.

Lemma 6.1. *Let $k \in \{0,1\}$. We have*

$$\deg f = k \deg h. \tag{6.4}$$

Proof. We emphasize the fact that here we assume $N \geq 2$, although the conclusion still holds when $N = 1$ (see below).

It will be convenient to assume that F satisfies (6.1) and (6.2); the general case follows by density.

The cases where $k = 0$ (respectively $d = 0$) are trivial via homotopy to $F \equiv 0$ (respectively $h \equiv C$). With no loss of generality, we assume that $d := \deg h > 0$ and $k = 1$.

Since f is constant outside $B_R(\sigma)$, it suffices to determine the degree of $f_{|B_R(\sigma)}$, and then we may as well work on the flat ball $B_R(0) \subset \mathbb{R}^N$. We will work in the class of maps

$$C^0_{\mathbf{C}}(\overline{B}_R(0); \mathbb{S}^N) := \{g : \overline{B}_R(0) \to \mathbb{S}^N;\ g = \mathbf{C} \text{ on } \partial B_R(0)\},$$

which have a well-defined degree (since they can be identified with maps in $C^0(\mathbb{S}^N; \mathbb{S}^N)$).

Step 1. Proof of (6.4) when $d = 1$ and $k = 1$

This case can be reduced by homotopy to the case $h = \mathrm{Id}$ and $F : [0,R] \to [0,\pi]$ is non decreasing. In this case, for almost every $\mathbf{s} \in \mathbb{S}^N$ the equation $f(\mathbf{t}) = \mathbf{s}$ has exactly one solution $\mathbf{t}$, and f is orientation preserving at $\mathbf{t}$. Thus $\deg f = 1$.

Step 2. Proof of (6.4) when $d > 1$ and $k = 1$

Consider smooth maps $h_1, h_2, \ldots, h_d : \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}$ of degree 1 which "live" in different regions $\omega_1, \ldots, \omega_d$ of $\mathbb{S}^{N-1}$, in the sense that $\overline{\omega_j} \cap \overline{\omega_k} = \emptyset$ when $j \neq k$ and $h_j = (0,0,\ldots,0,1)$ in $\mathbb{S}^{N-1} \setminus \omega_j$, $\forall j$. We glue these maps together and obtain a smooth map $\widetilde{h} : \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}$ of degree d. Since h and $\widetilde{h}$ are homotopic within $C^\infty(\mathbb{S}^{N-1}; \mathbb{S}^{N-1})$, the map f and the map $\widetilde{f}$ corresponding to $\widetilde{h}$ (via (6.3)) are homotopic within $C^\infty(B_R(0); \mathbb{S}^N)$. Thus $\deg f = \deg \widetilde{f}$.

On the other hand, let f_j be the map associated to h_j via (6.3). Set

$$\Omega_j := \{r\, y;\ y \in \omega_j, 0 < r < R\}.$$

Note that the Ω_j's are mutually disjoint.

If $x \in \overline{B}_R(0) \setminus \Omega_j$, then $f_j(x) \in \mathscr{C}$, where

$$\mathscr{C} := \{(0,0,\ldots,0,\sin\theta,\cos\theta);\ \theta \in \mathbb{R}\} \subset \mathbb{S}^N$$

(since for such x we have $h(x/|x|) = (0,0,\ldots,0,1)$). Similarly, if $x \in \overline{B}_R(0) \setminus \cup_j \Omega_j$, then $f(x) \in \mathscr{C}$.

Since $\mathscr{C}$ has null measure in $\mathbb{S}^N$ (here we use $N \geq 2$), we may find some value $z \in \mathbb{S}^N \setminus \mathscr{C}$ regular for f (and thus for each f_j). For such z, we have

$$\deg f = \sum_{x \in f^{-1}(z)} \operatorname{sgn} \operatorname{Jac} f(x) = \sum_j \sum_{x \in f^{-1}(z) \cap \Omega_j} \operatorname{sgn} \operatorname{Jac} f(x) = \sum_j \deg f_j = d,$$

the latter equality following from Step 1. □

The conclusion of Lemma 6.1 also holds for $N = 1$ and arbitrary k, but this requires a separate argument. When $N = 1$, we have $\mathbb{S}^{N-1} = \mathbb{S}^0 = \{-1,1\}$ and we have (modulo symmetry) only two maps $h : \mathbb{S}^0 \to \mathbb{S}^0$, namely

$$h_1(-1) = -1,\ h_1(1) = 1,$$
$$h_2(-1) = 1,\ h_2(1) = 1.$$

Then $\deg h_1 = 1$ and $\deg h_2 = 0$.

The associated maps f_1, f_2 defined on $B_R(0) = (-R, R)$ with values in $\mathbb{S}^1$ are

$$f_1(x) = \begin{cases} (\sin F(x), -\cos F(x)), & \text{if } x > 0 \\ (-\sin F(-x), -\cos F(-x)), & \text{if } x < 0 \end{cases},$$

$$f_2(x) = \begin{cases} (\sin F(x), -\cos F(x)), & \text{if } x > 0 \\ (\sin F(-x), -\cos F(-x)), & \text{if } x < 0 \end{cases}.$$

Clearly $f_1 = e^{\imath\varphi_1}$ and $f_2 = e^{\imath\varphi_2}$, where

$$\varphi_1(x) = \begin{cases} -\pi/2 + F(x), & \text{if } x > 0 \\ -\pi/2 - F(-x), & \text{if } x < 0 \end{cases},$$

$$\varphi_2(x) = \begin{cases} -\pi/2 + F(x), & \text{if } x > 0 \\ -\pi/2 + F(-x), & \text{if } x < 0 \end{cases}.$$

Thus

$$\deg f_1 = \frac{1}{2\pi}(\varphi_1(R) - \varphi_1(-R)) = \frac{2F(R)}{2\pi} = k$$

and

$$\deg f_2 = \frac{1}{2\pi}(\varphi_2(R) - \varphi_2(-R)) = 0.$$

For the record, we call the attention of the reader to the following generalization of Lemma 6.1

Lemma 6.2. *For every $k \in \mathbb{Z}$,*

$$\deg f = \begin{cases} k \deg h, & \textit{if } N \textit{ is odd} \\ \deg h, & \textit{if } N \textit{ is even and } k \textit{ is odd} \\ 0, & \textit{if } N \textit{ is even and } k \textit{ is even} \end{cases}. \tag{6.5}$$

Proof. Assume e.g. that $k \geq 2$. [The case $k < 0$ is handled similarly and is left to the reader.]

As explained in the proof of Lemma 6.1, we may work in the class $C^0_{\mathbf{C}}(\overline{B}_R(0); \mathbb{S}^N)$.

We may assume via homotopy that $F(r) = k\pi r/R$. Set $r_j = jR/k$, $j = 0, \ldots, k$. Consider the functions

$$F_j(r) := \begin{cases} 0, & \text{if } r < r_{j-1} \\ F(r) - (j-1)\pi, & \text{if } r_{j-1} \leq r < r_j \\ \pi, & \text{if } r \geq r_j \end{cases}, \ j = 1, \ldots, k.$$

Consider also the maps f_j corresponding to F_j via (6.3). Then f is obtained by gluing the maps $(-1)^{j-1} f_j$. By Lemma 6.1, we have

$$\deg f_j = \deg h, \ j = 1, \ldots, k. \tag{6.6}$$

We next note that

$$\text{for every } g \in C^0_{\mathbf{C}}(\overline{B}_R(0); \mathbb{S}^N), \ \deg(-g) = \begin{cases} \deg g, & \text{if } N \text{ is odd} \\ -\deg g, & \text{if } N \text{ is even} \end{cases}. \tag{6.7}$$

By (6.6) and (6.7), we have

$$\deg f = \sum_j \deg\left((-1)^{j-1} f_j\right) = \begin{cases} k \deg h, & \text{if } N \text{ is odd} \\ \deg h, & \text{if } N \text{ is even and } k \text{ is odd} \\ 0, & \text{if } N \text{ is even and } k \text{ is even} \end{cases}. \qquad \square$$

6.2 Proof of Theorem 4, item *2*

Step 1. Proof of the lower bound in (1.24)
Since we assume that

$$[s>0 \text{ and } sp>N] \text{ or } [s=N \text{ and } p=1], \tag{6.8}$$

the space $W^{s,p}$ is embedded continuously in the space of continuous functions, and there exists a constant $C_{N,s,p}$ such that

$$\left\| f - \fint_{\mathbb{S}^N} f \right\|_{L^\infty} \le C_{N,s,p} |f|_{W^{s,p}}, \ \forall f \in W^{s,p}. \tag{6.9}$$

Step 1 is a direct consequence of the next lemma.

Lemma 6.3. *In all spaces $W^{s,p}$ satisfying (6.8) we have, for all $f \in \mathscr{E}_{d_1}$, $g \in \mathscr{E}_{d_2}$, $d_1 \ne d_2$,*

$$d_{W^{s,p}}(f,g) \ge \frac{1}{C_{N,s,p}}, \tag{6.10}$$

where $C_{N,s,p}$ is the constant in (6.9).

Proof of Lemma 6.3. Recall (see (1.11)) that

$$\|f-g\|_{L^\infty} = 2. \tag{6.11}$$

From (6.9) we have

$$\left\| (f-g) - \fint_{\mathbb{S}^N} (f-g) \right\|_{L^\infty} \le C_{N,s,p} |f-g|_{W^{s,p}}, \tag{6.12}$$

so that

$$2 = \|f-g\|_{L^\infty} \le |A| + r, \tag{6.13}$$

where $A := \fint_{\mathbb{S}^N}(f-g)$ and $r := C_{N,s,p}|f-g|_{W^{s,p}}$.
We may assume that $A \ne 0$, otherwise (6.10) is clear. From (6.12) we have

$$f(\mathbb{S}^N) \subset \mathbb{S}^N + A + \overline{B}(0,r). \tag{6.14}$$

Clearly,

$$-\frac{A}{|A|} \not\in \mathbb{S}^N + A + \overline{B}(0,r) \text{ if } |A| > r,$$

and then f cannot be surjective – so that $\deg f = 0$. Similarly, we have $\deg g = 0$. This is impossible since $d_1 \ne d_2$, and therefore

$$|A| \le r = C_{N,s,p} |f-g|_{W^{s,p}}. \tag{6.15}$$

Combining (6.13) and (6.15) yields $1 \le C_{N,s,p}|f-g|_{W^{s,p}}$. □

Step 2. Proof of the upper bound in (1.24)

We will construct maps $f \in \mathscr{E}_{d_1}$, $g \in \mathscr{E}_{d_2}$, constant outside some small neighborhood $B_R(\mathbf{N})$ of the north pole $\mathbf{N} = (0,0,\dots,0,1)$ of $\mathbb{S}^N$, satisfying (1.24). We will use the setting described in Section 6.1.

We start with the case $d_1 = d$, $d_2 = 0$. Let $h : \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}$ be any smooth map of degree d. [Here we use the assumption $N \geq 2$. If $N = 1$, such an h does not exist when $|d| \geq 2$; see the discussion in Section 6.1 concerning the case $N = 1$.] Let $G : [0,R] \to \mathbb{R}$ be a smooth function such that

$$G(r) = \begin{cases} 0, & \text{if } r \leq R/4 \\ \pi/2, & \text{if } R/3 \leq r \leq 2R/3 \\ 0, & \text{if } 3R/4 \leq r \leq R \end{cases}.$$

Let $F : [0,R] \to \mathbb{R}$ be defined by

$$F(r) := \begin{cases} G(r), & \text{if } 0 \leq r < R/2 \\ \pi - G(r), & \text{if } R/2 \leq r \leq R \end{cases}.$$

Clearly, F and G satisfy assumptions (6.1) and (6.2).

We now define as in Section 6.1

$$f(x) = (\sin F(|x|)\, h(x/|x|), (-)^N \cos F(|x|)),$$
$$g(x) = (\sin G(|x|)\, h(x/|x|), (-1)^N \cos G(|x|)).$$

From Lemma 6.1 we have $\deg f = d$ and $\deg g = 0$. Clearly

$$\sin F(r) = \sin G(r),\ \forall\, r \in [0,R],$$

and thus

$$f(x) - g(x) = \begin{cases} 0, & \text{if } |x| < R/2 \\ (0,0,\dots,0,2(-1)^N \cos F(|x|)), & \text{if } R/2 \leq |x| < R \end{cases}.$$

In the case where $d_1 = d$ and $d_2 = 0$, the upper bound (1.24) follows from the fact that $f - g$ does not depend on d.

We next turn to the general case. Consider a map $m \in C^\infty(\mathbb{R}^N;\mathbb{S}^N)$ such that $m(x) = \mathbf{N}$ when $|x| > R/4$ and $\deg m = d_2$. Then, with $d := d_1 - d_2$ and with f and g as above, consider

$$\tilde{f}(x) = \begin{cases} m(x), & \text{if } |x| < R/4 \\ f(x), & \text{if } R/4 \leq |x| < R \end{cases}, \quad \tilde{g}(x) = \begin{cases} m(x), & \text{if } |x| < R/4 \\ g(x), & \text{if } R/4 \leq |x| < R \end{cases}.$$

Then $\tilde{f} \in \mathscr{E}_{d_1}$, $\tilde{g} \in \mathscr{E}_{d_2}$, and $\tilde{f} - \tilde{g} = f - g$, whence (1.24). □

6.3 Proof of Theorem 4, item *1*

Here $N \geq 1$. A key ingredient is the following

Lemma 6.4. *There are two families of smooth maps $f_\varepsilon, g_\varepsilon : \mathbb{S}^N \to \mathbb{S}^N$, defined for ε small, such that*

$$f_\varepsilon(\mathbf{s}) = g_\varepsilon(\mathbf{s}) = \mathbf{N},\ \forall\, \mathbf{s} \in B_{\varepsilon/4}(\mathbf{S}), \tag{6.16}$$
$$f_\varepsilon(\mathbf{s}) = \mathbf{S},\ \forall\, \mathbf{s} \in \mathbb{S}^N \setminus B_{\varepsilon^{1/2}}(\mathbf{S}), \tag{6.17}$$
$$g_\varepsilon(\mathbf{s}) = \mathbf{N},\ \forall\, \mathbf{s} \in \mathbb{S}^N \setminus B_{\varepsilon^{1/2}}(\mathbf{S}), \tag{6.18}$$
$$\deg f_\varepsilon = 1, \tag{6.19}$$
$$\deg g_\varepsilon = 0, \tag{6.20}$$
$$|f_\varepsilon - g_\varepsilon|_{W^{N/p,p}(\mathbb{S}^N)} \to 0 \text{ as } \varepsilon \to 0,\ \forall\, 1 < p < \infty. \tag{6.21}$$

Granted Lemma 6.4 we proceed with the

Proof of Theorem 4, item 1. Assume e.g. that $d := d_1 - d_2 > 0$. We fix d distinct points $\sigma_1, \dots, \sigma_d \in \mathbb{S}^N$. Note that $f_\varepsilon - \mathbf{S}$ has support in $B_{\varepsilon^{1/2}}(\mathbf{S})$. Therefore, for sufficiently small ε, we may glue d copies of f_ε centered at $\sigma_1, \dots, \sigma_d \in \mathbb{S}^N$. We denote by $\widetilde{f}_\varepsilon$ the resulting map. By construction $\widetilde{f}_\varepsilon - \mathbf{S}$ is supported in the union of mutually disjoint balls $B_{\varepsilon^{1/2}}(\sigma_i)$, $i = 1, \dots, d$. From (6.19) we have

$$\deg \widetilde{f}_\varepsilon = d. \tag{6.22}$$

Next we consider a family of smooth maps $h_\varepsilon : \mathbb{S}^N \to \mathbb{S}^N$ such that

$$\deg h_\varepsilon = d_2 \tag{6.23}$$

and

$$h_\varepsilon(\mathbf{s}) = \mathbf{N}, \ \forall\, \mathbf{s} \in \mathbb{S}^N \setminus B_{\varepsilon/8}(\sigma_1). \tag{6.24}$$

[The construction of h_ε is totally standard.]

We glue h_ε to $\widetilde{f}_\varepsilon$ by inserting it in $B_{\varepsilon/8}(\sigma_1)$ (here we use (6.16)). The resulting map is denoted by $\widehat{f}_\varepsilon$. From (6.22) and (6.23) we have

$$\deg \widehat{f}_\varepsilon = d + d_2 = d_1, \tag{6.25}$$

so that $\widehat{f}_\varepsilon \in \mathscr{E}_{d_1}$.

We proceed similarly with g_ε using the same points $\sigma_1, \dots, \sigma_d \in \mathbb{S}^N$. We first obtain $\widetilde{g}_\varepsilon$ such that, by (6.20),

$$\deg \widetilde{g}_\varepsilon = 0. \tag{6.26}$$

We then glue h_ε to g_ε as above and obtain some $\widehat{g}_\varepsilon$ such that, by (6.23) and (6.26),

$$\deg \widehat{g}_\varepsilon = 0 + d_2 = d_2, \tag{6.27}$$

so that $\widehat{g}_\varepsilon \in \mathscr{E}_{d_2}$.

Clearly $\widehat{f}_\varepsilon - \widehat{g}_\varepsilon$ consists of d glued copies of $f_\varepsilon - g_\varepsilon$. Therefore

$$|\widehat{f}_\varepsilon - \widehat{g}_\varepsilon|_{W^{N/p,p}} \le d\, |f_\varepsilon - g_\varepsilon|_{W^{N/p,p}}$$

and thus

$$\operatorname{dist}_{W^{N/p,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) \le |\widehat{f}_\varepsilon - \widehat{g}_\varepsilon|_{W^{N/p,p}} \to 0 \text{ as } \varepsilon \to 0. \qquad \square$$

We now turn to the

Proof of Lemma 6.4. Since the construction is localized on a small geodesic ball, we may as well work on the flat ball $B_R(0)$ centered at 0 in $\mathbb{R}^N$, with $R > \varepsilon^{1/2}$.

Fix a smooth nonincreasing function $K : \mathbb{R} \to [0,1]$ such that

$$K(t) = \begin{cases} 1, & \text{if } t \le 1/4 \\ 0, & \text{if } t \ge 3/4 \end{cases}. \tag{6.28}$$

Consider the family of radial functions $H_\varepsilon(x) = H_\varepsilon(|x|) : \mathbb{R}^N \to [0,1]$ defined by

$$H_\varepsilon(x) = H_\varepsilon(|x|) := \begin{cases} K\left(\dfrac{1}{4} - \dfrac{1}{2\ln 2} \ln\left(\dfrac{\ln 1/|x|}{\ln 1/\varepsilon}\right)\right), & \text{if } |x| < 1 \\ 0, & \text{if } |x| \ge 1 \end{cases}. \tag{6.29}$$

Here, ε is a parameter such that

$$0 < \varepsilon < 1/e^2. \tag{6.30}$$

We also consider the radial functions $F_\varepsilon(r)$ and $G_\varepsilon(r)$ defined by

$$F_\varepsilon(r) := \begin{cases} \pi(1 - K(r/\varepsilon))/2, & \text{if } r < \varepsilon \\ \pi(1 - H_\varepsilon(r)/2), & \text{if } \varepsilon \le r < R \end{cases} \tag{6.31}$$

and

$$G_\varepsilon(r) := \begin{cases} F_\varepsilon(r), & \text{if } r < \varepsilon \\ \pi - F_\varepsilon(r) = \pi H_\varepsilon(r)/2, & \text{if } \varepsilon \le r < R \end{cases}. \tag{6.32}$$

Note that F_ε and G_ε are smooth (this is clear in the regions $\{r < \varepsilon\}$ and $\{r > 3\varepsilon/4\}$).

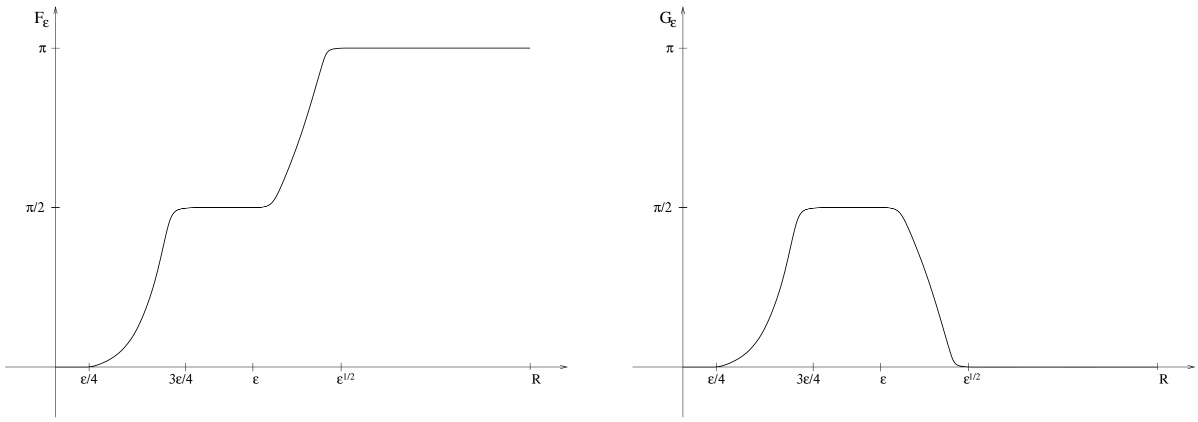

Figure 1: Plots of F_ε and G_ε given by (6.31) and (6.32)

As in Section 6.1 set

$$f_\varepsilon(x) = \left(\sin F_\varepsilon(|x|)\frac{x}{|x|}, (-1)^N \cos F_\varepsilon(|x|)\right), \ \forall x \in B_R(0),$$

$$g_\varepsilon(x) = \left(\sin G_\varepsilon(|x|)\frac{x}{|x|}, (-1)^N \cos G_\varepsilon(|x|)\right), \ \forall x \in B_R(0).$$

It is clear (using Lemma 6.1) that (6.16)–(6.20) hold. Moreover,

$$f_\varepsilon(x) - g_\varepsilon(x) = \left(0, 0, \ldots, 0, 2(-1)^{N+1}\cos\left(\frac{\pi}{2}H_\varepsilon(|x|)\right)\right), \ \forall x \in B_R(0),$$

(since $H_\varepsilon(r) = 1$ when $r < \varepsilon$ by (6.29)). Therefore

$$|f_\varepsilon - g_\varepsilon|_{W^{N/p,p}} = 2\left|\cos\left(\frac{\pi}{2}H_\varepsilon\right)\right|_{W^{N/p,p}}.$$

Consider the function

$$\widetilde{K}(r) = 1 - \cos\left(\frac{\pi}{2}K(r)\right), \ \forall r \in \mathbb{R}.$$

Clearly $\widetilde{K}$ satisfies (6.28). Consider the function $\widetilde{H}_\varepsilon$ derived from $\widetilde{K}$ via (6.29), so that

$$\widetilde{H}_\varepsilon(x) = 1 - \cos\left(\frac{\pi}{2}H_\varepsilon(x)\right), \ \forall x \in \mathbb{R}^N,$$

and therefore

$$|f_\varepsilon - g_\varepsilon|_{W^{N/p,p}(\mathbb{R}^N)} = 2\left|\widetilde{H}_\varepsilon\right|_{W^{N/p,p}(\mathbb{R}^N)} \to 0 \text{ as } \varepsilon \to 0$$

by (A.5) in Lemma A.1 (applied to $\widetilde{K}$). □

6.4 Proof of Theorem 5, item 1 (and of Theorem 3, item 2)

We rely on the following result, whose proof is postponed to the Appendix.

Lemma 6.5. *Let $N \geq 1$ and $1 \leq p < \infty$. Fix a geodesic ball $B \subset \mathbb{S}^N$ (of small radius). Then there exists a map $h : \mathbb{S}^N \to \mathbb{S}^N$ (depending on d) such that*

1. $\deg h = d$.
2. $h = (0, 0, \ldots, 0, 1)$ *outside* B.
3. $|h|_{W^{N/p,p}} \leq C_{N,p} |d|^{1/p}$.

Granted Lemma 6.5, we proceed as follows. Let $g \in \mathscr{E}_{d_2}$ be a smooth map such that g is constant in a neighborhood of some closed ball B. Such maps are dense in $\mathscr{E}_{d_2}$, and with no loss of generality we assume that $g = (0, 0, \ldots, 0, 1)$ near B. Let h be as in the above lemma, with $d := d_1 - d_2$, and set $f = \begin{cases} g, & \text{in } \mathbb{S}^N \setminus B \\ h, & \text{in } B \end{cases}$. Then clearly $f \in \mathscr{E}_{d_1}$ and

$$\text{dist}_{W^{N/p,p}}(g, \mathscr{E}_{d_1}) \leq |f - g|_{W^{N/p,p}} \leq C_{N,p} |d_1 - d_2|^{1/p}. \tag{6.33}$$

The validity of (6.33) for arbitrary $g \in \mathscr{E}_{d_2}$ follows by density. □

6.5 Proof of Theorem 5, item *2* (and of Theorem 3, item *3*)

This time the key construction is provided by the following

Lemma 6.6. *Let $N \geq 1$. Fix $d_1 \in \mathbb{Z}$. Then there exists a sequence of smooth maps $f_n : \mathbb{S}^N \to \mathbb{S}^N$ (with sufficiently large n) such that:*

1. $\deg f_n = d_1$.
2. *For every geodesic ball $B \subset \mathbb{S}^N$ of radius $1/n$, $f_n(B) = \mathbb{S}^N$.*

Granted Lemma 6.6, we claim that the sequence (f_n) satisfies

$$\text{dist}_{W^{s,p}}(f_n, \mathscr{E}_{d_2}) \geq C'_{s,p,N,\alpha} n^\alpha, \text{ with } C'_{s,p,N,\alpha} > 0, \tag{6.34}$$

for any $0 < \alpha \leq 1$ such that $W^{s,p} \hookrightarrow C^\alpha$. Clearly, the desired result follows from (6.34).

In order to prove (6.34), we argue by contradiction. Then, possibly along a subsequence still denoted f_n, there exist maps $g_n \in \mathscr{E}_{d_2}$ such that

$$|f_n - g_n|_{C^\alpha} = o(n^\alpha) \text{ as } n \to \infty; \tag{6.35}$$

here, we consider the C^α semi-norm

$$|f|_{C^\alpha} := \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^\alpha}; x, y \in \mathbb{S}^N, \ x \neq y \right\}.$$

By (6.11), for each n there exists a point $\mathbf{s} = \mathbf{s}_n$ such that $g_n(\mathbf{s}) = -f_n(\mathbf{s})$. With no loss of generality, we may assume that $f_n(\mathbf{s}) = (0, \ldots, 0, 1)$ and therefore $g_n(\mathbf{s}) = (0, \ldots, 0, -1)$. Let h_n denote the last component of $f_n - g_n$ and let B_n denote the ball of radius $1/n$ centered at $\mathbf{s}$. By (6.35), we have $h_n \geq 2 - o(1)$ in B_n. On the other hand, Lemma 6.6, item *2*, implies that there exists some $\mathbf{t} \in B_n$ such that $f_n(\mathbf{t}) = (0, \ldots, 0, -1)$. It follows that $h_n(\mathbf{t}) \leq 0$. This leads to a contradiction for large n, and thus (6.34) is proved. □

7 Some partial results towards Open Problems 2, 2′ and 3

7.1 Full answer to Open Problem 2′ when $N = 1$ or 2, $1 \le p \le 2$, and $d_1 d_2 \ge 0$

We start with the special cases $[N = 1,\ p = 2]$ and $[N = 2,\ p = 2]$. In this cases, we are able to determine the exact value of $\mathrm{Dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2})$ provided $d_2 > d_1 \ge 0$ (Propositions 7.1, 7.2 and their consequences in Proposition 7.3). This allows us to give a positive answer to Open Problem 2′ when $N = 2$ and $1 \le p \le 2$ under the extra assumption that $d_1 d_2 \ge 0$ (Corollary 7.4).

Proposition 7.1. *Assume that $N = 1$ and $d_2 > d_1 \ge 0$. Let $f(z) = z^{d_1}$, $z \in \mathbb{S}^1$. Then*

$$|f - g|^2_{H^{1/2}} \ge 4\pi^2 (d_2 - d_1),\ \forall g \in \mathscr{E}_{d_2}. \tag{7.1}$$

Proof. We will use the Fourier decomposition of $g \in H^{1/2}(\mathbb{S}^1;\mathbb{S}^1)$, given by $g(e^{\imath\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{\imath n\theta}$. Recall (see e.g. [8]) that the Gagliardo semi-norm (1.16) has a simple form

$$|g|^2_{H^{1/2}} = 4\pi^2 \sum_{n=-\infty}^{\infty} |n|\,|a_n|^2 \tag{7.2}$$

and that for every $g \in H^{1/2}(\mathbb{S}^1;\mathbb{S}^1)$,

$$\deg g = \sum_{n=-\infty}^{\infty} n\,|a_n|^2, \tag{7.3}$$

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = 1. \tag{7.4}$$

By (7.2) we have

$$\begin{aligned}\frac{1}{4\pi^2}|f - g|^2_{H^{1/2}} &= \sum_{\substack{n\in\mathbb{Z}\\ n\ne d_1}} |n|\,|a_n|^2 + d_1 |a_{d_1} - 1|^2 = \sum_{n\in\mathbb{Z}} |n|\,|a_n|^2 + d_1(|a_{d_1} - 1|^2 - |a_{d_1}|^2)\\ &= \sum_{n\in\mathbb{Z}} |n|\,|a_n|^2 + d_1(1 - 2\,\mathrm{Re}\,a_{d_1}) \ge d_2 - d_1,\end{aligned}$$

by (7.3) and (7.4). □

Proposition 7.2. *Assume that $N = 2$ and $d_2 > d_1 \ge 0$. Let $f \in \mathscr{E}_{d_1}$ be defined by $f(\mathbf{s}) = \mathscr{T}^{-1}\Big(\big(\mathscr{T}(\mathbf{s})\big)^{d_1}\Big)$ where $\mathscr{T} : \mathbb{S}^2 \to \mathbb{C}$ is the stereographic projection. Then*

$$|f - g|^2_{H^1} \ge 8\pi(d_2 - d_1),\ \forall g \in \mathscr{E}_{d_2}. \tag{7.5}$$

Proof. Recall that f is a harmonic map and that

$$\int_{\mathbb{S}^2} |\nabla f|^2 = 8\pi d_1; \tag{7.6}$$

see e.g. [6] and the references therein. For any $g \in \mathscr{E}_{d_2}$, write

$$\begin{aligned}|f - g|^2_{H^1} &= \int_{\mathbb{S}^2} |\nabla(f - g)|^2 = \int_{\mathbb{S}^2} |\nabla f|^2 - 2\int_{\mathbb{S}^2} |\nabla g|^2 (g\cdot f) + \int_{\mathbb{S}^2} |\nabla g|^2\\ &\ge \int_{\mathbb{S}^2} |\nabla g|^2 - \int_{\mathbb{S}^2} |\nabla f|^2 = \int_{\mathbb{S}^2} |\nabla g|^2 - 8\pi d_1 \ge 8\pi(d_2 - d_1),\end{aligned}$$

by (7.6) and Kronecker's formula (1.1). □

Proposition 7.3. *Let $d_1, d_2 \in \mathbb{Z}$ be such that $d_2 > d_1 \geq 0$.*

1. *When $N = 1$ we have*

$$\mathrm{Dist}_{H^{1/2}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) = (4\pi^2 |d_1 - d_2|)^{1/2}. \tag{7.7}$$

2. *When $N = 2$ we have*

$$\mathrm{Dist}_{H^1}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) = (8\pi |d_1 - d_2|)^{1/2}. \tag{7.8}$$

Proof. Formula (7.8) follows from (1.2) and (7.5).

On the other hand, (7.7) is a consequence of (7.1) and of the following one dimensional version of (1.2):

$$\text{Given } \varepsilon > 0 \text{ and } f \in \mathscr{E}_{d_1} \text{there exists some } g \in \mathscr{E}_{d_2} \text{such that } |f - g|^2_{H^{1/2}} \leq 4\pi^2 |d_1 - d_2| + \varepsilon. \tag{7.9}$$

Indeed, let $0 < \delta < 1$ and set $h_\delta(z) := \left(\dfrac{z - (1-\delta)}{(1-\delta)z - 1}\right)^{-d}$, with $d := d_1 - d_2$. Then $h_\delta \in \mathscr{E}_{-d}$, and thus $g_\delta := f h_\delta \in \mathscr{E}_{d_2}$. On the other hand, we clearly have $h_\delta \to 1$ a.e. as $\delta \to 0$. We claim that

$$|g_\delta - f|^2_{H^{1/2}} = |h_\delta|^2_{H^{1/2}} + o(1) \text{ as } \delta \to 0. \tag{7.10}$$

Indeed, we start from the identity

$$(g_\delta - f)(x) - (g_\delta - f)(y) = (h_\delta(x) - 1)(f(x) - f(y)) + (h_\delta(x) - h_\delta(y)) f(y),$$

which leads to the inequalities

$$|(g_\delta - f)(x) - (g_\delta - f)(y)| \geq |h_\delta(x) - h_\delta(y)| - |h_\delta(x) - 1|\,|f(x) - f(y)| \tag{7.11}$$

and

$$|(g_\delta - f)(x) - (g_\delta - f)(y)| \leq |h_\delta(x) - h_\delta(y)| + |h_\delta(x) - 1|\,|f(x) - f(y)|. \tag{7.12}$$

By dominated convergence, we have

$$\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{|h_\delta(x) - 1|^2 |f(x) - f(y)|^2}{|x - y|^2} = o(1) \text{ as } \delta \to 0. \tag{7.13}$$

Formula (7.10) is a consequence of (7.11)–(7.13).

Finally, (7.9) follows from (7.10) and the fact that $|h_\delta|^2_{H^{1/2}} = 4\pi^2 |d|$ [1, Corollary 3.2]. □

Corollary 7.4. *Assume that $N = 1$ or 2, $1 \leq p \leq 2$ and $d_1 d_2 \geq 0$. Then*

$$H - \mathrm{dist}_{W^{N/p,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) \geq C'_{p,N} |d_1 - d_2|^{1/p} \tag{7.14}$$

for some constant $C'_{p,N} > 0$.

Proof. We may assume that $d_2 > d_1 \geq 0$, and under this assumption we will prove that

$$\mathrm{Dist}_{W^{N/p,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) \geq C'_{p,N} |d_1 - d_2|^{1/p}. \tag{7.15}$$

The case $N = 1$, $p = 1$ follows from Theorem 3, item *1*.

The case where $N = 1$, $1 < p < 2$ follows from (7.1) and the trivial inequality

$$|f|^2_{H^{1/2}} \leq |f|^p_{W^{1/p,p}} (2\|f\|_{L^\infty})^{2-p}, \ \forall\, 1 < p < 2, \ \forall f.$$

The case where $N = 2$ and $1 \leq p < 2$ follows from (7.5) and the Gagliardo-Nirenberg inequality

$$|f|^2_{H^1} \leq C_{p,N} |f|^p_{W^{2/p,p}} \|f\|^{2-p}_{L^\infty}, \ \forall f.$$

□

7.2 Full answer to Open Problem 2 when $1 \le p \le N+1$ and $d_1 d_2 \le 0$

In this section we prove that the answer to Open Problem 2 is positive when $N \ge 1$, $1 \le p \le N+1$ and $d_1 d_2 \le 0$ (Proposition 7.5). This implies that the answer to Open Problem $2'$ is positive when $N = 1$ or 2 and $1 \le p \le 2$ (Corollary 7.6). We end with a review of some simple cases of special interest which are still open (see Remark 7.7).

Proposition 7.5. *Let $N \ge 1$ and $1 \le p \le N+1$. Let $d_1, d_2 \in \mathbb{Z}$ be such that $d_1 d_2 \le 0$. We have*

$$\mathrm{Dist}_{W^{N/p,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) \ge C'_{p,N} |d_1 - d_2|^{1/p}. \tag{7.16}$$

Proof. We rely on the following estimate, valid when $1 \le p \le N+1$:

$$|\deg f - \deg g| \le C_{p,N} |f-g|^{p/(N+1)}_{W^{N/p,p}} \left(|f|^{Np/(N+1)}_{W^{N/p,p}} + |g|^{Np/(N+1)}_{W^{N/p,p}} \right), \ \forall f, g \in W^{N/p,p}(\mathbb{S}^N; \mathbb{S}^N), \tag{7.17}$$

(see Proposition 7.9 below).

Fix a canonical $f_1 \in \mathscr{E}_{d_1}$ (for example $f_1(z) = z^{d_1}$ when $N = 1$ or the map given by Lemma 6.5 for $N \ge 1$).

This f_1 satisfies

$$|f_1|_{W^{N/p,p}} \le C_{p,N} |d_1|^{1/p}. \tag{7.18}$$

Therefore, with different constants $C_{p,N}$ depending on p and N, but not on d_1 or d_2, we have

$$\begin{aligned} |d_1 - d_2| \le & C_{p,N} |f_1 - g|^{p/(N+1)}_{W^{N/p,p}} \left(|d_1|^{N/(N+1)} + |g|^{Np/(N+1)}_{W^{N/p,p}} \right) \\ \le & C_{p,N} |f_1 - g|^{p/(N+1)}_{W^{N/p,p}} \left(|d_1|^{N/(N+1)} + |f_1|^{Np/(N+1)}_{W^{N/p,p}} + |f_1 - g|^{Np/(N+1)}_{W^{N/p,p}} \right) \\ \le & C_{p,N} |f_1 - g|^{p/(N+1)}_{W^{N/p,p}} \left(|d_1|^{N/(N+1)} + |f_1 - g|^{Np/(N+1)}_{W^{N/p,p}} \right), \ \forall g \in \mathscr{E}_{d_2}. \end{aligned} \tag{7.19}$$

Using (7.19) and the fact that $|d_1| \le |d_1 - d_2|$ (since $d_1 d_2 \le 0$), we find that

$$|f_1 - g|_{W^{N/p,p}} \ge C'_{p,N} |d_1 - d_2|^{1/p}, \ \forall g \in \mathscr{E}_{d_2},$$

whence (7.16). □

Corollary 7.4 and Proposition 7.5 lead to the following

Corollary 7.6. *Assume that $N = 1$ or 2 and $1 \le p \le 2$. Then*

$$H - \mathrm{dist}_{W^{1/p,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) \ge C'_p |d_1 - d_2|^{1/p}, \ \forall d_1, d_2 \in \mathbb{Z},$$

for some constant $C'_p > 0$.

Remark 7.7. We mention here a few cases of special interest not covered by the results in Section 7.1 and 7.2.

1. In view of Propositions 7.3, item *1*, and Proposition 7.5, we know that when $N = 1$ and $p = 2$ we have

   $$\mathrm{Dist}_{H^{1/2}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) \ge C' |d_1 - d_2|^{1/2}, \text{ if either } 0 \le d_1 < d_2 \text{ or } d_1 d_2 < 0. \tag{7.20}$$

   We do not know whether (7.20) holds in the case where $0 < d_2 < d_1$.

2. Let $N = 2$ and $p = 2$. We do not know whether the inequality

   $$\mathrm{Dist}_{H^1}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) \ge C' |d_1 - d_2|^{1/2} \tag{7.21}$$

   (valid when $0 \le d_1 < d_2$ or $d_2 d_1 < 0$ by Proposition 7.3, item *2*, and Proposition 7.5), still holds in the remaining cases. A more precise question is whether (7.21) holds with $C' = (8\pi)^{1/2}$.

7.3 A very partial answer in the general case

Proposition 7.8. *Let $N \ge 1$ and $1 \le p < \infty$. Then for every $d_1 \in \mathbb{Z}$ there exists some C'_{p,d_1} such that*

$$\mathrm{Dist}_{W^{N/p,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) \ge C'_{p,d_1} |d_1 - d_2|^{1/p}, \ \forall d_2 \in \mathbb{Z}. \tag{7.22}$$

Proof.

Step 1. Proof of (7.22) when $d_1 = 0$

Since any constant map belongs to $\mathscr{E}_0$ it suffices to show that

$$\inf_{g \in \mathscr{E}_{d_2}} |g|_{W^{N/p,p}} \ge C'_p |d_2|^{1/p}, \ \forall d_2 \in \mathbb{Z}. \tag{7.23}$$

When $p > N$ we rely on [3, Theorem 0.6]. The case $p = N$ follows from Kronecker's formula (1.1), which leads to

$$C'_N |d_2|^{1/N} \le |g|_{W^{1,N}}, \ \forall g \in \mathscr{E}_{d_2}. \tag{7.24}$$

The case $1 \le p < N$ is a consequence of (7.24) and of the Gagliardo-Nirenberg inequality

$$|g|_{W^{1,N}} \le C |g|^{p/N}_{W^{N/p,p}} \|g\|^{1-p/N}_{L^\infty} = C |g|^{p/N}_{W^{N/p,p}}, \ \forall g \in W^{N/p,p}(\mathbb{S}^N; \mathbb{S}^N).$$

Step 2. Proof of (7.22) when $d_1 \neq 0$

As in the proof of Proposition 7.5, we fix a canonical $f_1 \in \mathscr{E}_{d_1}$ satisfying (7.18).

Next we claim that for every $d_2 \in \mathbb{Z}$, $d_2 \neq d_1$,

$$\inf_{g \in \mathscr{E}_{d_2}} |f_1 - g|_{W^{N/p,p}} = \alpha(d_1, d_2) > 0. \tag{7.25}$$

Indeed, we know from Theorem 2.3 that

$$\inf_{g \in \mathscr{E}_{d_2}} |f_1 - g|_{W^{N/p,p}} = \alpha(f_1, d_2) > 0. \tag{7.26}$$

But since f_1 is a canonical map in $\mathscr{E}_{d_1}$ we obtain (7.25).

Write, with $g \in \mathscr{E}_{d_2}$,

$$|f_1 - g|_{W^{N/p,p}} \ge |g|_{W^{N/p,p}} - |f_1|_{W^{N/p,p}} \ge C'_p |d_2|^{1/p} - C_p |d_1|^{1/p}, \tag{7.27}$$

by (7.23) and (7.18). Clearly

$$C'_p |d_2|^{1/p} - C_p |d_1|^{1/p} \ge \frac{1}{2} C'_p |d_2 - d_1|^{1/p} \tag{7.28}$$

provided $|d_2|$ is sufficiently large, say $|d_2| \ge C(p, d_1)$. Finally we apply (7.25) for all values of d_2, $|d_2| < C(p, d_1)$, $d_2 \neq d_1$, and we obtain

$$\inf_{g \in \mathscr{E}_{d_2}} |f_1 - g|_{W^{N/p,p}} \ge D_{p,d_1} |d_2 - d_1|^{1/p} \tag{7.29}$$

with $D_{p,d_1} > 0$, for every $d_2 \in \mathbb{Z}$, $|d_2| < C(p, d_1)$. Combining (7.27)–(7.29) yields

$$\inf_{g \in \mathscr{E}_{d_2}} |f_1 - g|_{W^{N/p,p}} \ge C'_{p,d_1} |d_1 - d_2|^{1/p}, \ \forall d_2 \in \mathbb{Z},$$

with $C'_{p,d_1} := \min\{(1/2) C'_p, D_{p,d_1}\} > 0$. □

7.4 A partial solution to Open Problem 3

Proposition 7.9. *Assume that $N\ge 1$ and $1\le p\le N+1$. Then*

$$|\deg f-\deg g|\le C_{p,N}|f-g|_{W^{N/p,p}}^{p/(N+1)}\left(|f|_{W^{N/p,p}}^{Np/(N+1)}+|g|_{W^{N/p,p}}^{Np/(N+1)}\right),\ \forall f,g\in W^{N/p,p}(\mathbb{S}^N;\mathbb{S}^N). \tag{7.30}$$

Note that Proposition 7.9 provides a positive answer to Open Problem 3 when $N\ge 1$ and $1\le p\le N+1$.

Proof. Assuming the case $p=N+1$ proved, the other cases follow via Gagliardo-Nirenberg, with the exception of the case $N=1$, $p=1$. However, in that special case estimate (7.30) follows from Theorem 0. We may thus assume that $p=N+1$.

Let F, G denote respectively the harmonic extension of f, g to the unit ball B of $\mathbb{R}^{N+1}$. Then $F,G\in W^{1,N+1}(B;\mathbb{R}^{N+1})$ and (see e.g. [3])

$$\deg f=\fint_B \operatorname{Jac} F,\ \deg g=\fint_B \operatorname{Jac} G. \tag{7.31}$$

Since for any square matrices A, B of size $N+1$ we have

$$|\det A-\det B|\le C\sum_{j=1}^{N+1}\|\operatorname{col}^j(A)-\operatorname{col}^j(B)\|\left(\|A\|^N+\|B\|^N\right), \tag{7.32}$$

we find from (7.31) and (7.32) that

$$|\deg f-\deg g|\le C\,|F-G|_{W^{1,N+1}}(|F|_{W^{1,N+1}}^N+|G|_{W^{1,N+1}}^N). \tag{7.33}$$

Finally, we obtain (7.30) from (7.33) and the estimates

$$|F|_{W^{1,N+1}}\le C\,|f|_{W^{N/(N+1),N+1}} \text{ and } |G|_{W^{1,N+1}}\le C\,|g|_{W^{N/(N+1),N+1}}.$$ □

Appendix. Proofs of some auxiliary results

Let $K:\mathbb{R}\to[0,1]$ be a smooth non increasing function such that

$$K(t)=\begin{cases}1, & \text{if } t\le 1/4\\ 0, & \text{if } t\ge 3/4\end{cases}.$$

Consider the family of radial functions $H_\varepsilon(x)=H_\varepsilon(|x|):\mathbb{R}^N\to[0,1]$, $N\ge 1$, defined by

$$H_\varepsilon(x)=H_\varepsilon(|x|):=\begin{cases}K\left(\dfrac14-\dfrac{1}{2\ln 2}\ln\left(\dfrac{\ln 1/|x|}{\ln 1/\varepsilon}\right)\right), & \text{if } |x|<1\\ 0, & \text{if } |x|\ge 1\end{cases},$$

and ε is a parameter such that $0<\varepsilon<1/e^2$.

The following lemma collects some useful properties of H_ε.

Lemma A.1. The functions H_ε satisfy

$$H_\varepsilon \text{ is smooth on } \mathbb{R}^N,\ \forall\varepsilon. \tag{A.1}$$

$$H_\varepsilon(r)=1,\ \forall\, 0\le r\le\varepsilon,\ \forall\varepsilon. \tag{A.2}$$

$$H_\varepsilon(r)=0,\ \forall\, r\ge\varepsilon^{1/2},\ \forall\varepsilon. \tag{A.3}$$

$$H_\varepsilon(r) \text{ is non increasing on } (0,\infty). \tag{A.4}$$

$$\text{for every } 1<p<\infty,\ \ \|H_\varepsilon(x)\|_{W^{N/p,p}(\mathbb{R}^N)}\to 0 \text{ as } \varepsilon\to 0. \tag{A.5}$$

$$\text{for every } 1<p<\infty \text{ and every } j=1,2,\dots,N,\ \|x_jH_\varepsilon(x)\|_{W^{1+N/p,p}(\mathbb{R}^N)}\to 0 \text{ as } \varepsilon\to 0. \tag{A.6}$$

Lemma A.1 implies in particular that the $W^{s,p}$-capacity of a point in $\mathbb{R}^N$ is zero when $sp \le N$ and $1 < p < \infty$. The above construction is inspired by some standard techniques related to capacity estimates.

Proof. Properties (A.2)–(A.4) are obvious. The smoothness of H_ε is clear (from its definition) in the region $\{|x| < 1\}$. It is even clearer from (A.3) in the region $\{|x| > \varepsilon^{1/2}\}$ and thus H_ε is smooth on $\mathbb{R}^N$ since $\varepsilon^{1/2} < 1$.

Consider the function $f : \mathbb{R}^N \to [0,\infty]$ defined by

$$f(x) = \begin{cases} \ln(\ln 1/|x|), & \text{if } |x| < 1/e \\ 0, & \text{if } |x| \ge 1/e \end{cases}. \tag{A.7}$$

We claim that

$$H_\varepsilon(x) = K(\alpha f(x) + \beta_\varepsilon),\ \forall x \in \mathbb{R}^N, \tag{A.8}$$

where

$$\alpha = -\frac{1}{2\ln 2} \text{ and } \beta_\varepsilon = \frac{1}{4} + \frac{1}{2\ln 2}\ln(\ln 1/\varepsilon).$$

Indeed, (A.8) is clear when $|x| < 1/e$. In the region $|x| \ge 1/e$ we have $H_\varepsilon(x) = 0$ by (A.3) (since $1/e \ge \varepsilon^{1/2}$); on the other hand for such x we have $K(\alpha f(x) + \beta_\varepsilon) = 0$ since $\beta_\varepsilon \ge 3/4$ (again thanks to the property $1/e \ge \varepsilon^{1/2}$).

For the proofs of (A.5) and (A.6) it is convenient to distinguish the cases $N = 1$ and $N \ge 2$.

Case 1: $N = 1$. We must show that

$$|H_\varepsilon(x)|_{W^{1/p,p}(\mathbb{R})} \to 0 \text{ as } \varepsilon \to 0 \tag{A.9}$$

and

$$|x H_\varepsilon(x)|_{W^{1+1/p,p}(\mathbb{R})} \to 0 \text{ as } \varepsilon \to 0. \tag{A.10}$$

We claim that

$$f \in W^{1/p,p}(\mathbb{R}),\ \forall\, 1 < p < \infty. \tag{A.11}$$

Clearly, it suffices to establish that

$$\iint_{0<y<x<e^{-1}} \frac{|f(x) - f(y)|^p}{(x-y)^2}\, dx dy < \infty,\ \forall\, 1 < p < \infty. \tag{A.12}$$

With the change of variables $x = e^{-s}$, $y = e^{-s-t}$, $s > 1$, $t > 0$, inequality (A.12) amounts to

$$\int_0^\infty \int_1^\infty \frac{[\ln(1+t/s)]^p}{(e^{-s} - e^{-s-t})^2} e^{-2s-t}\, dt ds = \int_0^\infty \int_1^\infty \frac{[\ln(1+t/s)]^p}{(e^{t/2} - e^{-t/2})^2}\, dt ds < \infty. \tag{A.13}$$

In order to prove (A.13), we invoke the inequality $\ln(1+t/s) \le t/s$ and the convergence of the integrals $\displaystyle\int_0^\infty \frac{t^p}{(e^{t/2} - e^{-t/2})^2}\, dt$, respectively $\displaystyle\int_1^\infty \frac{1}{s^p}\, ds$.

Next, we deduce from (A.8) that

$$\frac{|H_\varepsilon(x) - H_\varepsilon(y)|^p}{|x-y|^2} \le C \frac{|f(x) - f(y)|^p}{|x-y|^2},\ \forall x, y \in \mathbb{R}. \tag{A.14}$$

Dominated convergence, (A.14) and (A.3) imply that

$$|H_\varepsilon|_{W^{1/p,p}(\mathbb{R})} = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|H_\varepsilon(x)-H_\varepsilon(y)|^p}{|x-y|^2}\,dxdy \to 0 \text{ as } \varepsilon\to 0.$$

In view of (A.9), property (A.10) amounts to

$$|xH_\varepsilon'(x)|_{W^{1/p,p}(\mathbb{R})} \to 0 \text{ as } \varepsilon \to 0. \tag{A.15}$$

Clearly

$$xH_\varepsilon'(x) = |\alpha| \frac{K'(\alpha f(x)+\beta_\varepsilon)}{\ln 1/|x|},\ \forall x\in\mathbb{R}, \tag{A.16}$$

and thus

$$xH_\varepsilon'(x) = |\alpha| \frac{K'(\alpha f(x)+\beta_\varepsilon)}{e^{f(x)}},\ \forall x\in\mathbb{R} \tag{A.17}$$

(note that $xH_\varepsilon'(x)=0$ in the region $|x|\ge 1/e$, while $f(x)=\ln(\ln 1/|x|)$ in the region $|x|<1/e$).

Hence we may write

$$xH_\varepsilon'(x) = Q_\varepsilon(\alpha f(x)+\beta_\varepsilon),\ \forall x\in\mathbb{R}, \tag{A.18}$$

where

$$Q_\varepsilon(t) = |\alpha| \frac{K'(t)}{e^{(t-\beta_\varepsilon)/\alpha}} = \frac{C}{\ln 1/\varepsilon}\frac{K'(t)}{e^{t/\alpha}},\ \forall t\in\mathbb{R}, \tag{A.19}$$

and C is a universal constant. Clearly $K'(t)e^{-t/\alpha}$ belongs to $C_c^\infty(\mathbb{R})$ and thus is Lipschitz. We deduce from (A.11),(A.18) and (A.19) that

$$|xH_\varepsilon'(x)|_{W^{1/p,p}(\mathbb{R})} \le \frac{C}{\ln 1/\varepsilon}|f|_{W^{1/p,p}(\mathbb{R})} \to 0 \text{ as } \varepsilon\to 0.$$

Case 2: $N\ge 2$. We must show that for every $1<p<\infty$,

$$\|H_\varepsilon(x)\|_{W^{N/p,p}(\mathbb{R}^N)} \to 0 \text{ as } \varepsilon\to 0 \tag{A.20}$$

and

$$\|x_j\nabla H_\varepsilon(x)\|_{W^{N/p,p}(\mathbb{R}^N)} \to 0 \text{ as } \varepsilon\to 0. \tag{A.21}$$

We claim that

$$\|H_\varepsilon\|_{W^{1,N}(\mathbb{R}^N)} \le \frac{C}{(\ln 1/\varepsilon)^{(N-1)/N}} \to 0 \text{ as } \varepsilon\to 0 \tag{A.22}$$

and

$$\|H_\varepsilon\|_{W^{N,1}(\mathbb{R}^N)} \le C \text{ as } \varepsilon\to 0. \tag{A.23}$$

Assertion (A.20) with $p>N$ (respectively $p<N$) follows from Gagliardo-Nirenberg, (A.22) and $\|H_\varepsilon\|_{L^\infty}=1$ (respectively Gagliardo-Nirenberg, (A.22) and (A.23)).

For the verification of (A.22) and (A.23) note that

$$|\partial^\gamma H_\varepsilon(x)| \le \frac{C_k}{\ln 1/\varepsilon}\frac{1}{|x|^k}\mathbb{1}_{M_\varepsilon}(x),\ \forall x\in\mathbb{R}^N, \tag{A.24}$$

for every multi-index γ of length $k:=|\gamma|\ge 1$, where

$$M_\varepsilon := \{x\in\mathbb{R}^N;\ \varepsilon<|x|<\varepsilon^{1/2}\}.$$

Assertion (A.21) is proved in a similar manner using the fact that

$$\|x_j\nabla H_\varepsilon(x)\|_{L^\infty(\mathbb{R}^N)} \le \frac{C}{\ln 1/\varepsilon}.$$

□

Proof of Lemma 6.5. We may as well work in a ball B in $\mathbb{R}^N$. We may assume $d > 0$. Fix d points $P_1, \ldots, P_d$ in B. Consider a smooth map $T : \mathbb{R}^N \to \mathbb{S}^N$ such that $T(x) = (1, 0, \ldots, 0)$ when $|x| \ge 1$ and $\deg T = 1$. For large n, let

$$h(x) = \begin{cases} T(n(x - P_j)), & \text{if } |x - P_j| < 1/n \text{ for some } j \\ (1, 0, \ldots, 0), & \text{otherwise} \end{cases}.$$

Clearly, h satisfies properties *1* and *2*. We claim that h also satisfies *3*. Indeed, this is clear for $p = 1$ (by scaling). When $N \ge 2$, the general case follows from Gagliardo-Nirenberg.

When $N = 1$, item *3* still holds, but not the above argument, since we do not have $W^{1,1} \hookrightarrow W^{1/p,p}$ when $1 < p < \infty$. In order to establish item *3* in $W^{1/p,p}$ with $1 < p < \infty$, we fix a small $\delta > 0$. Consider the intervals $I_1, \ldots, I_d$ of length δ centered at $P_1, \ldots, P_d$ and set $I_{d+1} := B \setminus (I_1 \cup \cdots \cup I_d)$. By straightforward calculations, we have, as $n \to \infty$:

$$\int_{I_j} \int_{I_k} \frac{|h(x) - h(y)|^p}{|y - x|^{1+(1/p)p}} \, dx dy = \begin{cases} C_p + o(1), & \text{if } 1 \le j = k \le d \\ o(1), & \text{otherwise} \end{cases}; \tag{A.25}$$

this implies that $|h|^p_{W^{1/p,p}} = C_p \, d + o(1)$ and completes the proof of the lemma when $N = 1$. □

Proof of Lemma 6.6. We may assume that $d_1 \ge 0$. Consider a maximal family $(B_j)_{1 \le j \le J}$ of disjoint balls in $\mathbb{S}^N$ of radius $1/(3n)$. For large n we have $J \ge d_1$. Consider a smooth map $f_n : \mathbb{S}^N \to \mathbb{S}^N$ such that:

1. $f_n = (1, 0, \ldots, 0)$ outside $\cup B_j$.
2. $\deg f_n = 1$ on each $B_1, \ldots, B_{d_1}$.
3. $\deg f_n = 0$ and f_n is onto on each $B_{d_1+1}, \ldots, B_J$.

Then clearly f_n has all the required properties. □

Finally, we present the

Proof of Lemma 2.2. We work on a ball B containing the origin, instead of $\mathbb{S}^N$, and when the given point is the origin. It suffices to establish the conclusion of the lemma when $f \in W^{s,p}(B; \mathbb{R})$ is smooth in $\overline{B}$ and satisfies $f(0) = 0$. By the Sobolev embeddings, we may assume that $1 < p < \infty$ and $s = 1 + N/p$.

Write $f = \sum_{j=1}^N x_j g_j$, with g_j smooth. This is possible since $f(0) = 0$. Then

$$\partial_k [(1 - H_\varepsilon) f - f] = -H_\varepsilon \partial_k f - \sum_{j=1}^N x_j \partial_k H_\varepsilon \, g_j \to 0 \text{ in } W^{N/p,p} \text{ as } \delta \to 0; \tag{A.26}$$

this follows from properties (A.5) and (A.6) of H_ε and from the fact that the multiplication with a fixed smooth function is continuous in $W^{N/p,p}$.

Using (A.26), we immediately obtain that $(1 - H_\varepsilon) f \to f$ in $W^{1+N/p,p}$ as $\varepsilon \to 0$. On the other hand, $(1 - H_\varepsilon) f$ vanishes near the origin. □

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